the same transformation gives

$$\mathbf{M}(\Phi) = \int \mathbf{A}^* \mathbf{x}^* \mathbf{x} \mathbf{A} p\{\mathbf{x}'\}\, d\mathbf{x}',$$

$$= \mathbf{A}^* \mathbf{M}(\Phi') \mathbf{A}$$

which for $A = \Phi^{-\frac{1}{2}}$ leaves us with

$$\mathbf{M}(\Phi) = (\Phi')^{-1}$$

because $\mathbf{M}(1) = 1$.

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THE DISTRIBUTION OF A DEFINITE QUADRATIC FORM

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Let $X_1, \cdots, X_n$ be independent normal variates with zero means and unit variances, let $a_1, \cdots, a_n$ be positive constants and define

(1) $$U_n = \frac{a_1}{2} X_1^2 + \cdots + \frac{a_n}{2} X_n^2,$$

(2) $$F_n(x) = \Pr [U_n \leq x], \qquad f_n(x) = F_n'(x).$$

Setting

(3) $$a = (a_1 \cdots a_n)^{1/n}$$

and using the convolution formula we may show by induction that for $x > 0$,

(4) $$f_n(x) = a^{-\frac{1}{2}n} x^{\frac{1}{2}n-1} \sum_{k=0}^{\infty} \frac{c_k(-x)^k}{\Gamma(\frac{1}{2}n + k)},$$

(5) $$F_n(x) = a^{-\frac{1}{2}n} x^{\frac{1}{2}n} \sum_{k=0}^{\infty} \frac{c_k(-x)^k}{\Gamma(\frac{1}{2}n + k + 1)},$$

where for $k = 0, 1, \cdots$

(6) $$c_k = \pi^{-\frac{1}{2}n} \sum_{i_1+\cdots+i_n=k} \frac{\Gamma(i_1 + \frac{1}{2}) \cdots \Gamma(i_n + \frac{1}{2})}{i_1! \cdots i_n!\, a_1^{i_1} \cdots a_n^{i_n}} > 0.$$

In particular, if $a_1 = \cdots = a_n = 2$, then using the known distribution of χ^2 with n degrees of freedom we have

$$\bar{f}_n(x) = \frac{x^{\frac{1}{2}n-1}e^{-\frac{1}{2}x}}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} = \frac{x^{\frac{1}{2}n-1}}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}\sum_{k=0}^{\infty}\frac{(-x)^k}{2^k k!} = \frac{x^{\frac{1}{2}n-1}}{2^{\frac{1}{2}n}}\sum_{k=0}^{\infty}\frac{\bar{c}_k(-x)^k}{\Gamma(\frac{1}{2}n+k)},$$

so that

$$\bar{c}_k = \frac{\Gamma(\frac{1}{2}n+k)}{2^k k!\,\Gamma(\frac{1}{2}n)} = \frac{\pi^{-\frac{1}{2}n}}{2^k}\sum_{i_1+\cdots+i_n=k}\frac{\Gamma(i_1+\frac{1}{2})\cdots\Gamma(i_n+\frac{1}{2})}{i_1!\cdots i_n!},$$

and therefore

$$\sum_{i_1+\cdots+i_n=k}\frac{\Gamma(i_1+\frac{1}{2})\cdots\Gamma(i_n+\frac{1}{2})}{i_1!\cdots i_n!} = \frac{\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+k)}{k!\,\Gamma(\frac{1}{2}n)}. \tag{7}$$

Now in the general case let

$$\alpha = \min\{a_1, \cdots, a_n\}; \tag{8}$$

then from (6) and (7) we deduce that

$$\left|\frac{c_k(-x)^k}{\Gamma(\frac{1}{2}n+k)}\right| \le \frac{(x/\alpha)^k}{\Gamma(\frac{1}{2}n)k!}, \tag{9}$$

with a similar inequality for the general term of (5).

It is difficult to evaluate numerically the coefficient c_k by a direct application of the definition (6). We shall therefore give a method by which the c_k may be computed easily. We shall assume in what follows that the a_i are distinct.

