A DOUBLE SAMPLE TEST PROCEDURE[1]

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1. Summary and introduction. Three different testing procedures which involve a minimum of modification of the usual single sample tests of the hypotheses considered are given here. Tests are made by taking samples at two stages for testing the mean of a normal distribution. A known standard deviation is assumed, but an extension to the case where the standard deviation is unknown is also given. Special examples show that tests can be chosen so that the expected number of observations is less than the number required for the ordinary single sample test and indeed can give considerable savings. The tests in Sections 3 and 4 give the greater savings, but the powers are more difficult to evaluate than the power for the test of Section 2. Also, it is a little more work to apply the test in Section 4. Wald in [9] has discussed a sequential test where the observations are taken in groups. The tests given here could be considered very special cases of this where the number of observations is truncated after two groups. Romig in [7] has set up a double sampling procedure for sampling from a finite population that is approximately normal where the rejection points are determined by preassigned engineering or specifications limits and not by the normal distribution itself as is done for the first sample of the double sample tests given below. Bowker and Goode in [1] give tests similar to those given by Romig. Chapman in [2] and Stein in [8] have discussed two sample tests where the object is to obtain tests with the power independent of an unknown variance and where there is no upper limit on the number of observations required. There is a definite ceiling on the number of observations required for the tests presented here and they have many interesting properties that make them very desirable from the standpoint of saving of observations and simplicity.

2. First test procedure. A double sample test for the hypothesis $H:m = m_0$ against $\tilde{H}:m < m_0$ where m is the mean of a normal random variable, X, with known standard deviation, σ, will be constructed. Extensions to tests of other hypotheses will be clearly possible. Assume that the number of observations, n, of a single sample test has been determined in accordance with the methods mentioned in [3] so as to have a given probability of Type II error for $m = m_1$ where $m_1 < m_0$. Let $G(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$ and let h be defined by $G(-h) = \alpha < \frac{1}{2}$. Let n_1 be the number of observations in the first sample of the double

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sample test where $n_1 < n$. Let $p = n_1/n$ and let θ be a positive constant defined by equation (1) below.

For the double sample test, n_1 observations, $x_1, \cdots, x_{n_1}$, are taken and $\bar{x}_1 = \sum_{i=1}^{n_1} x_i/n_1$ and $u_1 = \sqrt{n_1}(\bar{x}_1 - m_0)/\sigma$ are computed. If $u_1 < -\sqrt{p}h - \theta$ reject H; if $u_1 > -\sqrt{p}h + \theta$ accept H; and if $-\sqrt{p}h - \theta \leqq u_1 \leqq -\sqrt{p}h + \theta$ take n additional observations.

For $n_1 + n$ observations, $\bar{x}_2 = \sum_{i=n_1+1}^{n_1+n} x_i/n$ and $u_2 = \sqrt{n}(\bar{x}_2 - m_0)/\sigma$ are computed. If $u_2 \leqq -h$ reject H and if $u_2 > -h$ accept H.

The Type I error of the double sample test is equal to

$$G(-\sqrt{p}h - \theta) + \alpha[1 - G(-\sqrt{p}h - \theta) - G(\sqrt{p}h - \theta)].$$

If the Type I error of the double sample test is to be equal to α, then

$$\alpha^{-1}(1 - \alpha)G(-\sqrt{p}h - \theta) = G(\sqrt{p}h - \theta) \tag{1}$$

holds, which is the equation that defines θ.

THEOREM 2.1. *For each given p and α there exists one and only one θ which satisfies equation* (1).

PROOF. Set $Y = \alpha^{-1}(1 - \alpha)G(-\sqrt{p}h - \theta) - G(\sqrt{p}h - \theta)$ Then to show that Y has only one positive zero, note that

$$\frac{dY}{d\theta} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{p}h-\theta)^2}\left[1 - \frac{1-\alpha}{\alpha} e^{-2\sqrt{p}h\theta}\right]$$

and hence that Y is positive decreasing at $\theta = 0$, and negative increasing as θ approaches infinity. Since Y has only one critical point, Y has only one zero.

The power of the double sample test is given by

$$G(-\sqrt{p}h - \theta + \sqrt{p}w) + [G(-h + w)][1 - G(-\sqrt{p}h - \theta + \sqrt{p}w) - G(\sqrt{p}h - \theta - \sqrt{p}w)]$$

where $w = \sqrt{n}(m_0 - m)/\sigma$. Define

$$\phi(w) = [1 - G(-h + w)][G(-\sqrt{p}h - \theta + \sqrt{p}w)] - [G(-h + w)][G(\sqrt{p}h - \theta - \sqrt{p}w)];$$

that is, $\phi(w)$ is equal to the power of the double sample test minus the power of the single sample test based on n observations.

