# GENERALIZED HIT PROBABILITIES WITH A GAUSSIAN TARGET[1]

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**1. Summary.** A general discrete distribution is obtained whose random variable is the number of "hits" on a target. The target is $k$-dimensional and Gaussian diffuse, that is, the probability of a hit is given to within a constant factor by a Gaussian probability density function of the position of the "trajectory" in $k$ dimensions. For a series of $n$ rounds, the $n$ positions of the trajectory have a multivariate Gaussian distribution. An expression is given, using Theorems 1 and 2 or 1 and 3, for the probability of $r$ hits as a linear combination of probabilities of all hits on each possible set of rounds. Theorems 4, 5, and 6, with Theorem 1, give three limiting distributions as $n$, the number of rounds, tends to infinity. Theorems 7, 8, and 9, with Theorem 1, present three other limiting cases, and Theorems 10 and 1 give a time average result.

**2. The problem.** In [1], L. B. C. Cunningham and H. R. B. Hynd proposed a problem in multivariate statistics: to find the probability of at least one hit when an automatic gun is used against a moving target. Because of inability in aiming, the point of aim, by which we mean the centre of the distribution of the shell trajectory, will not always be the centre of the target. In fact, while the gun is being fired, the point of aim is found to wander back and forth across the target. The main complication in the problem arises in taking account of the dependence between the successive points of aim at the instants of firing.

In [1] the problem is given an approximate solution covering a partial range of parameter values and assuming the target has a circular outline.

Here the problem is modified by using a Gaussian diffuse target, a target for which the probability of a hit is given to within a constant factor by a Gaussian probability density function of the position of the trajectory. From a target which is essentially two-dimensional for aiming, the problem is generalized to a target in $k$ dimensions, having in mind the possibility of application to other problems.

If we assume the target to be a Gaussian diffuse target and the position of the trajectory to be distributed according to a two-dimensional Gaussian distribution about the point of aim, then the probability of a "hit" as a function of the point of aim also has the form of a Gaussian diffuse target; that is, it is a constant times a Gaussian pdf of the point of aim. This will be discussed in a later paper, where the general theory will be applied to the two-dimensional problem as proposed by Cunningham and Hynd and a method of numerical evaluation considered and applied to an example.

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For the general theory we shall start with the Gaussian diffuse target in terms of the point of aim, consider it in $k$ dimensions, and call it a "success function." The point of aim is a random variable in $k$ dimensions and will be called a prediction. If the prediction yields a hit we shall speak of a successful prediction.

The abstracted statistical problem may be stated as follows. In a series of $n$ predictions having a joint distribution, find the probability distribution of the number $R$ of successful predictions. Let the $i$th prediction be $\bar{X}_i = (X_{1i}, X_{2i}, \cdots, X_{ki}) = \{X_{\mu i}\}$ where $\mu$ ranges over the set $(1, 2, \cdots, k)$. A prediction $\bar{X}_i = \bar{x}_i$ becomes a successful prediction with probability given by the success function $s_i(\bar{x}_i)$, that is,

$$Pr\{\text{Successful prediction} \mid \bar{X}_i = \bar{x}_i\} = s_i(\bar{x}_i),$$

where $0 \leq s_i(\bar{x}_i) \leq 1$.

In the following theory the problem is solved when the predictions have a Gaussian distribution and the success functions have the form of a Gaussian diffuse target.

In the original problem of Cunningham and Hynd it was found that the horizontal and vertical components of the point of aim were reasonably independent. Consequently we shall assume independence between the values of the $\mu$th coordinate for the $n$ predictions and values of the $\nu$th coordinate, where $\mu \neq \nu = 1, 2, \cdots, k$. The generalization omitting this independence provides little additional complication.

