A NOTE ON THE BEHRENS-FISHER PROBLEM

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A commonly occurring problem of statistical inference is the comparison of the means of two normal universes when the ratio of their variances is unknown. Let $(x_1, \cdots, x_m)$ be a random sample from one normal population with mean α and variance μ, and $(y_1, \cdots, y_n)$, a random sample from another with mean β and variance ν. The problem is then that of making statistical inferences about the difference δ of the means, $\delta = \alpha - \beta$, when the ratio μ/ν is unknown. Convenient tests and confidence intervals are available if one can find a linear form L and a quadratic form Q in the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with coefficients independent of the unknown parameters α, β, μ, ν, such that for some positive integer k, the quotient

$$(L - \delta)/(Q/k)^{\frac{1}{2}} \tag{1}$$

has the t-distribution with k degrees of freedom. For this it is sufficient that the following conditions be satisfied for *all* values of the parameters: (*i*) *L and Q are independently distributed*, (*ii*)[1] $E(L) = \delta$, (*iii*) *Q/σ^2 has the χ^2-distribution with k degrees of freedom, where σ^2 is the variance of L.*

In a recent paper [1] the author investigated the Behrens-Fisher problem as delimited by the above three conditions,[2] and among other results arrived at a simple solution. This solution however does not have the property that the quotient (1) is symmetric, whereby in this note we shall mean the following: A function of the samples and parameters will be called symmetric if it is invariant under permutations of the x's among themselves and of the y's among themselves. Let us therefore formulate condition (*iv*): *the quotient* (1) *is symmetric.* Since (*iv*) would be extremely desirable, both for practical and theoretical reasons, and since the author has received several inquiries on this matter, it is considered worth while to outline a proof that conditions (*ii*) and (*iii*) imply that (*iv*) cannot be satisfied, in other words, there exists no "symmetric solution" of the Behrens-Fisher problem within the framework we have imposed. Perhaps this is a simple example of a larger class of problems in which an approach, natural in the light of past developments, forces us to an asymmetric solution.

Suppose (*iv*) is satisfied. By substituting special values for the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ and then making permutations allowed by (*iv*) we find that L and Q must be of the form

$$L = c_1 \sum_i x_i + c_2 \sum_j y_j, \tag{2}$$

$$Q = c_3 \sum_i x_i^2 + c_4 \sum_{i \neq i'} x_i x_{i'} + c_5 \sum_j y_j^2 + c_6 \sum_{j \neq j'} y_j y_{j'} + c_7 \sum_{i,j} x_i y_j, \tag{3}$$

[1] $E(f)$ denotes the expected value of f.

[2] Although these conditions appear simpler than those in [1] they may be shown equivalent.

where all c's are independent of parameters, and the range of indices i, i' is from 1 to m, the range of j, j' from 1 to n.

Condition (*ii*) requires that

$$E(L) = \alpha - \beta. \tag{4}$$

But from (2),

$$E(L) = c_1 m\alpha + c_2 n\beta. \tag{5}$$

Since (4) and (5) must be satisfied identically in α, β, it follows that $c_1 = 1/m$, $c_2 = -1/n$, and thus the variance of L is

$$\sigma^2 = \mu/m + \nu/n. \tag{6}$$

Because of condition (*iii*) we must have $E(Q/\sigma^2) = k$, and combining this with (6), we have

$$E(Q) = k(\mu/m + \nu/n). \tag{7}$$

However, from (3),

$$E(Q) = c_3 m(\mu + \alpha^2) + c_4(m^2 - m)\alpha^2 + c_5 n(\nu + \beta^2) + c_6(n^2 - n)\beta^2 + c_7 mn\alpha\beta. \tag{8}$$

Equating (7) and (8) gives us an identity in α, β, μ, ν from which we can determine the c's, and after putting these back in (3) we find that the result may be written

$$Q = k[S_x/(m^2 - m) + S_y/(n^2 - n)], \tag{9}$$

where

$$S_x = \sum_i (x_i - \bar{x})^2, \qquad \bar{x} = \sum_i x_i/m,$$

$$S_y = \sum_j (y_j - \bar{y})^2, \qquad \bar{y} = \sum_j y_j/n.$$

The last step of the proof consists of showing that Q defined by (9) violates (*iii*). Write $u_1 = S_x/\mu$, $u_2 = S_y/\nu$. Then u_1 and u_2 have independent χ^2-distributions. Now (9) states that $u = Q/\sigma^2$ is of the form $u = a_1u_1 + a_2u_2$, where a_1 and a_2 are constants. Let $\phi(t)$, $\phi_1(t)$, $\phi_2(t)$ be the respective characteristic functions of u, u_1, u_2. Then $\phi(t) = \phi_1(a_1t)\phi_2(a_2t)$ because u_1 and u_2 are statistically independent. Since the characteristic function of a χ^2-variable with r degrees of freedom is $(1 - 2it)^{-\frac{1}{2}r}$, it is evident that u has a χ^2-distribution if and only if $a_1 = a_2 = 1$. From (9),

$$a_1/a_2 = [\mu/(m^2 - m)]/[\nu/(n^2 - n)],$$

and thus a necessary condition (it is also sufficient) for Q/σ^2 to have a χ^2-distribution is

$$\mu/\nu = (m^2 - m)/(n^2 - n). \tag{10}$$

But (*iii*) states that Q/σ^2 has a χ^2-distribution for *all* parameter values. This contradiction completes the proof.

We remark in closing that we have at hand a counter example of practical interest to the statement found in several statistics texts that if z is a normal variable with zero mean and v is an independent unbiased quadratic estimate of the variance of z, then $z/v^{\frac{1}{2}}$ has a t-distribution. The counter example consists of taking $z = \bar{x} - \bar{y} - \delta$ and $v = Q/k$ defined by (9). It may be shown that $z/v^{\frac{1}{2}}$ does not have a t-distribution except in the trivial case (10).

REFERENCE

[1] H. SCHEFFÉ, *Annals of Math. Stat.*, Vol. 15 (1943), pp. 35–44.

ON MULTIPLE MATCHING WITH ONE VARIABLE DECK

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The problem of card matching has been considered by a number of writers. A complete bibliography has been given by Battin [1], who also published the most general treatment of the subject to date, dealing with the simultaneous matching of any number of decks of arbitrary composition. He considers, however, only the case in which the order of every deck is variable, all possible permutations being equally likely. Some interest attaches to the case in which all the decks but one have fixed orders in relation to one another, especially in connection with radio experiments in telepathy, where a large number of subjects simultaneously attempt to call the same target.

The simplest case is that in which the target for each trial is chosen at random, independently of the other trials. If the target is to be selected from s possibilities, and if p_i denotes the probability that the ith possibility will be chosen as the target, while m_i denotes the number of subjects who call the ith possibility, then the mean value of h, the number of correct calls is, of course,

$$M_h = \sum_{i=1}^{s} p_i m_i , \tag{1}$$

and the variance is

$$V_h = \sum_{i=1}^{s} p_i m_i^2 - M_h^2 . \tag{2}$$

Evidently, the mean number of hits for a succession of trials is the sum of the means for the individual trials, and the variance is the sum of the variances.

A slightly more difficult problem is presented when the target series is a true "deck": that is, when its composition is determined in advance, only the order being left to chance. Let n denote the number of trials and $n_i (i = 1, 2, \cdots , s)$