SOME GENERALIZATIONS OF THE LOGARITHMIC MEAN AND OF SIMILAR MEANS OF TWO VARIATES WHICH BECOME INDETERMINATE WHEN THE TWO VARIATES ARE EQUAL

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1. Introduction. The logarithmic mean m of positive numbers, x and y, as given by

$$m = \frac{y - x}{\log_e y - \log_e x} = \frac{y - x}{\log_e (y/x)} \tag{1}$$

is of considerable importance in problems[1] relating to the flow of heat.

The logarithmic mean arises, moreover, in less technical problems such as the following: Given that incomes t in the interval, $x \leqq t \leqq y$, are distributed with frequency inversely proportional to t. That is, with $k = a$ positive constant,

$$\phi(t)\, dt = (k/t)\, dt \tag{2}$$

is the number of individuals with incomes lying between t and $t + dt$. Then, with $x > 0$, the total number f of individual incomes is

$$f = \int_x^y \phi(t)\, dt = k(\log y - \log x). \tag{3}$$

The combined income g of the group is

$$g = \int_x^y t\phi(t)\, dt = k(y - x). \tag{4}$$

And thus the *logarithmic* mean g/f of the *two* numbers x and y in (1) is the *arithmetic mean of all* the incomes; that is, the *average income*—at least to a close approximation if the group is large enough that integration may replace summation.

Now m in (1) becomes *indeterminate*, if $x = y$. Nevertheless, if $c > 0$, and $x \to c$ and $y \to c$, then $m \to c$. Thus, we may properly speak of m as a mean of these two variates, x and y.

This logarithmic mean is one of a set of means studied by Renzo Cisbani[2], the general form being

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[1] See Walker, Lewis, and McAdams, *Principles of Chemical Engineering*, McGraw Hill & Co., Part IV, Logarithmic mean temperature difference.

[2] R. Cisbani, "Contributi alla teoria delle medie." *Metron*, Vol. 13(1938), pp. 23–34.

(5) $$z = \left[\frac{b^{x+j} - a^{x+j}}{(x/j + 1)(b^j - a^j)}\right]^{1/x}$$

and the logarithmic mean appearing when $x = 1, j \to 0$.

In a chart between pages 28 and 29 Cisbani exhibits thirty varieties of these means (5). It will be noticed that z is *indeterminate* if $a = b$.

Some methods for dealing with means which may become indeterminate forms I have indicated in a recent paper.[3]

Now a generalization from a mean of *two* variates to a mean of three or more variates may sometimes *seem* to be *immediate.* However, for the arithmetic mean $(x + y)/2$ of two variates x and y, the function [min. (x, y, z) + max. (x, y, z)]/2 is as much a generalization as is the arithmetic mean $(x + y + z)/3$. *Actually,* the direction in which generalization is to take place is *arbitrary.* However, it is natural to expect the generalization to arise from a problem somewhat similar to one that may give rise to the original mean. And it is desirable that to the generalization should be carried over as many properties or characteristics of the original as is possible.

In the foregoing illustration, we considered a *single* interval $x \leqq t \leqq y$ in which incomes are distributed in accordance with a *relative* frequency proportional to $\phi(t)$. And the *arithmetic* mean of *all* these incomes was obtained as a *logarithmic* mean of the *two* range limits x and y, at least approximately, allowing integration to take the place of summation. If $\phi(t)$ had been $kt^{-3/2}$, instead of kt^{-1}, then the average of *all* the incomes would have been the *geometric* mean of the *two* range limits x and y.

To effect a *first* generalization, we shall now suppose an original interval x_0 to x_n, to be divided into n subintervals by points x_r such that

(6) $$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n .$$

For each subinterval x_{r-1} to x_r the *same* function $\phi(t)$ will be used to describe the *relative* frequency; but the *total population* for this subinterval will be controlled by a positive constant k_r, in general *different* for the different subintervals. This may be described as *stratification.* To make this more concrete, let us suppose, as before, that $\phi(t) = k/t$. Then, with $x_0 > 0$, the mean M, which will be described more in detail in the next section, will take the form

(7) $$M = \frac{\sum_1^n k_r(x_r - x_{r-1})}{\sum_1^n k_r \log (x_r/x_{r-1})}.$$

Applied to incomes, M would, like m in (1), give average income. To get some idea of the significance of k_r, let us imagine that in some community there are f_r individuals in the income bracket x_{r-1} to x_r, say from \$1001 to \$2000. Let us suppose now that f_r other individuals with incomes between \$1001 and \$2000 distributed in exactly the same manner move into this same community.

[3] "The substitutive mean and certain subclasses of this general mean." *Annals of Math. Stat.*, Vol. 11(1940), pp. 163–176. See p. 171.