For each value of $\mu$, let the set $\{X_{\mu i}\}$ have a Gaussian distribution with means $\{m_{\mu i}\}$ and covariance matrix $\| \sigma_{ij}^{(\mu)} \|$ which is positive definite. Because we shall want the probability density function for any subset of the predictions, we introduce the following notation for the inverse of the covariance matrix corresponding to a subset. For a typical subset $(i_1, i_2, \cdots, i_r)$ of the integers $(1, 2, \cdots, n)$ we shall use the symbol $\beta_r$. Then if $p, q$ range over this subset, we have

$$\| \sigma_{pq}^{(\mu)} \|^{-1} = \| \sigma_{\beta_r(\mu)}^{pq} \| \tag{2.1}$$

as the inverse of the covariance matrix for the $\mu$th coordinates of the subset $\beta_r$ of the predictions. Therefore they will have probability density element

$$\frac{| \sigma_{\beta_r(\mu)}^{pq} |^{\frac{1}{2}}}{(2\pi)^{r/2}} \exp\left[-\tfrac{1}{2}\left\{ \sum_{p,q\varepsilon\beta_r} \sigma_{\beta_r(\mu)}^{pq}(x_{\mu p} - m_{\mu p})(x_{\mu q} - m_{\mu q})\right\}\right] \prod_{p\varepsilon\beta_r} dx_{\mu p}. \tag{2.2}$$

Let the success function of the $i$th prediction have the following form:

$$s_i(\bar{x}_i) = C_i \exp\left[-\tfrac{1}{2} \sum_{\mu,\nu} \tau_{(i)}^{\mu\nu} x_{\mu i} x_{\nu i}\right], \tag{2.3}$$

where $0 \leq C_i \leq 1$, $\| \tau_{(i)}^{\mu\nu} \|$ is positive definite, and $\mu, \nu$ range over the set $\{1, 2, \cdots, k\}$. There is no essential restriction in assuming that the success function is centered at the origin, since a change of origin in each $k$-dimensional space to center the success functions would only adjust the values $m_{\mu i}$.

**3. Probabilities from expectations.** To describe the distribution of $R$ we need the probabilities of 0, 1, 2, $\cdots$ , $n$ successful predictions, that is, $Pr\{R = r\}$ for $r = 0, 1, \cdots , n$. These will not be given in the main theorems, but rather an expression for $E_r$ defined below from which the probabilities can be calculated by well known formulas which are given in Theorem 1.

$$E_r = \sum_{i_1 < \cdots < i_r} E_{i_1 i_2 \cdots i_r} = \sum_{\beta_r} E_{\beta_r}, \tag{3.1}$$

where the summation is over all sets of $r$ integers chosen from the set $(1, 2, \cdots, n)$. $E_{i_1 i_2 \cdots i_r}$ is the probability that predictions $i_1, i_2, \cdots, i_r$ will be successful predictions. $E_r$ can be interpreted as the expected number of sets of $r$ successful predictions, counted with overlapping, in our series of $n$ predictions. This is easily seen since $E_r$ is the sum of the probabilities for all the possible sets of $r$ predictions.

Theorem 1. *If $E_r$ is defined by equation* (3.1) *and following, then the probability of* 0 *successes is*

$$Pr\{R = 0\} = 1 - E_1 + E_2 - \cdots + (-1)^n E_n, \tag{3.2}$$

*and the probability of $r$ successes is*

$$\begin{aligned} Pr\{R = r\} &= \frac{1}{r!}\left\{r!E_r - \frac{(r+1)!}{1!} E_{r+1} + \cdots + (-1)^s \frac{(r+s)!}{s!} E_{r+s} + \cdots + (-1)^{n-r} \frac{n!}{(n-r)!} E_n\right\} \\ &= E_r - \binom{r+1}{r} E_{r+1} + \cdots + (-1)^s \binom{r+s}{r} E_{r+s} + \cdots + (-1)^{n-r} \binom{n}{r} E_n . \end{aligned} \tag{3.3}$$

These are well known formulas of probability theory.