THEOREM 2.2. *The function, $\phi(w)$, has the following zeros: $w = -\infty$, $w = 0$, $w = h$, $w = 2h$, and $w = +\infty$.*

PROOF. Substitution in $\phi(w)$ verifies the theorem if equation (1) is kept in mind for $w = 0$ and $w = 2h$. Note that uniqueness is not claimed although this is apparently the case.

THEOREM 2.3. *If $w = u + h$, then $\phi(u + h) = -\phi(-u + h)$.*

PROOF. $\phi(u + h) = [1 - G(u)][G(-\theta + \sqrt{p}u)] - [G(u)][G(-\theta - \sqrt{p}u)]$, and since $G(-u) = 1 - G(u)$, $\phi(u + h) = [G(-u)][G(-\theta + \sqrt{p}u)] - [1 - G(-u)][G(-\theta - \sqrt{p}u)] = -\phi(-u + h)$.

It is convenient now to define

$$R(w) = G(-\sqrt{p}h - \theta + \sqrt{p}w) + G(\sqrt{p}h - \theta - \sqrt{p}w),$$

that is, $R(w)$ is the sum of the probability of rejecting H and the probability of accepting H at the first step of the double sample test. The expected number of observations, which is discussed below, is a function of $R(w)$.

THEOREM 2.4. *The function, $R(w)$, has a minimum with respect to w when $w = h$ and $R(w)$ is a decreasing function for $w < h$ and an increasing function of w for $w > h$.*

PROOF. This follows immediately from the derivatives of $R(w)$.

THEOREM 2.5. *The power of the double sample test is an increasing function of w.*

PROOF. The power is an increasing function up to $w = h$ since the power is equal to $G(-\sqrt{p}h - \theta + \sqrt{p}w) + G(-h + w)[1 - R(w)]$ and $G(-\sqrt{p}h - \theta + \sqrt{p}w)$ and $G(-h + w)$ are increasing functions of w and $R(w)$ is a nonincreasing function up to $w = h$ by Theorem 2.4. But if $w = u + h$, this means that $G(-u) + \phi(-u + h)$ is a decreasing function of u for all positive u. Hence the power which is equal to $\phi(u + h) + G(u)$, which is equal to $1 - \phi(-u + h) - G(-u)$ by Theorem 2.3, is an increasing function for all u and hence for all w.

THEOREM 2.6. *The function, $\phi(w)$, is an increasing function of w at $w = h$, provided $p \geqq (2\pi)^{-1}$ and $h \geqq 0.468$.*

OUTLINE OF PROOF. $(d\phi/dw)_h = (2\pi)^{-\frac{1}{2}}[\sqrt{p}e^{-\frac{1}{2}\theta^2} - 2G(-\theta)]$. To show that $(d\phi/dw)_h$ is positive for $p \geqq (2\pi)^{-1}$ and $h \geqq 0.468$ show that it is positive for $p = (2\pi)^{-1}$, positive decreasing as $p \rightarrow 1$, and has only one critical point for $(2\pi)^{-1} < p < 1$. It is easy to show that $(d\phi/dw)_h \rightarrow 0$ through positive values as $p \rightarrow 1$ by consideration of the derivatives of $\sqrt{p}\, e^{-\frac{1}{2}\theta^2}$ and $2G(-\theta)$ as $p \rightarrow 1$. Next it can be shown that there exists one and only one value of $z = \theta\sqrt{p}$ which corresponds to any critical points of $(d\phi/dw)_h$. Then for all $p > (2\pi)^{-1}$ there exists one and only one p which will give any particular value of $z = \theta\sqrt{p}$. Next it can be shown that for $p = (2\pi)^{-1}$ and $\theta \geqq 1.572$ $(d\phi/dw)_h$ is positive, and then for all $h \geqq 0.468$ and $p = (2\pi)^{-1}$, $\theta \geqq 1.572$.

