(3.6) $$P(t; \theta, \varphi; \theta', \varphi') = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \exp(-k(k+1)t) Y_k^{(m)}(\theta, \varphi) Y_k^{(m)}(\theta', \varphi')$$

is absolutely and uniformly convergent on S^3. We will show that this P is the required (unique) Brownian motion on S^3.

The proof may be given in three steps. i) We see by (3.2) and (3.6), that $\int_{S_3} f(y)P(t, y, x)\, dx$ satisfies (2.15) if

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{m=-k}^{k} d_k^{(m)} Y_k^{(m)}(x), \qquad \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \exp(-k(k+1)t)k(k+1)\, d_k^{(m)} Y_k^{(m)}(x)$$

are both absolutely and uniformly convergent. By the completeness of $\{Y_k^{(m)}(x)\}$, such $f(x)$ are dense in $L_1(S)$.

ii) Because of (3.3) we see that (3.6) satisfies the spacial homogeneity (1.4).

iii) (1.3) is obvious by the orthonormality of $\{Y_k^{(m)}(x)\}$ and the constancy on S^3 of $Y_0^{(0)}(x)$. Next, for the solution $f(t, x)$ of (2.15)–(2.16), let $f(x) = f(0, x)$ be non-negative on S^3, then $g_\epsilon(t, x) = \exp(-\epsilon t)f(t, x)$, $(\epsilon > 0)$, satisfies

$$\frac{\partial g_\epsilon(t, x)}{\partial t} = \Lambda \cdot g_\epsilon(t, x) - \epsilon g_\epsilon(t, x), \qquad (t > 0),$$

$$g_\epsilon(0, x) = f(x) \geqq 0 \qquad \text{(on } S^3).$$

Thus $g_\epsilon(t, x) \geqq 0$ on S^3, since $g_\epsilon(t, x)$ cannot have a negative minimum on the product space $[t_1, t_2] \times S^3$, for any $t_2 > t_1 > 0$. For at such minimizing point we must have

$$\frac{\partial g_\epsilon}{\partial t} = 0, \qquad \frac{\partial g_\epsilon}{\partial \theta} = 0, \qquad \frac{\partial g_\epsilon}{\partial \varphi} = 0, \qquad \frac{\partial^2 g_\epsilon}{\partial \theta^2} \geqq 0, \qquad \frac{\partial^2 g_\epsilon}{\partial \varphi^2} \geqq 0.$$

Therefore, since $\epsilon > 0$, $t_2 > t_1 > 0$ were arbitrary, we conclude that $f(t, x) \geqq 0$ on S^3 for $t > 0$ if $f(x) = 0$ on S^3. This proves (1.2). The same argument simultaneously shows us that the solution P of (2.15)–(2.16) and (1.2)–(1.3) is unique.

ON THE STRONG STABILITY OF A SEQUENCE OF EVENTS

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1. Summary. M. Loève [3] has found conditions under which a sequence of events which may be interdependent in an arbitrary manner is strongly stable. In this note it is established that considerably weaker conditions imply the strong stability.

2. Introduction. Let

(1) $$A_1, A_2, \cdots, A_n, \cdots$$

be a sequence of events, which may depend on each other in any way whatsoever, defined on the same set of trials.

Let R_n be the *repetition* function of (1), i.e. R_n is the number of those among the first n events: $A_1, A_2, \cdots, A_n$ which were realized, and put $f_n = R_n/n$. The random variable f_n is called the *frequency* function of (1).

Denoting by $E\{x\} = \bar{x}$ the expected value of x it is evident that

$$\bar{R}_n = E\{R_n\} = \sum_{\nu=1}^{n} \Pr(A_i), \qquad \bar{f}_n = E\{f_n\} = \frac{1}{n} E\{R_n\}.$$

Following Loève [3, p. 252] we say that (1) is *strongly stable* if the sequence $\varphi_n = f_n - \bar{f}_n$ $(n = 1, 2, \cdots)$ is strongly stable in the usual Kolmogoroff sense [1, p. 58], i.e. if

$$\lim_{n\to\infty} \Pr(\sup_{\nu > n} |\varphi_\nu| > \epsilon) = 0 \tag{2}$$

for every $\epsilon > 0$.

Putting[1]

$$\beta_n = \frac{1}{n}\sum_{i=1}^{n} \Pr(A_i), \qquad \gamma_n = \frac{2}{n(n-1)} \sum_{1\le\mu<\nu\le n} \Pr(A_\mu A_\nu)$$

and introducing the abbreviation[2]

$$\delta_n = \gamma_n - \beta_n^2,$$

Loève's result [3, pp. 257–9] is the following:

If $n\delta_n$ is bounded then (1) *is strongly stable.*

This, even when specialized to sequences of independent events, includes the Bernoulli and Poisson cases.

Here the following stronger result will be established.

THEOREM. *If $\Sigma\, \delta_n/n$ is convergent then* (1) *is strongly stable.*

In particular, if for some $\epsilon > 0$ the sequence $n^\epsilon \delta_n$ is bounded then (1) is strongly stable.

3. A lemma. The new tool here used is the following simple result on series of positive terms.

LEMMA. *Let $a_n \ge 0$ for $n = 1, 2, \cdots$ and*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \tag{3}$$

be convergent. Then there exists a sequence n_i of integers satisfying

$$0 < n_{i+1} - n_i = o(n_i) \qquad (i \to \infty), \tag{4}$$

and such that the series $\Sigma_{i=1}^{\infty} a_{n_i}$ is convergent.