## 4. The main theorem.

Theorem 2. *Given that the success functions are Gaussian diffuse as given by* (2.3), *and that the $n$ predictions have a Gaussian joint distribution as given by* (2.1) *and* (2.2), *then*

$$E_{\beta_r} = (\prod_p C_p) \mid \delta_{pq}\delta_{\mu\nu} + \sigma_{pq}^{(\mu)} \tau_{(p)}^{\mu\nu} \mid^{-\frac{1}{2}} \tag{4.1}$$

*when all the $m_{\mu i} = 0$, and a more general formula is given by* (4.4) *and* (4.5) *below.*

**Proof.** Consider the following expression for $E_{\beta_r}$:

$$
\begin{aligned}
E_{\beta_r} &= Pr\{\text{predictions } i_1, i_2, \cdots, i_r \text{ will be successful predictions}\} \\
&= E\{\prod_p [C_p \exp(-\tfrac{1}{2} \sum_{\mu,\nu} \tau^{\mu\nu}_{(p)} x_{\mu p} x_{\nu p})]\} \\
&= (\prod_p C_p) \frac{\prod_\mu | \sigma^{pq}_{\beta_r(\mu)}|^{\frac{1}{2}}}{(2\pi)^{kr/2}} \\
&\quad \cdot \int \exp[-\tfrac{1}{2} \sum_{\mu,\nu,p,q} \{\tau^{\mu\nu}_{(p)} \delta_{pq} x_{\mu p} x_{\nu q} + \sigma^{pq}_{\beta_r(\mu)} \delta_{\mu\nu} (x_{\mu p} - m_{\mu p})(x_{\nu q} - m_{\nu q})\}] \prod_{\mu,p} dx_{\mu p} \\
&= (\prod_p C_p) \frac{|A|^{\frac{1}{2}}}{(2\pi)^{kr/2}} \int \exp[-\tfrac{1}{2}\{y'Ty + (y-m)'A(y-m)\}]\, dy.
\end{aligned}
$$

The matrices in the last expression are defined by

$$
\begin{aligned}
&(4.2) \qquad & A &= \| \sigma^{pq}_{\beta_r(\mu)} \delta_{\mu\nu} \|, \\
&(4.3) \qquad & T &= \| \tau^{\mu\nu}_{(p)} \delta_{pq} \|, \\
& & y &= \| x_{\mu p} \|, \\
& & m &= \| m_{\mu p} \|.
\end{aligned}
$$

The matrices are $kr$ by $kr$ or $kr$ by 1, with $\mu$, $p$ indicating rows and $\nu$, $q$ indicating columns.

$$
\begin{aligned}
E_{\beta_r} &= (\prod_p C_p) \frac{|A|^{\frac{1}{2}}}{(2\pi)^{kr/2}} \\
&\quad \cdot \int \exp[-\tfrac{1}{2}\{y'(T+A)y - y'Am - m'Ay + m'A(T+A)^{-1}Am\}]\, dy \\
&\qquad\qquad \cdot \exp[-\tfrac{1}{2}\{m'Am - m'A(T+A)^{-1}Am\}].
\end{aligned}
$$

We have completed the quadratic form by removing an appropriate factor from under the sign of integration. Integrating over the whole space, we find

$$
\begin{aligned}
E_{\beta_r} &= (\prod_p C_p) \frac{|A|^{\frac{1}{2}}}{(A+T)^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(m'Am - m'A(A+T)^{-1}Am)] \\
(4.4) \qquad &= (\prod_p C_p) \, | I + A^{-1}T |^{-\frac{1}{2}} \exp[-\tfrac{1}{2} m'Bm] \\
&= (\prod_p C_p) \, | \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{pq} \tau^{\mu\nu}_{(p)} |^{-\frac{1}{2}} \exp[-\tfrac{1}{2} \sum_{\mu,\nu,p,q} m_{\mu p} B_{\mu,p,\nu,q} m_{\nu q}],
\end{aligned}
$$

where

$$
\begin{aligned}
B &= \| B_{\mu,p,\nu,q} \| \\
&= A - A(A+T)^{-1}A \\
(4.5) \qquad &= A[I - (I + A^{-1}T)^{-1}] \\
&= \| \sigma^{(\mu)}_{pq} \delta_{\mu\nu} \| \cdot [\, \| \delta_{pq}\delta_{\mu\nu} \| - \| \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{pq} \tau^{\mu\nu}_{(p)} \|^{-1}].
\end{aligned}
$$

**5. Simplifications.** There are two important cases given by Theorem 3 and its corollary in which we obtain a simplification in the formula for $E_{\beta_r}$.

