ON AN EXTENSION OF THE CONCEPT OF MOMENT WITH APPLICATIONS TO MEASURES OF VARIABILITY, GENERAL SIMILARITY, AND OVERLAPPING[1]

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1. Introduction. Given a frequency distribution D: $[X_i, F_i]$ $(i = 1, 2, 3, \cdots, n)$, we shall call the expression

$$M_r(D, X_j) = \sum_{i=1}^{n} (X_i - X_j)^r F_i$$

the rth total moment of D about the origin X_j. We shall consider the weighted sum

$$\mathfrak{M}_r = \Sigma_j W_j M_r(D, X_j)$$

where W_j denotes the weight corresponding to the particular origin X_j, and the summation is over a field ϕ. In particular, if ϕ is the set of all values assumed in D by the variate X_i, and if $W_j = F_j$, we shall call the quantity the rth *complete total moment* of D. If, on the contrary, W_j is the frequency F_j' of the value X_j' in a second frequency distribution D': $[X_j', F_j']$ and ϕ' is the set of all values assumed by the variate X_j' in D', $\mathfrak{M}_r$ will be called the rth *aggregate moment* of D and D'. A modification of this procedure leads to what we shall call the *moment of transvariation* of D and D'.

The consideration of complete moments draws attention to certain previously known measures of variability which are independent of the origin selected, and also provides simple methods of computation which are useful for data given in the form of a frequency distribution. The investigation of aggregate moments and moments of transvariation gives rise to certain measures of general similarity between two distributions, as well as measures of the amount of overlapping.

2. Sliding and complete moments of a frequency distribution.

2.1. We shall give the name *sliding total moments* of order r to the successive values, for particular values of j, of the expression

$$M_r(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j)^r F_i]. \tag{2.11}$$

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[1] The Portuguese original of this paper was written in Brazil, in August 1943. Its translation into English was entirely revised by Dr. T. Greville, Bureau of the Census, who proposed also many simplifications in the derivation of formulae. For his painstaking labor and interest I wish to express my very sincere appreciation. I also wish to thank Dr. W. Edwards Deming for reading the manuscript and making several valuable suggestions.

The expression for the complete total moment, written out in full, is

$$\mathfrak{M}_r = \sum_{j=1}^{n} M_r(X_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} [(X_i - X_j)^r F_i F_j]. \tag{2.12}$$

It is readily seen that the complete moment is independent of the choice of origin.

2.2. If $r = 0$, we have

$$M_0(X_j) = F_j \sum_{i=1}^{n} F_i .$$

The complete total moment of order zero will therefore be

$$\mathfrak{M}_0 = \sum_{j=1}^{n} F_j \sum_{i=1}^{n} F_i = M_0^2 \tag{2.21}$$

where M_0 stands for the total moment of order zero about the origin of the X', that is,

$$M_0 = N\nu_0' .$$

2.3. If $r = 1$, we shall have

$$M_1(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j) F_i].$$

Using M_1 to denote the total moment of order one about the origin of the X, we obtain

$$M_1(X_j) = F_j \sum_{1}^{n} X_i F_i - X_j F_j \sum_{1}^{n} F_i = F_j M_1 - X_j F_j M_0 .$$

Making j vary from 1 to n and summing, we have

$$\begin{aligned} \mathfrak{M}_1 &= \sum_{j=1}^{n} F_j M_1 - \sum_{j=1}^{n} X_j F_j M_0 \\ &= M_0 M_1 - M_1 M_0 = 0. \end{aligned} \tag{2.31}$$

This result is due to the fact that we took the deviations $X_i - X_j$ with their proper signs. We may, however, calculate the value which the complete moment of first order would have if using absolute values. Thus, the sliding total moment thus modified becomes

$$|M_1(X_j)| = F_j \left[\sum_{i=1}^{j-1} (X_j - X_i) F_i + \sum_{i=j}^{n} (X_i - X_j) F_i \right]$$

which may be put in the form

$$|M_1(X_j)| = F_j X_j \left[\sum_{i=1}^{j-1} F_i - \sum_{i=j}^{n} F_i \right] - F_j \left[\sum_{i=1}^{j-1} F_i X_i - \sum_{i=j}^{n} F_i X_i \right]. \tag{2.32}$$