Then k_r would be changed to $k_r' = 2k_r$. But, of course, among the entire $2f_r$ individuals the *relative* distribution of incomes is *exactly* the *same* as among the original f_r individuals.

In *this interpretation* k_r is a *weight* for a *bracket of items*. But, taking M in (7) just as it stands, k_r is the *weight* for the *consecutive pair* of *numbers* x_{r-1} and x_r.

2. The first generalization. When t is in some interval, $I = (a, a')$, finite or infinite, let $\phi(t)$ be a non-negative, integrable function of t.

And in I let the points at which $\phi(t) = 0$, if any, form a null-set. Then, with t in I, write

$$\Phi(t) = \int_a^t \phi(t)\, dt. \tag{8}$$

And, supposing that in (6), $a < x_0$, $a_n < a'$, set

$$f_r = \int_{x_{r-1}}^{x_r} \phi(t)\, dt = \Phi(x_r) - \Phi(x_{r-1}); \qquad r = 1, 2, \cdots, n. \tag{9}$$

Then $f_r > 0$; since $\phi(t) > 0$ and is continuous almost everywhere in (x_{r-1}, x_r). Since in any finite subinterval of I, $t\phi(t)$ is integrable, we may set

$$\Psi(t) = \int_a^t \psi(t)\, dt = \int_a^t t\phi(t)\, dt. \tag{10}$$

$$g_r = \int_{x_{r-1}}^{x_r} \psi(t)\, dt = \Psi(x_r) - \Psi(x_{r-1}). \tag{11}$$

Now, by a mean value theorem, there exists a number t_r' such that

$$g_r/f_r = t_r', \qquad x_{r-1} < t_r' < x_r. \tag{12}$$

Taking *positive* numbers k_r, the weighted arithmetic mean of g_r/f_r, with weights $k_r f_r$ is then

$$M = \frac{\sum_1^n k_r g_r}{\sum_1^n k_r f_r} = \frac{\sum_1^n k_r[\Psi(x_r) - \Psi(x_{r-1})]}{\sum_1^n k_r[\Phi(x_r) - \Phi(x_{r-1})]}. \tag{13}$$

If $\phi(t) = k/t$, this becomes the mean (7) associated with the logarithmic mean. Now, since for (13) the weights $k_r f_r$ are *positive*, it follows from (12) that

$$x_0 < t_1' \leqq M \leqq t_n' < x_n. \tag{14}$$

Suppose, now, that b lies in I, and that subject to (6) each $x_r \to b$. Then, by (14), $M \to b$. And thus M is an *internal mean* of $x_0, x_1, \cdots, x_n$, although with the x's all equal, M assumes an indeterminate form.

In (13) the *weights* k_r are applied to *pairs* of numbers, either to $\Psi(x_r) - \Psi(x_{r-1})$ or to $\Phi(x_r) - \Phi(x_{r-1})$, whereas in *most* weighted means, the weights are applied to *individual* numbers. We consider now a form equivalent to (13), but in which the weights c_r are attached to the *individual* numbers. It seemed possible to get a more general mean than (13) by abandoning certain conditions upon the weights c_r which first arose. But such relaxing of restrictions leads to difficulties, as will be shown. By setting

$$(15) \qquad c_0 = -k_1, \qquad c_n = k_n; \qquad c_r = k_r - k_{r+1}, \qquad r = 1, 2, \cdots, n-1,$$

we may write M in the form;

$$(16) \qquad M = \frac{\sum_0^n c_r \Psi(x_r)}{\sum_0^n c_r \Phi(x_r)}.$$

On the other hand, if we choose c's subject to

$$(17) \qquad c_0 < 0, \qquad c_r < -(c_0 + c_1 + \cdots + c_{r-1}) \quad \text{for } 0 < r < n,$$

$$(18) \qquad c_n = -\sum_0^{n-1} c_r;$$

then positive k's can be found to pass from (16) back to (13).

The question arises whether if the conditions (17) are abandoned, and with the c_r not all zero, (18) is retained as

$$(19) \qquad \sum_0^n c_r = 0; \qquad \text{Some } c_r \neq 0,$$

M in (16) will continue to be a mean of $x_0, x_1, \cdots, x_n$, possibly, an external mean.

It may be noted that the condition $\sum c_r = 0$ arises from the fact that when parentheses are removed from (13), each k_r is matched by $-k_r$.