THEOREM 3. *Given the conditions of Theorem 2 and the condition that* $||\tau_{(i)}^{\mu\nu}||$ *is diagonal for each* $i$, *the following expression is obtained for* $E_{\beta_r}$:

$$E_{\beta_r} = \left(\prod_p C_p\right) \prod_\mu | \delta_{pq} + \sigma_{pq}^{(\mu)} \tau_{(p)}^{\mu\mu} |^{-\frac{1}{2}} \tag{5.1}$$

*when* $m_{\mu i} \equiv 0$.

PROOF. We note that $|| \tau_{(i)}^{\mu\nu} || = || \tau_{(i)}^{\mu\mu} \delta_{\mu\nu} ||$ and hence the determinant in (4.1) consists of diagonal blocks with zeros elsewhere. When we expand, (5.1) is obtained.

COROLLARY. *Assuming the conditions of Theorem 2 and the condition that* $|| \tau_{(i)}^{\mu\nu} ||$ *has for each* $i$ *the same principal axes and* $|| \sigma_{ij}^{(\mu)} || = || \sigma_{ij} ||$ *independent of* $\mu$, *then*

$$E_{\beta_r} = \left(\prod_p C_p\right) \prod_\mu | \delta_{pq} + \sigma_{pq} \lambda_{(p)}^{\mu} |^{-\frac{1}{2}} \tag{5.2}$$

*when* $m_{\mu i} \equiv 0$. $\{\lambda_{(i)}^{\mu}\}$ *are the characteristic roots of the matrix* $|| \tau_{(i)}^{\mu\nu} ||$ *and the superscripts* $\mu$ *yield corresponding roots in the* $k$*-dimensional spaces for the* $n$ *values of* $i$.

The proof is obtained by rotating each $k$-dimensional space to diagonalize the matrices. The same rotation will diagonalize for each $i$. Because $|| \sigma_{ij}^{(\mu)} ||$ is independent of $\mu$ the covariance matrix for the predictions will be unchanged.

**6. Limiting distribution (number of predictions $n \to \infty$).** Because the expression for $E_r$ is a sum of $\binom{n}{r}$ terms, the numerical calculations for large values of $n$ would be prodigious. Consequently we introduce several limiting distributions obtained by letting $n$ increase indefinitely, subject to suitable conditions. The limiting conditions in each case should indicate the applicability in particular situations.

Concerning the question of the existence of the different limiting distributions, a sufficient condition would be the convergence of the series for $Pr\{R = r\}$,

$$Pr\{R = r\} = \sum_{s=0}^{\infty} (-1)^s \binom{r+s}{r} E_{r+s},$$

obtained by having

$$\lim_{r \to \infty} \frac{r! E_r}{m^r} \leq 1 \tag{6.1}$$

for some value of $m$.

When $n$ becomes large, the enumeration of predictions is unwieldy. Therefore, they will be given in terms of time, a convenient parameter for intuitive consideration. Thus we write

$$\sigma_{ij}^{(\mu)} = \sigma_{(t_i, t_j)}^{(\mu)},$$

where this is the covariance between the $\mu$th coordinates of the predictions at times $t_i$ and $t_j$. Also

$$\tau^{\mu\nu}_{(i)'} = \tau^{\mu\nu}_{(t_i)},$$

an element of the success function matrix at time $t_i$.

THEOREM 4. TYPE I. *Assuming the conditions of Theorem 2 and letting $n \to \infty$ so that the predictions are uniformly spaced from 0 to $T$ and the success functions approach 0 as $1/n$ with $D(t) = nC(t)$ bounded and independent of $n$, then*

$$(6.2)\quad E_r = \frac{1}{r!T^r}\int_0^T \cdots \int_0^T \left(\prod_1^r D(t_p)\right) |\delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)}\tau^{\mu\nu}_{(t_p)}|^{-\frac{1}{2}}\, dt_1\, dt_2 \cdots dt_p,$$