Summing with respect to j and employing the substitutions

$$\sum_{i=j}^{n} F_i = M_0 - \sum_{i=1}^{j-1} F_i \tag{2.33}$$
$$\sum_{i=j}^{n} F_i X_i = M_1 - \sum_{i=1}^{j-1} F_i X_i$$

gives for the complete total moment

$$|\mathfrak{M}_1| = 2\sum_{j=1}^{n}\left[F_j X_j \sum_{i=1}^{j-1} F_i\right] - 2\sum_{j=1}^{n}\left[F_j \sum_{i=1}^{j-1} F_i X_i\right]. \tag{2.34}$$

The quotient

$$m_1 = \frac{|\mathfrak{M}_1|}{\mathfrak{M}_0} \tag{2.35}$$

of the complete total moment of order one by the complete total moment of order zero we shall call the *complete unit moment* of order one, or simply the complete moment of order one, when no confusion would result.

The complete unit moment is a measure of variability, identical with that already considered by Andrae and Helmert, respectively in 1869 and in 1876, and which C. Gini, in 1912, called mean difference with repetition.[2]

The numerator of m_1 is easily computed if we observe that the upper limit $j - 1$ of the F_i summation, for example, means that each product $X_j F_j$ must be multiplied by the cumulative frequency corresponding to the class immediately preceding. We only have to shift the cumulative frequencies column by one class in the proper direction; the second term is similarly dealt with.

2.4. The second order sliding total moment is

$$M_2(X_j) = F_j \sum_{i=1}^{n} [(X_i - X_j)^2 F_i] = F_j M_2 - 2F_j X_j M_1 + F_j X_j^2 M_0$$

where M_2 is the total moment of order two. Summing with respect to j gives the complete total moment of order two

$$\mathfrak{M}_2 = \sum_{j=1}^{n} M_2(X_j) = 2(M_2 M_0 - M_1^2). \tag{2.41}$$

The complete unit moment of order two is therefore

$$\begin{aligned} m_2 &= 2\left[\frac{M_2}{M_0} - \left(\frac{M_1}{M_0}\right)^2\right] = 2(\nu_2' - \nu_1'^2) \\ &= 2\sigma^2 \end{aligned} \tag{2.42}$$

[2] Apud Czuber, *Wahrscheinlichkeitsrechnung*, Vol. 2, (1932), p. 316. C. Gini, *Variabilità e Mutabilità*, Cagliari, 1912.

where ν' stands for a unit moment about the origin of the X, namely

$$\nu'_r = \frac{\Sigma X^r F}{\Sigma F},$$

m_2 is also a measure of variability, independent of the choice of origin. It is equal to the square of Gauss's "Präzisionsmass", and to the double of Fisher's variance; like m_1 it was defined by Andrae and Helmert, and was called by Gini the mean square difference with repetition.

2.5. If $r = 3$ we have for the sliding moments,

$$M_3(X_j) = F_j \sum_{i=1}^{n} (X_i - X_j)^3 F_i$$
$$= F_j M_3 - 3F_j X_j M_2 + 3F_j X_j^2 M_1 - F_j X_j^3 M_0 .$$