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[1] $A_\mu A_\nu$ denotes the event: both A_μ and A_ν.

[2] Our β_n, γ_n and δ_n correspond to Loève's $p_1(n)$, $p_2(n)$ and d_n^2 respectively.

PROOF. Since (3) is convergent it is well known[3] that there exists a sequence of numbers $l_n (n = 1, 2, \cdots)$ satisfying

$$l_{n+1} \geq l_n, \qquad \lim_{n=\infty} l_n = \infty \tag{5}$$

having the property that

$$\sum_{n=1}^{\infty} l_n \frac{a_n}{n} < \infty. \tag{6}$$

We define inductively a sequence of integers $m(i)$ through

$$m(1) = 1, \qquad m(i+1) = m(i) + 1 + \left[\frac{m(i)}{l_{m(i)}}\right], \tag{7}$$

the square brackets denoting the integral part. Clearly

$$0 < m(i+1) - m(i) = o(m(i)). \tag{8}$$

Now for every i we choose n_i so that

$$m(i) \leq n_i < m(i+1) \quad \text{and} \quad a_{n_i} = \min_{m(i) \leq \nu < m(i+1)} a_\nu .$$

These n_i satisfy the requirements of the lemma.

Indeed, (4) holds in virtue of (8) while applying (5) and (7) we obtain

$$s_i = \sum_{\nu=m(i)}^{m(i+1)-1} l_\nu \frac{a_\nu}{\nu} \geq (m(i+1) - m(i)) l_{m(i)} \frac{a_{n_i}}{m(i+1)} \geq \frac{m(i)}{m(i+1)} a_{n_i} .$$

Since $\Sigma\, s_i$ converges by (6) it follows from the preceding inequality and (8) that $\Sigma\, a_{n_i} < \infty$ as required.

COROLLARY. *The conclusion of the lemma remains valid if the condition $a_n \geq 0$ is dropped provided* (3) *is absolutely convergent.*

4. Proof of the theorem. An easy calculation [3, p. 253] gives

$$\sigma_n^2 = E\{(f_n - \bar{f}_n)^2\} = \delta_n + \frac{\beta_n - \gamma_n}{n}.$$

Since both β_n and γ_n are between zero and one we have

$$-\frac{1}{n} < \sigma_n^2 - \delta_n < \frac{1}{n}.$$

Therefore it follows from the assumption of the theorem that $\Sigma\, (\sigma_n^2/n)$ is convergent. Hence by the lemma there exists a sequence of integers n_i satisfying (4) and such that $\Sigma\, \sigma_{n_i}^2$ converges.

[3] Take e.g. $l_n = (\Sigma_{\nu > n}\, \nu^{-1} a_\nu)^{-1/2}$ (cf. [2, p. 299]).

Applying Tchebytcheff's inequality to $\varphi_{n_\nu} = f_{n_\nu} - \bar{f}_{n_\nu}$ and adding for $\nu \geq i$ we have for every $\epsilon > 0$

$$\Pr\left(\sup_{\nu \geq i} |\varphi_{n_\nu}| > \epsilon\right) \leq \frac{1}{\epsilon^2} \sum_{\nu=i}^{\infty} \sigma_{n_\nu}^2. \tag{9}$$

If $n_i \leq n < n_{i+1}$ then

$$|f_n - f_{n_i}| = \left| \frac{R_n}{n} - \frac{R_{n_i}}{n_i} \right| < \frac{n_{i+1} - n_i}{n_i}.$$

Denoting the last term of this inequality by ϵ_i and putting $\bar{\epsilon}_i = \max_{\nu \geq i} \epsilon_\nu$, we have from (9)

$$\Pr\left(\sup_{n \geq n_i} |\varphi_n| > \epsilon + 2\bar{\epsilon}_i\right) \leq \frac{1}{\epsilon^2} \sum_{\nu=1}^{\infty} \sigma_{n_\nu}^2.$$

As $\bar{\epsilon}_i \to 0$ and the right hand term is the remainder of a convergent series, (2) follows and the theorem is proved.

5. Remarks. 1. The lemma used here can also be applied to the study of the order of magnitude of φ_n in the almost certain sense.

2. If the terms of (3) are decreasing then the existence of a convergent subseries of $\Sigma\, a_n$ satisfying (4) implies $\Sigma_{i=1}^{\infty} a_{2^i} < \infty$. But this is equivalent to the convergence of the series with monotone terms (3) (cf. e.g. [2, p. 130]). Hence in this case the convergence of (3) is *necessary as well as sufficient* for the validity of the lemma. It may be possible to use this remark in order to establish in some special cases, where the interdependence of the variables decreases steadily in a suitable sense, necessary and sufficient conditions for strong stability.

3. The sequence of δ_n is of course, of very specialized structure. Thus, since the stability of (1) is equivalent [3, p. 255] to $\delta_n \to 0$ and is implied by strong stability, it follows that $\delta_n \to 0$ whenever $\Sigma\, (\delta_n/n)$ is convergent.

Added in proof: Since this paper was submitted I heard from Professor M. Loève that he has independently obtained the theorem of section 2 by another method.

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