By an example, it will be shown that under (19) alone, M in (16) may fail to be a mean. In (8) and (10) take $a = 0$. Then with $n = 2$, $\phi(t) = t$, take $c_0 = 1$, $c_1 = -2$, $c_2 = 1$ in (16). Then

$$(20) \qquad M = \frac{x_0^2 - 2x_1^2 + x_2^2}{2(x_0 - 2x_1 + x_2)}.$$

If $b > 0$, $\epsilon = x_0 - b$, $\eta = x_1 - b$, and $\xi = x_2 - b$, then

$$(21) \qquad M = b + \frac{1}{2}\,\frac{\epsilon^2 - 2\eta^2 + \xi^2}{\epsilon - 2\eta + \xi}.$$

If now $\eta = 2\epsilon$, and $\xi = 3\epsilon + \epsilon^2$, then

$$(22) \qquad M = b + (2 + 6\epsilon + \epsilon^2)/2 \to b + 1, \qquad \text{as } \epsilon \to 0.$$

Since M does not approach b here, when x_0, x_1, and $x_2 \to b$, in the manner specified, M in (20) is *not* a mean of x_0, x_1, and x_2.

We may enquire, further, whether the function M in (16) could be a mean if, discarding (13), (17) and (18), we put upon c_r the single restriction $c_r > 0$. In that case, if $x_0 < t < x_n$, then, since $\Phi(t)$ and $\Psi(t)$ are continuous functions of t—see (8), (10)—it would follow that if each $x_r \to t$, then $M \to \Psi(t)/\Phi(t)$. But if M is to be a mean of $x_0, x_1, \cdots, x_n$, then $M \to t$ when each $x_r \to t$. Thus we are led to $\Psi(t) = t\Phi(t)$. Except possibly for points of a null set, $\Phi(t)$ and $\Psi(t)$ have derivatives $\phi(t)$ and $\psi(t)$; and thus

$$\psi(t) = \Psi'(t) = t\Phi'(t) + \Phi(t) = t\phi(t) + \Phi(t). \tag{23}$$

But then, since $\psi(t) = t\phi(t)$—see (10)—it would follow that $\Phi(t) = 0$ almost everywhere in I; but $\Phi(t) > 0$, if $t > a$. Hence the assumption $c_r > 0$ is not sufficient to make the function in (16) a mean of $x_0, x_1, \cdots, x_n$.

In the simple case of $n = 1$, M becomes

$$M = \frac{\Psi(x_1) - \Psi(x_0)}{\Phi(x_1) - \Phi(x_0)}; \tag{24}$$

and this is a *symmetrical* function of x_0 and x_1.

The question arises whether if $n > 1$, M in (13) or (16) can be a symmetrical function of $x_0, x_1, \cdots, x_n$. Assume, if possible, that with $x < y < z$,

$$H(x, y, z) \equiv \frac{c_0\Psi(x) + c_1\Psi(y) + c_2\Psi(z)}{c_0\Phi(x) + c_1\Phi(y) + c_2\Phi(z)} \tag{25}$$

is a symmetrical function of x, y and z. Now if $a/b = c/d$, and $b - d \neq 0$, it is well known that $a/b = (a - c)/(b - d)$.

Hence, if $H(x, y, z) = H(z, y, x)$, and $c_0 \neq c_2$, then

$$H(x, y, z) \equiv \frac{(c_0 - c_2)[\Psi(x) - \Psi(z)]}{(c_0 - c_2)[\Phi(x) - \Phi(z)]}, \tag{26}$$

which is not symmetrical in the three variables. Then H is not symmetrical in x, y and z, unless, possibly, when $c_0 = c_2$.

Likewise from $H(x, y, z) = H(x, z, y)$, we are led to the conclusion that H is *not* a symmetrical function of x, y, and z, unless possibly when $c_1 = c_2$. But $c_0 = c_1 = c_2$ substituted into (15) makes $k_1 = k_2 = 0$, which is contrary to hypothesis that $k_r > 0$. Then in (25) the constants c_0, c_1 and c_2 can not be chosen in conformity with (15) so as to make $H(x, y, z)$ a symmetrical function of the *three* variables.

Symmetry in *two* variables will appear, however, if the mean (13) *reduces* to a mean of just *two* variables as it does when each $k_r = k$, constant, in which case,

$$M = \frac{\Psi(x_n) - \Psi(x_0)}{\Phi(x_n) - \Phi(x_0)}. \tag{27}$$

Although in the generalization (13) *symmetry* is thus lost, another property, *homogeneity* is retained in what seem to be the most important cases.

Most means $\Omega(x, y, \cdots, w)$ in common use are *homogeneous* functions of their arguments. That is, if c is a constant, and $\Omega(x, y, \cdots, w)$ and $\Omega(cx, cy, \cdots, cw)$ are both defined when $x, y, \cdots, w$ lie in some interval J, then

$$\Omega(cx, cy, \cdots, cw) = c\Omega(x, y, \cdots, w). \tag{28}$$

This *homogeneity* is associated geometrically with ruled surfaces, in particular with *cones*.

