Finally, it is interesting to point out the relation of this note to some work on the problem of finding upper bounds to the roots. In fact, the inequalities $\lambda \leqq N(A)$ and $\lambda \leqq R(A)$, which are consequences of (6), are Theorem 2 of Farnell [4] and Theorem 3 of Barankin [5] respectively.

REFERENCES

[1] HAROLD HOTELLING, "Some new methods in matrix calculation," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 1–34.
[2] T. RELLA, "Über den absoluten Betrag von Matrizen," International Congress of Mathematicians at Oslo (1936).
[3] FRANZ L. ALT, "Multiplication of matrices," *Math. Tables and Aids to Compution*, Vol. 2 (1946), pp. 12–13.
[4] A. B. FARNELL, "Limits for the characteristic roots of a matrix," *Bull. Amer. Math. Soc.*, Vol. 50 (1944), pp. 789–794.
[5] EDWARD W. BARANKIN, "Bounds for the characteristic roots of a matrix," *Bull. Amer. Math. Soc.*, Vol. 51 (1945), pp. 767–770.

DEFINITION OF THE PROBABLE DEVIATION

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The probable deviation has recently been defined by E. J. Gumbel [1], [2] as the smallest of the intervals corresponding to the probability $\frac{1}{2}$. It so happened that the author was led to an equivalent definition starting from a general idea which may be applied to absolutely general cases and which, for this reason, might be of interest.

In recent years, the author has been occupied with a study of random elements of any nature (curves, surfaces, functions, qualitative elements), a study whose future seems promising, [3]. I gave a definition of the mean of such an element expressed by an abstract integral which, however, is only defined if the random element is situated in a metric vectorial (Wiener-Banach) space.[1] But[2] a still more general definition is valid if the random element is placed in any metric space. It consists of taking, as mean position of the random element X, a fixed (non-statistical) element $b = \bar{X}$ such that the function of a which represents the mean $M(X, a)^2$ of the squared distance of X to the fixed element a, is minimum for $a = b$. (In the case where X and a are numbers, and where $M(X)^2$ is finite, we know that this minimum is reached and that there is one, and only one, determination b of a). This definition has the advantage of also defining the equiprobable position of X. This is a fixed element $c = \bar{\bar{X}}$ such that $M(X, a)$ is minimum for $c = a$. (If X and a are numbers, we know that this minimum is still reached, but may be so reached by several values of $\bar{\bar{X}}$).

Since reading Gumbel's paper, a still more general definition suggested itself.

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[1] For the definition of metric vectorial spaces see [4].
[2] See Note 2, p. 503 of [4].

The expressions $M(X, a)$ and $\sqrt{M(X, a)^2}$ themselves may be considered as distances, but as distances of two random elements *taken together*. To each of these distances corresponds as minimum, when a varies, a different "typical" function $\bar{\bar{X}}$ or $\bar{X}$ $\cdots$. Thus, without supposing anything about the space into which the different trials place X, we assume that we have defined a "deviation" of two random elements X, Y taken together. We represent this function of two random variables by $(([X], [Y]))$, a notation which differs from the representation of the distance (X, Y) of the two positions X and Y with respect to a single trial. The lower boundary of the deviation $(([X], [a]))$, a function of a, which is reached for $a = \widehat{X}$ defines a "typical" position $\widehat{X}$. Moreover, the value of this $(([X], [\widehat{X}]))$ may be considered as a measure or, at least, as a numerical ranging point of the dispersion of X.

Let us abandon these generalities. They hold especially if the element X is a real valued random variable. Among the possible and reasonable[3] expressions for the deviation $(([X], [a]))$ of the numerical variate X from a fixed number a, we may use the equiprobable value of $| X - a |$ which may be called the equiprobable deviation of X from a. Thus we have, on one side, a new "typical value" of X which will be a value of a such that the equiprobable deviation of X from a is minimum, and a new measure of dispersion which is the value of this minimum and which might be called simply the equiprobable deviation of X.