From the foregoing theorems it appears that if $h \geqq 0.468$and $p \geqq (2\pi)^{-1}$, the power of the double sample test is less than that of the corresponding single sample test based on n observations, $G(-h + w)$, for $0 < w < h$ and $w > 2h$ and is greater for $w < 0$ and $h < w < 2h$. Since uniqueness of the zeros has not been shown, this is, of course, conjecture, but Example 2.1 below shows that this is true in case $p = \frac{1}{2}$ and $\alpha = 0.05$. This would make the double sample test more desirable than the single sample test based on n observations if, as is frequently the case, it is more important to reject less often for small values of w or more often for large values of w. In the special examples that have been computed $\phi(w)$ has had consistently small values (less than 0.01 for $w < 0$ and $w > 2h$), that is, the discrepancy between the powers of the single and double sample tests has been negligible in the tails.

The expected number of observations for the double sample test is given by

$E_w(N) = n[1 + p - R(w)]$. For a fixed p, $E_w(N)$ is a maximum when $w = h$ by Theorem 2.4. When the hypothesis H is true, the expected number of observations is $E_0(N) = n[1 + p - \alpha^{-1}G(-\sqrt{p}h - \theta)]$. The minimum of $E_0(N)$ with respect to p for $\alpha = 0.01$, $\alpha = 0.05$, and $\alpha = 0.10$ is for $p = 0.505$, $p = 0.524$ and $p = 0.5003$, approximately, respectively.

TABLE 2.1

p	$-\sqrt{p}h - \theta$	$-\sqrt{p}h + \theta$	θ	k	τ
			For $\alpha = 0.05$		
0.20	−2.3006	+0.8293	1.5650	0.009118	−1.71
0.25	−2.1331	+0.4882	1.3107	0.008988	−1.74
0.30	−2.0174	+0.2155	1.1165	0.008566	−1.80
0.40	−1.8707	−0.2099	0.8304	0.007402	−1.88
0.50	−1.7847	−0.5415	0.6216	0.006210	−1.96
0.60	−1.7308	−0.8175	0.4566	0.005158	−2.00
0.70	−1.6956	−1.0569	0.3193	0.004277	−2.05
0.75	−1.6826	−1.1664	0.2581	0.003899	−2.06
0.80	−1.6720	−1.2705	0.2007	0.003557	−2.06
0.90	−1.6559	−1.4651	0.0954	0.002972	−2.01
			For $\alpha = 0.01$		
0.20	−2.8387	+0.7580	1.7984	0.0013253	
0.25	−2.6837	+0.3573	1.5205	0.0012304	
0.30	−2.5816	+0.0331	1.3073	0.0010936	
0.40	−2.4616	−0.4809	0.9903	0.0008073	
0.50	−2.3991	−0.8909	0.7541	0.0005710	
0.60	−2.3651	−1.2388	0.5632	0.0003983	
0.70	−2.3461	−1.5466	0.3998	0.0002767	
0.75	−2.3400	−1.6891	0.3255	0.0002314	
0.80	−2.3356	−1.8258	0.2549	0.0001940	
0.90	−2.3296	−2.0842	0.1227	0.0001379	

This test is obviously not as efficient as it could be since when it is necessary to take the second sample no use of the first sample is made in the second test. Even so, Example 2.1 below shows that for p equal to one-half, the expected number of observations is considerably less than the number required for the single sample test based on n observations and the power has some desirable properties over the power of the single sample test. In sections 3 and 4 the above procedure is modified so that the test at the second stage makes use of the first set of observations.

Table 2.1 is a tabulation of the rejection and acceptance points for various values of p and $\alpha = 0.05$ and 0.01, together with a tabulation of θ. The quantity, k, is defined in Section 4. For $\alpha = 0.05$, k has a maximum when $p = 0.2090$, $\theta = 1.5129$ and $k = 0.009126$. The quantity, τ, is defined in Section 3.

EXAMPLE 2.1. If $\alpha = 0.05$ and $p = \frac{1}{2}$, then $\theta = 0.6216$. For purposes of comparison the powers of various single sample tests are listed in Table 2.2 beside the power of the double sample test. The column headed power is the power of the double sample test as outlined in this section. $G_1 = G(-h + w)$ is the power of a single sample test based on n observations. $G_2 = G(-h + 0.9828w)$ is the power of a single sample test based on $0.9658n$ observations, that is, on the maximum expected number of observations of the double sample test. $G_3 = G(-h + 0.8700w)$ is the power of a single sample test based on $0.7569n$ observations, that is, on the expected number of observations of the double sample