Summation over j gives

$$\text{(2.51)} \quad \mathfrak{M}_3 = \sum_{j=1}^{n} M_3(X_j) = M_0 M_3 - 3M_1 M_2 + 3M_2 M_1 - M_3 M_0 = 0,$$

a result which is easily shown to hold for any complete moment of odd order. We may calculate the value of the complete moment of order three using absolute values of the deviations $X_i - X_j$ by a process similar to that previously described for the calculation of $|\mathfrak{M}_1|$. This gives

$$\text{(2.52)} \quad |\mathfrak{M}_3| = 2\Bigg[\sum_{j=1}^{n} F_j X_j^3 \sum_{i=1}^{j-1} F_i - 3 \sum_{j=1}^{n} F_j X_j^2 \sum_{i=1}^{j-1} F_i X_i + 3 \sum_{j=1}^{n} F_j X_j \sum_{i=1}^{j-1} F_i X_i^2 - \sum_{j=1}^{n} F_j \sum_{i=1}^{j-1} F_i X_i^3\Bigg].$$

2.6. The sliding moments of order four are

$$M_4(X_j) = F_j M_4 - 4F_j X_j M_3 + 6F_j X_j^2 M_2 - 4F_j X_j^3 M_1 + F_j X_j^4 M_0 .$$

Summing with respect to j and simplifying, we have

$$\text{(2.61)} \quad \mathfrak{M}_4 = M_0 M_4 - 4M_1 M_3 + 6M_2^2 - 4M_3 M_1 + M_4 M_0$$
$$= 2(M_0 M_4 - 4M_1 M_3 + 3M_2^2).$$

Dividing both sides by $\mathfrak{M}_0$ in order to obtain the complete moment on a unit basis, we have

$$m_4 = 2\left[\frac{M_4}{M_0} - 4\frac{M_1}{M_0}\frac{M_3}{M_0} + 3\left(\frac{M_2}{M_0}\right)^2\right] = 2\,(\nu'_4 - 4\nu'_1\nu'_3 + 3\nu'^2_2).$$

But, if ν indicates a moment about the mean

$$\nu_4 = \nu'_4 - 4\nu'_1\nu'_3 + 6\nu'^2_1\nu'_2 - 3\nu'^4_1.$$

By substitution, therefore

$$
\begin{aligned}
m_4 &= 2(\nu_4 + 3\nu_2'^2 - 6\nu_1'^2\nu_2' + 3\nu_1'^4) \\
&= 2[\nu_4 + 3(\nu_2' - \nu_1'^2)^2] \qquad (2.62)\\
&= 2(\nu_4 + 3\nu_2^2).
\end{aligned}
$$

This complete moment gives rise to a measure of kurtosis independent of the choice of origin

$$k = \frac{m_4}{m_2^2} = \frac{\nu_4}{2\nu_2^2} + \frac{3}{2}.$$

In case of mesokurtosis this reduces to 3, since for the normal curve $\nu^4/\nu_2^2 = 3$; leptokurtosis and platikurtosis occur for the same ranges as in the case of Pearson's measure β_2.

3. Aggregate moments of two frequency distributions.

3.1. Given two frequency distributions, D:$[X_i, F_i](i = 1, 2, 3, \cdots, n)$ and D': $[X_j', F_j'](j = 1, 2, 3, \cdots, p)$ and a fixed point X_j' belonging to the second distribution, we shall call the expression

$$M_r(D, X_j') = F_j' \sum_{i=1}^{n} (X_i - X_j')^r F_i \qquad (3.11)$$

the rth *aggregate sliding total moment* of the first distribution about the element X_j' of the second. Summation over j gives

$$^c\mathfrak{M}_r = \sum_{j=1}^{p} \sum_{i=1}^{n} F_j'(X_i - X_j')^r F_i. \qquad (3.12)$$

We shall call $^c\mathfrak{M}_r$ the *aggregate complete total moment* or, simply, the aggregate total moment of D about D'. It is clear that this is a symmetric function of the two distributions, except for a change of sign in the case of odd moments.