In the case where X has everywhere a continuous and finite density of probability $w(X)$ we find, as typical value, what Gumbel calls the "midvalue" and represents by $\mathfrak{z}$, and, as equiprobable deviation, what Gumbel calls the "probable deviation" and represents by $\mathfrak{z}$.

We may also consider the discontinuous case, which was given as a problem to candidates of the "Certificat d'Etudes Supérieures de Calcul des Probabilités, Option Statistique Mathématique, Session May-June, 1944." They had to solve various questions of which I cite the beginning below:

"Consider n real numbers $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ and represent, by E_a, a median value of the deviations $| x_k - a |$ of the numbers x_k and a. If a varies, E_a has a minimum E which is reached by one or several values A of a.

1) Explain, in a few words, the meaning of the values E and A.

2) For simplicity's sake, suppose that n is odd $(n = 2r + 1)$. How should E and A be calculated practically? (To find the answer, investigate first how E_a varies if a varies only slightly).

3) In the case where $n = 4s + 3$ (s is an integer equal to, or larger than, zero) show that $E \leqq \frac{(q_3 - q_1)}{2}$

where $$q_1 = x_{s+1}, \quad q_3 = x_{n-s}."$$

The study of this new typical value and of this new equiprobable deviation has the advantage that their determination is very rapid and requires hardly

[3] See the *Remark* at end of note.

any calculations. However, we have to note an important inferiority of the equiprobable deviation of X compared to the mean and the standard deviations of X. If one or the other of the last two deviations is zero, X is a fixed number (except for the case of the probability zero). This property seems requested by the intuitive meaning which we attribute to the dispersion, and to every measure or any mark of it. Now, the equiprobable deviation lacks this property. If, for instance, X has only three values: 0, 2, 1, the first two with the probability 0.249, and the last with the probability 0.502, the equiprobable deviation of X will be zero, whereas X will be equal to its typical value 1 only with a probability of 0.502, and not with a probability equal to unity. The same holds for any distribution for which there is a point with probability exceeding $\frac{1}{2}$.

Remark. The definitions of the mean and of the equiprobable position become meaningless in the case that $M(X, a)$, or $M(X, a)^2$, is infinite. However, we succeeded in surmounting the difficulty, and to reach definitions which are valid even in this case. If X is a number, the new definitions become equivalent to the classical definitions of the mean and equiprobable value. The proofs are given in two recent articles [5], [6].

REFERENCES

[1] E. J. Gumbel, "Definition of the probable error," *Annals of Math. Stat.*, Vol. 13 (1942)' p. 110–111.

[2] E. J. Gumbel, "Probable deviation," *Stat. Jour.*, City College, (N. Y.), Vol. 6 (1943), pp. 25–26.

[3] M. Fréchet, "L'intégrale abstraite d'une fonction abstraite d'une variable abstraite et son application à la moyenne d'un élément aléatoire de nature quelconque," *Revue Scientifique*, Vol. 82 (1944), pp. 483–512.

[4] M. Fréchet, *Les Espaces Abstraites*, Gauthier-Villars, Paris, 1928, pp. 125–141.

[5] M. Fréchet, "Les éléments aléatoires de nature quelconque," *Ann. Inst. H. Poincaré*, 1947.

[6] M. Fréchet, "Nouvelles définitions de la valeur moyenne et des valeurs probables d'un nombre aléatoire," *Ann. Univ. de Lyon*, 1947.

THE GENERAL RELATION BETWEEN THE MEAN AND THE MODE FOR A DISCONTINUOUS VARIATE

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Dr. Gumbel has pointed out that one of the author's arguments employed in several particular cases (see [1]) can be employed in a general case which includes them and leads to the following result: If a statistical variate R has only positive entire values differing from zero, and if its mean value $\bar{R}$ is smaller than, or equal to, unity, the same holds for its equiprobable value $\bar{\bar{R}}$ and its mode $\tilde{R}$. There are two generalizations of this result which might be of interest:

1) On the one hand, the author has shown [2] that, if a variate R can only have values (entire or not) equal to, or larger than, zero, its equiprobable value