*where $m_{\mu i} \equiv 0$.*

PROOF. The minimum value of $|\delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)}P^{\mu\nu}_{(t_p)}|$, which is the determinant of a positive definite matrix with 1's added down the diagonal, will be greater than 1. If in addition $D(t)$ is bounded, we have

$$\lim_{r\to\infty} \frac{r!E_r}{[\sup D(t)]^r} \le 1,$$

and this is sufficient to assure the existence of the limiting distribution.

$$E_r = \lim_{n\to\infty}\sum_{\beta_r}\left\{\left(\prod_p \frac{D(t_p)}{n}\right)\middle|\, \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)}\tau^{\mu\nu}_{(t_p)}\right|^{-\frac{1}{2}}$$

$$= \lim_{n\to\infty}\frac{(1-1/n)\cdots(1-r/n)}{r!}$$

$$\cdot\sum_{\substack{\text{all permutations}\\ r \text{ from } n}} \frac{(\prod_p D(t_p)\ |\ \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)}\tau^{\mu\nu}_{(t_p)}\ |^{-\frac{1}{2}}}{n(n-1)\cdots(n-r+1)}$$

$$= \frac{1}{r!T^r}\int_0^T \cdots \int_0^T \left(\prod_{p=1}^r D(t_p)\right)\left|\, \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)}\tau^{\mu\nu}_{(t_p)}\right|^{-\frac{1}{2}} \prod_{p=1}^r dt_p.$$

This completes the proof.

The applicability of this distribution as an approximation for large values of $n$ will be discussed for the Cunningham and Hynd problem in a later paper.

THEOREM 5. TYPE II. *Assuming the conditions of Theorem 2 and letting $n \to \infty$ and the scale of the success functions decrease such that $\tau^{\mu\nu}_{(t_i)} = n^{2/k}\tau'^{\mu\nu}_{(t_i)}$, then*

$$(6.3)\quad E_r = \frac{1}{r!T^r}\int_0^T \cdots \int_0^T \left(\prod_{p=1}^r C(t_p)\ |\ T'^{\mu\nu}_{(t_p)}\ |^{-\frac{1}{2}}\right)\prod_\mu\ |\ \sigma^{(\mu)}_{(t_p,t_q)}\ |^{-\frac{1}{2}} \prod_{p=1}^r dt_p$$

*if $m_{\mu i} \equiv 0$, $|\ \tau'^{\mu\nu}_{(t)}\ |$ is bounded from 0, and*

$$\int_0 \cdots \int_0^T \prod_\mu\ |\ \sigma^{(\mu)}_{(t_p,t_q)}\ |^{-\frac{1}{2}} \prod_{p=1}^r dt_p$$

*exists and* $\leq m^r$ *where* $m$ *is independent of* $r$.

PROOF. The existence of the limiting distribution is guaranteed by these last conditions, which insure that the set $\{E_r\}$ satisfies the condition given by formula (6.1).

$$
\begin{aligned}
E_r &= \lim_{n\to\infty} \sum_{\beta_r} \Big\{\Big(\prod_p C(t_p)\Big) \,\Big|\, \delta_{pq}\delta_{\mu\nu} + \sigma^{(\mu)}_{(t_p,t_q)} n^{2/k} \tau'^{\mu\nu}_{(t_p)} \Big|^{-\frac{1}{2}}\Big\} \\
&= \lim_{n\to\infty} \sum_{\beta_r} \Big\{\Big(\prod_p C(t_p)\Big)\Big(\prod_\mu \big| \sigma^{(\mu)}_{(t_p,t_q)} \big|^{-\frac{1}{2}}\Big) \,\Big|\, \sigma^{(t_p,t_q)}_{\beta_r(\mu)} \delta_{\mu\nu} + \tau'^{\mu\nu}_{(t_p)} n^{2/k} \delta_{pq} \Big|^{-\frac{1}{2}}\Big\} \\
&= \lim_{n\to\infty} \sum_{\beta_r} \left\{\frac{\Big(\prod_p C(t_p)\Big)\Big(\prod_\mu \big| \sigma^{(\mu)}_{(t_p t_q)} \big|^{-\frac{1}{2}}\Big) \prod_p \big| \tau'^{\mu\nu}_{(t_p)} \big|^{-\frac{1}{2}}(1 + O(n^{-2/k}))}{n^r}\right\} \\
&= \frac{1}{r!T^r} \int_0^T \cdots \int_0^T \prod_p \Big(C(t_p) \,\big|\, \tau'^{\mu\nu}_{(t_p)} \big|^{-\frac{1}{2}}\Big) \prod_\mu \big| \sigma^{(\mu)}_{(t_p,t_q)} \big|^{-\frac{1}{2}} \prod_{p=1}^{r} dt_p .
\end{aligned}
$$