3.2. If $r = 0$, we have

$$M_0(D, X_j') = F_j' \sum_{i=1}^{n} F_i \qquad (3.21)$$

$$^c\mathfrak{M}_0 = \sum_{j=1}^{p} F_j' \sum_{i=1}^{n} F_i = M_0 M_0'. \qquad (3.22)$$

3.3. If $r = 1$, we have

$$M_1(D, X_j') = F_j' M_1 - F_j' X_j' M_0 \qquad (3.31)$$

$$^c\mathfrak{M}_1 = M_1 M_0' - M_0 M_1'. \qquad (3.32)$$

We shall call the quotient

$$^c m_1 = \frac{^c\mathfrak{M}_1}{^c\mathfrak{M}_0} \qquad (3.33)$$

the aggregate *unit* moment of order r (or the aggregate moment coefficient), or simply the aggregate moment of order r whenever the simpler name will not cause confusion.

It is obvious that the aggregate moments are measures of general similarity, as to form and position, between D and D'. This similarity will be an identity in case the two distributions coincide perfectly; on the other hand, it is clear that there is no limit to the degree of non-similarity which may be encountered. We shall take unity to represent the maximum and zero the minimum of similarity, and thus define a provisional similarity index

$$S = \frac{m_1 m_1'}{{}^c m_1^2}. \tag{3.34}$$

But

$${}^c m_1 = \frac{M_1 M_0' - M_0 M_1'}{M_0 M_0'} = A - A'$$

where A and A' stand for the arithmetic means of D and D', respectively. Now it will be seen that if $A = A'$, $S = \infty$. This result is due to the fact that in the calculation of m_1 and m_1' we took the absolute values of the deviations $X_i - X_j'$, while in the calculation of ${}^c m_1$ we retained the algebraic signs. In order to make the two terms of the fraction in (3.34) comparable, we can either: 1) calculate ${}^c m_1$ also using absolute values; or 2) take only the positive or only the negative part of both numerator and denominator of S. In any case, $A = A'$ is a necessary condition for the maximum of S.

3.4. We shall employ the first method suggested above, although we shall return to the second in the third part of the paper. As long as D and D' do not overlap, all the $X_i - X_j'$ deviations have the same sign and this is the same as that of the difference $A - A'$. If, however, there is some overlapping this will not be the case, some deviations having different signs from that of $A - A'$. This brings us to Gini's concept of "transvariation". He applies this term to any deviation $X_i - X_j'$ which does not have the same sign as $\bar{X} - \bar{X}'$, these symbols denoting averages of any previously specified type; and he calls the magnitude of the deviation its "intensity".

In computing the complete moment of the first order using absolute values, in order to simplify the algebra we shall assume the same origin for X and X' and therefore drop the stroke from the X, but not of course from the F. If certain values of X occur in one distribution and not in the other, we can merely consider the frequency as zero in the second distribution. In this way the two distributions can be regarded as extending over the same total range. If X_1 and X_m denote the extreme values, the sliding total moment is

$$\begin{aligned} |M_1(D, X_j)| &= F_j'\left[\sum_{i=1}^{j-1}(X_j - X_i)F_i + \sum_{i=j}^{m}(X_i - X_j)F_i\right] \\ &= F_j' X_j\left(\sum_{i=1}^{j-1}F_i - \sum_{i=j}^{m}F_1\right) - F_j'\left(\sum_{i=1}^{j-1}F_iX_i - \sum_{i=j}^{m}F_iX_i\right). \end{aligned}$$

Summing with respect to j and at the same time employing the substitutions (2.33) or their transposed form, we obtain the following alternative expressions for the complete aggregate moment:

$$(3.41)\quad |{}^c\mathfrak{M}_1| = M_1M_0' - M_0M_1' + 2\sum_{j=1}^{m}\left[F_j'X_j\sum_{i=1}^{j-1}F_i\right] - 2\sum_{j=1}^{m}\left[F_j'\sum_{i=1}^{j-1}F_iX_i\right]$$

$$(3.42)\quad |{}^c\mathfrak{M}_1| = M_0M_1' - M_1M_0' - 2\sum_{j=1}^{m}\left[F_j'X_j\sum_{i=j}^{m}F_i\right] + 2\sum_{j=1}^{m}\left[F_j'\sum_{i=j}^{m}F_iX_i\right].$$

Note the similarity of the first of these forms to formula (2.34) which is in fact a particular case of formula (3.41). Alternatively, we may obtain from formula (3.42) the particular case

$$(2.34a)\qquad |\mathfrak{M}_1| = 2\sum_{j=1}^{n}\left(F_j\sum_{i=j}^{n}F_iX_i\right) - 2\sum_{j=1}^{n}\left(F_jX_j\sum_{i=j}^{n}F_i\right)$$

which is equivalent to (2.34).