Let $Y_1, \cdots, Y_n$ be another set of variates with the same joint distribution as the X_i and independent of the X_i, and set

$$V_{2n} = \frac{a_1}{2}X_1^2 + \cdots + \frac{a_n}{2}X_n^2 + \frac{a_1}{2}Y_1^2 + \cdots + \frac{a_n}{2}Y_n^2, \tag{10}$$

$$G_{2n}(x) = \Pr[V_{2n} \le x], \qquad g_{2n}(x) = G'_{2n}(x). \tag{11}$$

Then by the convolution formula,

$$g_{2n}(x) = \int_0^x f_n(x-y)f_n(y)\,dy = a^{-n}x^{n-1}\sum_{k=0}^{\infty}\left\{\sum_{i=0}^{k}c_i c_{k-i}\right\}\frac{(-x)^k}{\Gamma(k+n)}. \tag{12}$$

But we may show directly that, setting

$$q_i = \prod_{j\ne i}(a_j - a_i)^{-1} \qquad (i = 1, \cdots, n), \tag{13}$$

we have

$$g_{2n}(x) = (-1)^{n-1}\sum_{i=1}^{n}q_i a_i^{n-2}e^{-x/a_i} = (-1)^{n-1}\sum_{k=0}^{\infty}\left\{\sum_{i=1}^{n}q_i a_i^{n-k-2}\right\}\frac{(-x)^k}{k!}. \tag{14}$$

Equating coefficients in (12) and (14) we derive the fundamental formula

$$\sum_{i=0}^{k}c_i c_{k-i} = a^n\sum_{i=1}^{n}q_i a_i^{-(k+1)} \qquad (k = 0, 1, \cdots). \tag{15}$$

Thus, defining

$$2P_k = a^n \sum_{i=1}^{n} q_i a_i^{-(k+1)}, \tag{16}$$

we may write

$$\sum_{i=0}^{k} c_i c_{k-i} = 2P_k. \tag{17}$$

From (6) or (17) it follows that

$$c_0 = 1. \tag{18}$$

Thus we may solve (17) successively for the c_k in terms of the P_k: for $j = 0, 1, \cdots$

$$\begin{aligned} c_{2j} &= P_{2j} - \left\{ c_1 c_{2j-1} + c_2 c_{2j-2} + \cdots + c_{j-1} c_{j+1} + \frac{c_j^2}{2} \right\}, \\ c_{2j+1} &= P_{2j+1} - \{c_1 c_{2j} + c_2 c_{2j-1} + \cdots + c_j c_{j+1}\}. \end{aligned} \tag{19}$$

When the n constants $q_1, \cdots, q_n$ have been computed we may compute the P_k by (16) and then the c_k by (19) successively as far as desired. The inequality (9) gives an indication of the number of terms of the alternating series (4) or (5) which are necessary to secure a desired accuracy. A sharper result than (9) should certainly be possible when the a_i are well separated.

The foregoing method may be modified to cover cases in which some of the a_i are equal, the formulas (16) being replaced by the appropriate limits as the a_i approach equality.

We shall now derive an expansion of $f_n(x)$ and $F_n(x)$ in terms of χ^2 distributions. Let us set for $x > 0$,

$$f_n(x) = \sum_{k=0}^{\infty} (-1)^k d_k \cdot \frac{x^{\frac{1}{2}n+k-1} e^{-x/a}}{a^{\frac{1}{2}n+k} \Gamma(\frac{1}{2}n + k)}, \tag{20}$$

or, equivalently,

$$\begin{aligned} \frac{a}{2} f_n\left(\frac{a}{2} x\right) &= \sum_{k=0}^{\infty} (-1)^k d_k \frac{x^{\frac{1}{2}n+k-1} e^{-\frac{1}{2}x}}{2^{\frac{1}{2}n+k} \Gamma(\frac{1}{2}n + k)} \\ &= \frac{x^{\frac{1}{2}n-1} e^{-\frac{1}{2}x}}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)} \cdot \sum_{k=0}^{\infty} (-1)^k d_k \frac{(\frac{1}{2}x)^k \Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n + k)}, \end{aligned} \tag{21}$$

where the d_k are to be determined. It follows, after some reduction, that

$$g_{2n}(x) = a^{-n} x^{n-1} e^{-x/a} \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} d_i d_{k-i} \right\} \frac{(-x)^k}{a^k \Gamma(\frac{1}{2}n + k)}. \tag{22}$$

But we may write (14) in the form

$$g_{2n}(x) = a^{-n} x^{n-1} e^{-x/a} \sum_{k=0}^{\infty} \left\{ \sum_{i=1}^{n} a q_i a_i^{n-2-k} (a - a_i)^k \right\} \frac{(-x)^{k-n+1}}{a^{k-n+1} \Gamma(k + 1)}. \tag{23}$$