With reference to (8) and (10), let us write

$$F(x, y) = \frac{\Psi(y) - \Psi(x)}{\Phi(y) - \Phi(x)}. \tag{29}$$

And now, let us consider a special variety of means obtained by taking in (8)

$$\phi(t) = t^q, \tag{30}$$

where q is any real number. Then $F(x, y)$ is a homogeneous mean; that is,

$$F(cx, cy) = cF(x, y). \tag{31}$$

This is valid, indeed, even in the special cases, $q = 0, -1$, and -2, which lead, respectively to the arithmetic mean, the logarithmic mean (1) and to a second variety of logarithmic mean

$$m = \frac{xy \log (y/x)}{y - x}, \tag{32}$$

exhibited by Cisbani. It may be noted that $q = -3/2$ leads to the geometric mean, and $q = -3$ to the harmonic mean of x and y.

It is conceivable that for $\phi(t)$ other functions than t^q—functions not equivalent to t^q in integration—might be used to lead to a homogeneous $F(x, y)$ in (29). But such functions, if any, would hardly seem to be in common use.

The M in (13) retains the property of homogeneity, at least for $\phi(t) = t^q$; and so will also the more general means exhibited in the next section.

3. Further generalization. The means of Cisbani (5) suggest the following generalization. Let p be an integer or the reciprocal of an odd integer. With the notation of (13), take $k_r > 0$, and

$$F_p = \textstyle\sum_1^n k_r f_r^p, \qquad G_p = \sum_1^n k_r g_r^p, \tag{33}$$

$$M_p = [G_p/F_p]^{1/p}. \tag{34}$$

Indeed, if in (8) and (10), $a \geqq 0$, then $g_r > 0$; and we may take for p any real number except zero. Now, M_p^p may be described as the weighted arithmetic mean of $(g_r/f_r)^p$ with *positive* weights $k_r f_r^p$. And hence M_p is an *internal* mean of $x_0, x_1, \cdots, x_n$; that is

$$x_0 \leqq M_p \leqq x_n. \tag{35}$$

Furthermore, if in (8), $\phi(t) = t^q$, where q is any real number, then M_p is a homogeneous mean of $x_0, x_1, \cdots, x_n$.

Another generalization may be obtained by writing

$$(36) \qquad m_r = g_r/f_r ,$$

$$(37) \qquad M'_p = [\Sigma k_r m_r^p / \Sigma k_r]^{1/p}.$$

And still another

$$(38) \qquad M'' = [m_1^{k_1} \cdot m_2^{k_2} \cdots m_n^{k_n}]^{1/\Sigma k_r}.$$

These means (37) and (38) are internal; and they are homogeneous, if $F(x, y)$ in (29) is homogeneous.

The foregoing means are not, for $n > 1$, symmetrical functions of $x_1, x_2, \cdots, x_n$. Now the mere abandonment of (6) may lead to functions like (20) which are not means at all. But symmetry may be introduced as follows. First, lay aside (6), but suppose that the x_r are all different. Then let

$$(39) \qquad f_{r,s} = \int_{x_r}^{x_s} \phi(t)\, dt, \qquad g_{r,s} = \int_{x_r}^{x_s} t\phi(t)\, dt;$$

where $r = 0, 1, \cdots, (n - 1)$; $r < s \leqq n$. Then, let

$$(40) \qquad U = \Sigma f_{r,s}^2 , \qquad V = \Sigma g_{r,s}^2 ;$$

where U and V is each a sum of $n(n - 1)/2$ terms: Let W be the double-valued mean

$$(41) \qquad W = \pm[V/U]^{1/2}.$$

Then W is a symmetric function of $x_0, x_1, \cdots, x_n$. If, in (8), $a' \leqq 0$, then in (12) each $g_r/f_r < 0$; and in (41) the negative value of W is an internal mean. But the positive radical is *external*. On the other hand, if $a \geqq 0$; then $g_r/f_r > 0$; and the positive radical in (41) is internal. In this case, it may be well to use for W *only the positive value of* W.

In the more general case where $a < 0$ and $a' > 0$, the fractions g_r/f_r may have different signs. But, in all cases, *at least one* of the two radicals (41) is an *internal* mean of $x_0, x_1, \cdots, x_n$. Moreover, W is homogeneous, if in (8), $\phi(t) = t^q$.

Finally, let

$$(42) \qquad m_{r,s} = g_{r,s}/f_{r,s} ,$$

$$(43) \qquad Z = \pm\{[\Sigma m_{r,s}^2]/n(n - 1)\}^{1/2}.$$

Then Z is symmetric; and at least one value is internal. If $a > 0$, we would naturally take $Z > 0$; and this Z is then an internal mean. Moreover, Z is homogeneous if the $m_{r,s}$ are homogeneous; that is, if $F(x, y)$ in (29) is homogeneous for every x and y in I.