If the two distributions do not overlap, $|{}^c\mathfrak{M}_1|$ does not differ numerically from ${}^c\mathfrak{M}_1$. Let us consider the case in which there is actual overlapping, the range of non-zero frequencies extending from X_1 to X_{n+p} for D and from X_{n+1} to X_m for D'. Then formula (3.42) becomes, upon merely dropping all vanishing terms

$$(3.43)\quad \begin{aligned} |{}^c\mathfrak{M}_1| &= M_0M_1' - M_1M_0' \\ &\quad - 2\sum_{j=n+1}^{n+p}\left[F_j'X_j\sum_{i=n+1}^{j-1}F_i\right] + 2\sum_{j=n+1}^{n+p}\left[F_j'\sum_{i=j}^{n+p}F_iX_i\right]. \end{aligned}$$

On the other hand, formula (3.41) reduces, under the same circumstances, to a much less simple expression, which upon making the substitutions (2.33) and simplifying reduces to

$$(3.44)\quad \begin{aligned} |{}^c\mathfrak{M}_1| &= M_0M_1' - M_1M_0' + 2\sum_{j=n+1}^{n+p}\left[F_j'X_j\sum_{i=n+1}^{j-1}F_i\right] \\ &\quad - 2\sum_{j=n+1}^{n+p}\left[F_j'\sum_{i=j}^{n+p}F_iX_i\right] \\ &\quad - 2\sum_{j=n+1}^{n+p}F_j'X_j\sum_{i=n+1}^{n+p}F_i + 2\sum_{j=n+1}^{n+p}F_j'\sum_{i=n+1}^{n+p}F_iX_i. \end{aligned}$$

This result may be arrived at somewhat more easily by merely making the substitutions (2.33) directly in formula (3.43). It may be noted that formula (3.44) at once reduces to the form (2.34) if the two distributions are identical, since the additional terms all cancel. It is, however a less satisfactory result than formula (3.43) because of the larger number of terms it contains. In order to obtain a formula which resembles (2.34) more closely, we may reverse the order of summation in formula (3.43). Observing that the terms for $j = i$ collectively vanish, we see that

$$
\begin{aligned}
|{}^c\mathfrak{M}_1| &= M_0 M_1' - M_1 M_0' \\
&\quad - 2 \sum_{i=n+1}^{n+p} \left[F_i \sum_{j=n+1}^{i-1} F_j' X_j \right] + 2 \sum_{i=n+1}^{n+p} \left[F_i X_i \sum_{j=n+1}^{i-1} F_j' \right].
\end{aligned} \tag{3.45}
$$

It will be seen that the simple method of numerical computation described in section 2.8 is immediately applicable to all the formulas (3.41) to (3.45). Dividing any of these expressions by ${}^c\mathfrak{M}_0$ gives $|{}^c m_1|$. For example, if formula (3.43) is used, we have

$$
\begin{aligned}
|{}^c m_1| &= A' - A \\
&\quad - \frac{2}{M_0 M_0'} \left\{ \sum_{j=n+1}^{n+p} \left[F_j' X_j \sum_{i=j}^{n+p} F_i \right] - \sum_{i=j}^{n+p} \left[F_j' \sum_{i=j}^{n+p} F_i X_i \right] \right\}.
\end{aligned} \tag{3.46}
$$

Substituting this value in equation (3.34), we have

$$S_1 = \frac{m_1 m_1'}{|{}^c m_1|^2} \tag{3.47}$$

a quantity which we shall call the "mean coefficient of similarity."