Equating coefficients in (22) and (23) and setting

$$2Q_k = a \sum_{i=1}^{n} q_i a_i^{-(k+1)} (a - a_i)^{n+k-1}, \tag{24}$$

we obtain the relations $d_0 = 1$ and

$$\sum_{i=0}^{k} d_i d_{k-i} = 2Q_k \qquad (k = 0, 1, \cdots), \tag{25}$$

from which the d_k may be computed as in (19). Equation (20) or (21) then gives the expansion of $f_n(x)$ in a series of χ^2 frequency functions. The corresponding expansion of $F_n(x)$ is then

$$F_n(x) = \sum_{k=0}^{\infty} (-1)^k d_k \cdot \int_0^x \frac{t^{\frac{1}{2}n+k-1} e^{-t/a}}{a^{\frac{1}{2}n+k} \Gamma(\frac{1}{2}n + k)} dt, \tag{26}$$

or

$$F_n\left(\frac{a}{2} x\right) = \sum_{k=0}^{\infty} (-1)^k d_k \cdot \int_0^x \frac{t^{\frac{1}{2}n+k-1} e^{-t/2}}{2^{\frac{1}{2}n+k} \Gamma(\frac{1}{2}n + k)} dt. \tag{27}$$

By direct comparison of (4) and (20) we may establish the following relations among the c_k and d_k:

$$\begin{aligned} d_k &= (-1)^k \sum_{j=0}^{k} (-a)^j \binom{\frac{1}{2}n + k - 1}{k - j} c_j, \\ c_k &= a^{-k} \sum_{j=0}^{k} \binom{\frac{1}{2}n + k - 1}{k - j} d_j. \end{aligned} \tag{28}$$

These may be used if both the power series and the χ^2 series are desired.

From (6) we see directly that

$$c_1 = \frac{1}{2} \sum_{i=1}^{n} a_i^{-1}, \tag{29}$$

and from (28) it follows that

$$d_1 = \tfrac{1}{2} a n b_1, \tag{30}$$

where we have set

$$b_1 = \left\{ \frac{1}{n} \sum_{i=1}^{n} a_i^{-1} \right\} - (a_1 \cdots a_n)^{-1/n}. \tag{31}$$

Since by a well known inequality $b_1 \geq 0$ it follows that $d_1 \geq 0$, with equality only if all the a_i are equal. If we denote by $h_n(x)$ the frequency function of $\frac{1}{2}a(X_1^2 + \cdots + X_n^2)$ then

$$\frac{a}{2} h_n\left(\frac{a}{2} x\right) = \frac{x^{\frac{1}{2}n-1} e^{-\frac{1}{2}x}}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)}, \tag{32}$$

and hence (21) becomes

$$\frac{f_n(x)}{h_n(x)} = 1 - b_1 x + \cdots, \tag{33}$$

which is significant for x near 0.

EXACT LOWER MOMENTS OF ORDER STATISTICS IN SMALL SAMPLES FROM A NORMAL DISTRIBUTON

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1. Summary. Exact means in samples of size ≤ 3, and exact second moments and product-moments in samples of size ≤ 4, are given in Table 1 in terms of π for order statistics selected from the normal distribution $N(0, 1)$. The derivation employs multiple integration and some general properties of the moments.

TABLE 1

Expected values of lower moments of order statistics, $x_i \geq x_{i+1}$, in samples of size n from the normal distribution N(0, 1).

Moment	$n = 2$	$n = 3$	$n = 4$
$E[x_1]$	$1/\sqrt{\pi}$	$3/(2\sqrt{\pi})$	
$E[x_2]$		0	
$E[x_1^2]$	1	$1 + \sqrt{3}/(2\pi)$	$1 + \sqrt{3}/\pi$
$E[x_2^2]$		$1 - \sqrt{3}/\pi$	$1 - \sqrt{3}/\pi$
$E[x_1x_2]$	0	$\sqrt{3}/(2\pi)$	$\sqrt{3}/\pi$
$E[x_1x_3]$		$-\sqrt{3}/\pi$	$-(2\sqrt{3} - 3)/\pi$
$E[x_1x_4]$			$-3/\pi$
$E[x_2x_3]$			$(2\sqrt{3} - 3)/\pi$
σ_1^2	$1 - 1/\pi$	$1 - (9 - 2\sqrt{3})/(4\pi)$	
σ_2^2		$1 - \sqrt{3}/\pi$	
σ_{12}	$1/\pi$	$\sqrt{3}/(2\pi)$	
σ_{13}		$(9 - 4\sqrt{3})/(4\pi)$	

2. Introduction. The usefulness of the lower moments of order statistics for determining the moments of the range and for other purposes is well established. In small samples, however, computation of the moments by quadrature is laborious [1]. The values shown in Table 1 should therefore be helpful in problems requiring the use of these moments for samples of size ≤ 4, since the constant π has been evaluated to several hundred decimal places. Some of the methods used to obtain these results may also be useful in approximating or verifying the moments in larger samples.