This proves the theorem.

THEOREM 6. TYPE III. *Assuming the conditions of Theorem 2 and letting* $n \to \infty$ *with the scale of the success function increasing, and its density at any trial decreasing according to*

$$\text{(a)}\quad \tau^{\mu\nu}_{(t)} = \tau'^{\mu\nu}_{(t)} n^{-\alpha^2},$$

$$\text{(b)}\quad C(t) = \frac{1}{n} D(t),$$

*then*

$$E_r = \frac{1}{r!}\left[\frac{1}{T}\int_0^T D(t)\, dt\right]^r, \tag{6.4}$$

*where* $m_{\mu i} \equiv 0$.

The proof of this theorem is similar to that of Theorem 4. Note that the condition for a limiting distribution is satisfied so long as $D(t)$ is bounded.

This distribution is the Poisson distribution with the usual Poisson parameter

$$m = \frac{1}{T}\int_0^T D(t)\, dt.$$

## 7. Limiting distributions for a fixed number of predictions $n$.

THEOREM 7. *When the scale of the success function increases, the distribution approaches the simple generalization of the binomial where the probability of successful predictions need not be the same for each trial, and*

$$E_r = \sum_{\beta_r} \prod_{p\epsilon\beta_r} C_p, \tag{7.1}$$

*where* $m_{\mu i} \equiv 0$.

THEOREM 8. *When the correlation between the values of particular coordinates of* $\bar{X}_i$ *approaches* 0, *then the simple binomial generalization is obtained, with*

$$E_r = \sum_{\beta_r} \prod_{p\epsilon\beta_r} \{C_p \mid \delta_{\mu\nu} + \sigma^{(\mu)}_{pp}\tau^{\mu\nu}_{(p)} \mid^{-\frac{1}{2}}\}. \tag{7.2}$$

THEOREM 9. *When the correlation between the values at different trials of particular coordinates of $\bar{X}_i$ approaches* 1, *then*

$$E_r = \sum_{\beta_r} \left(\prod_p C_p\right) \mid \delta_{pq} \delta_{\mu\nu} + \sqrt{\sigma_{pp}^{(\mu)}} \sqrt{\sigma_{qq}^{(\mu)}} \, \tau_{(p)}^{\mu\nu} \mid^{-\frac{1}{2}}. \tag{7.3}$$

The proofs for Theorems 7, 8, and 9 follow routine lines.

**8. Time average for a fixed number of predictions $n$.** If the conditions for our generalized distribution vary with time, then an expression for the probabilities obtained as a time average would be appropriate.

THEOREM 10. *Assuming the time interval between predictions is $h$ and that the first prediction occurs at an undetermined time in the interval* $(0, T')$, *then the general distribution has its probabilities determined by*

$$E_r = \sum_{\beta_r} \frac{1}{T'} \int_{0-h}^{T'-h} \left(\prod_p C(t + ph)\right) \mid \delta_{pq} \delta_{\mu\nu} + \sigma_{(t+ph,t+qh)}^{(\mu)} \tau_{(t+ph)}^{\mu\nu} \mid^{-\frac{1}{2}} dt. \tag{8.1}$$

PROOF. Assuming that the time of the first prediction is uniformly distributed on the interval $(0, T')$, (8.1) follows from Theorem 2.

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## REFERENCE

[1] L. B. C. CUNNINGHAM AND W. R. B. HYND, "Random processes in air warfare," *Jour. Roy. Stat. Soc., Suppl.*, Vol. 8 (1946), p. 62–85.