We now observe that S_1 is a general measure of similarity whose magnitude is affected by differences in either form or position. It may, however, be desirable to eliminate the position element, in order to isolate the form aspect. To do this it will suffice to relate the value which $|{}^c m_1|$ would have for $A = A'$, to the product $m_1 m_1'$. This value of $|{}^c m_1|$ is, in fact, its minimum; denoting it by ${}^c\mu_1$ we obtain the index

$$\mathfrak{S}_1 = \frac{m_1 m_1'}{{}^c\mu_1^2} \tag{3.48}$$

which we shall call the mean similarity ratio.

It is clear that all the above mentioned indices measure overlapping as well as similarity. Overlapping between two distributions will be greatest when their similarity is greatest, or when $|{}^c m_1|$ is a minimum. In order to bring out more clearly the overlapping aspect we may follow Gini's procedure of contrasting the actual value of a measure with its maximum value. As already pointed out, if the form of the two distributions is held constant, but their relative position is varied, the degree of overlapping, as measured by the mean similarity ratio, is greatest when the arithmetic means coincide. This method of procedure is embodied in the index

$$\mathfrak{T}_1 = \frac{{}^c\mu_1}{{}^c m_1} \tag{3.49}$$

which we shall call the "intensity of transvariation or overlapping." To calculate ${}^c\mu_1$ we may, for example, merely add the difference $A' - A = c$ to the X

values, in order to move D along the X-axis a distance of c, and then proceed to calculate $|\,{}^c m_1\,|$ in the usual manner from the adjusted X values.

3.5. If, in (3.11), $r = 2$, we have

$$M_2(D, X_j) = F'_j \sum_{i=1}^{n} (X_i - X_j)^2 F_i$$
$$= F'_j M_2 - 2X'_j F'_j M_1 + X'^2_j F'_j M_0 .$$

Summing for j then gives

$$(3.51) \qquad {}^c\mathfrak{M}_2 = M'_0 M_2 - 2M'_1 M_1 + M'_2 M_0 .$$

If we define the second aggregate unit moment as

$${}^c m_2 = \frac{{}^c\mathfrak{M}_2}{{}^c\mathfrak{M}_0}$$

then

$$(3.52) \qquad \begin{aligned} {}^c m_2 &= \frac{M_2}{M_0} - 2\frac{M_1}{M_0}\frac{M'_1}{M_0} + \frac{M'_2}{M_0} \\ &= \sigma^2 + \sigma'^2 + (A - A')^2, \end{aligned}$$

where the σ and the A stand for the standard deviations and the arithmetic means of the respective distributions. Now we define the "mean square coefficient of similarity" as the value of

$$(3.53) \qquad \begin{aligned} S_2 &= \frac{m_2 m'_2}{{}^c m_2^2} \\ &= \frac{4\sigma^2\sigma'^2}{[\sigma^2 + \sigma'^2 + (A - A')^2]^2}. \end{aligned}$$

It is obvious that a minimum value of S_2 requires that $A = A'$ as a necessary condition for the maximum degree of overlapping. Maximum similarity requires, in addition, $\sigma = \sigma'$, in which case $S_2 = 1$.

For a measure of similarity which is independent of difference in position between the two distributions, we define.

$$(3.54) \qquad \mathfrak{S}_2 = \frac{m_2 m'_2}{{}^c\mu_2^2},$$

where ${}^c\mu_2$ is the minimum value of ${}^c m_2$ for all positions of the two distributions, without changing their form. This is obtained by merely taking

$$(3.55) \qquad {}^c\mu_2 = \sigma^2 + \sigma'^2.$$

For a measure of overlapping we can follow Gini in contrasting the actual value of ${}^c m_2$ with its minimum ${}^c\mu_2$, since the maximum of overlapping corresponds to the minimum value of ${}^c m_2$. We thus se.

$$\mathfrak{T}_2 = \frac{{}^c\mu_2}{{}^c m_2} = \frac{\sigma^2 + \sigma'^2}{\sigma^2 + \sigma'^2 + (A - A')^2} \tag{3.56}$$

a measure which we shall call the "density of overlapping". Its maximum value is unity.

It may be remarked that all the indices proposed in this paragraph are easier to calculate than those of paragraph 3.4. The individual terms are all functions of only one of the two distributions; yet the resulting indices are independent of the origin chosen, and therefore free from any criticism based on doubt as to the representativeness of the arithmetic mean, in cases of marked skewness.

4. Positive and negative moments, and moments of transvariation.

4.1. The aggregate sliding total moment of two frequency distributions D and D' may be expressed in the form

$$M_r(D, X'_j) = F'_j \sum_{i=1}^{j-1} (X_i - X_j)^r F_i + F'_j \sum_{i=j+1}^{m} (X_i - X_j)^r F_i \tag{4.11}$$

when both distributions have been artificially extended, if necessary, to cover the same total range, as previously described in section 3.4. We shall characterize the second term in the right member of (4.11) as the positive sliding moment, and the absolute value of the first term as the negative sliding moment. We shall denote these moments by ${}^+M_r(D, X_j)$ and ${}^-M_r(D, X_j)$. The complete moments obtained by summing these separate terms over the range of values of j we shall call the positive and negative aggregate complete moments. Thus the positive complete moment is

$${}^{+c}\mathfrak{M}_r = \sum_{j=1}^{m} \left[F'_j \sum_{i=j+1}^{m} (X_i - X_j)^r F_i \right] \tag{4.12}$$

and the negative complete moment is

$${}^{-c}\mathfrak{M}_r = \sum_{j=1}^{m} \left[F'_j \sum_{i=1}^{j-1} (X_j - X_i)^r F_i \right]. \tag{4.13}$$

That one of these two partial moments which is obtained from differences $X_i - X'_j$ having the opposite sense to that of the difference $\bar{X} - \bar{X}'$ will be called the moment of transvariation of the two distributions and will be denoted by the symbol ${}^T\mathfrak{M}_r$. Here, as in section 3.4, $\bar{X}$ and $\bar{X}'$ denote averages of any previously selected type. For example, if the arithmetic means are the averages selected, and if $A - A'$ is positive, then the negative aggregate moment is the moment of transvariation, and vice-versa.

In the trivial case in which the two distributions are identical, the positive and negative complete moments are equal, and both reduce to merely one half the aggregate complete moment (computed by the use of absolute values in the case of moments of odd order).

The unit moment of transvariation will be defined as

$$ {}^{T}m_r = \frac{{}^{T}\mathfrak{M}_r}{{}^{T}\mathfrak{M}_0} \tag{4.14} $$

4.2. It is evident that the moments of transvariation can be considered as measures of overlapping. Any such moment equals zero when there is no overlapping and becomes greatest when the two distributions coincide. Taking unity to represent the maximum and zero the minimum of overlapping, we may choose as a general measure of overlapping,

$$ T_r = \frac{4\,{}^{T}m_r^2}{|m_r|\,|m_r'|} = \frac{4\,{}^{T}\mathfrak{M}_r^2}{|\mathfrak{M}_r|\,|\mathfrak{M}_r'|}. \tag{4.21} $$

It will be seen that this quantity always equals zero when there is no overlapping, and equals unity when there is complete overlapping: that is when the two distributions are identical.

5. Need for further developments. All of the measures above described were defined for the case of finite sets of magnitudes, expressed as frequency distributions D and D'. Now these sets of magnitudes may be thought of as samples drawn out of their corresponding universes. The consideration of these universes would lead to more general representations under the form of frequency functions, and the above measures would be expressed as definite integrals rather than summations. This draws attention to the need for tests of significance of the magnitude of all the above measures, especially those of overlapping, in order to allow for sampling fluctuation. Obviously, when the frequency functions are of the asymptotic type some amount of overlapping will always exist.