TABLE 2.2

w	$E_w(N)$	Power	G_1	G_2	G_3	G_4
−2	$0.5245n$	0.0007	0.0001	0.0002	0.0004	0.0010
−1	$0.5995n$	0.0068	0.0041	0.0043	0.0060	0.0078
0	$0.7569n$	0.0500	0.0500	0.0500	0.0500	0.0500
0.5	$0.8493n$	0.1203	0.1261	0.1244	0.1132	0.1182
1	$0.9252n$	0.2509	0.2595	0.2540	0.2192	0.2473
1.6449	$0.9658n$	0.5000	0.5000	0.4887	0.4154	0.4887
2	$0.9531n$	0.6449	0.6387	0.6258	0.5379	0.6208
3.2898	$0.7569n$	0.9500	0.9500	0.9439	0.8882	0.8882
4	$0.6372n$	0.9876	0.9907	0.9889	0.9668	0.9392
5	$0.5386n$	0.9986	0.9996	0.9995	0.9966	0.9785

test when the hypothesis H is true. $G_4 = G[-h + \delta(w)]$ gives the values of the various single sample power curves based on the expected number of observations for the double sample test for the particular alternative at hand, that is, $\delta(w) = w\sqrt{E_w(N)/n}$, where $E_w(N)$ is the expected number of observations for the double sample test. Note that the power of the double sample test is everywhere better than the power, $G[-h + \delta(w)]$. Hence if power is lost for any alternative where w is positive it is not lost in greater measure than is caused by taking fewer observations on the average.

The choice of the rejection and acceptance intervals was an intuitive one in the first place, and although investigation into the optimum such choice indicates that the one that was made is probably the best that can be made from the standpoint of minimum expected number of observations balanced against a uniformly powerful test, no conclusive results have been obtained in this direction.

3. Second (τ) test procedure. The first part of the tests in this and the following section is the same as the test in Section 2. That is, n_1 observations are taken

($n_1 < n$) and $\bar{x}_1 = \sum_{i=1}^{n_1} x_i/n_1$, $p = n_1/n$, and $u_1 = \sqrt{n_1}(\bar{x}_1 - m_0)/\sigma$ are computed. If $u_1 < -\sqrt{p}h - \theta$ reject H; if $u_1 > -\sqrt{p}h + \theta$ accept H; and if $-\sqrt{p}h - \theta \leqq u_1 \leqq -\sqrt{p}h + \theta$ take n_2 additional observations, where $n_2 = n - n_1$.

For n observations in all $\bar{x}_\tau = \sum_{i=1}^{n} x_i/n$, and $u_\tau = \sqrt{n}(\bar{x}_\tau - m_0)/\sigma$ are computed. If $u_\tau \leqq \tau$ reject H, and if $u_\tau > \tau$ accept H, where τ is determined from the equation

$$\frac{1}{2\pi\sqrt{1-p}} \int_{-\infty}^{\tau} \int_{-\sqrt{p}h-\theta}^{-\sqrt{p}h+\theta} \exp\left(-\frac{u^2 - 2\sqrt{p}uv + v^2}{2(1-p)}\right) du\, dv$$
$$= \alpha[1 - G(-\sqrt{p}h - \theta) - G(\sqrt{p}h - \theta)]$$

since the joint distribution of the random variables U_1 and U_τ is bivariate normal with zero means and unit variances and correlation equal to $\sqrt{p}$. A similar test procedure is given in [1]. The quantity, τ, is tabulated for $\alpha = 0.05$ in Table 2.1 and was obtained by interpolation in the bivariate normal table given in [6]. For $\alpha = 0.01$ and other significance levels, however, the tables given in [6] are not extensive enough to obtain τ for most values of p. Section 4 gives an alternative procedure which can be used in this case.

The power of the test given in this section is equal to

$$G(-\sqrt{p}h - \theta + \sqrt{p}w) + \frac{1}{2\pi\sqrt{1-p}} \int_{-\infty}^{\tau+w} \int_{-\sqrt{p}h-\theta+\sqrt{p}w}^{-\sqrt{p}h+\theta+\sqrt{p}w}$$
$$\exp\left(-\frac{u^2 - 2\sqrt{p}uv + v^2}{2(1-p)}\, du\, dv\right),$$

and may be obtained in many cases from the bivariate normal table given in [6].

The expected number of observations for this double sample test is given by $E_w(N) = n[1 - (1-p)R(w)]$. For a fixed p, $E_w(N)$ is a maximum when $w = h$ by Theorem 2.4. When the hypothesis H is true,

$$E_0(N) = n[1 - \alpha^{-1}(1-p)G(-\sqrt{p}h - \theta)].$$

The minimum of $E_0(N)$ with respect to p for $\alpha = 0.01$, 0.05, and 0.10 is approximately for $p = 0.443$, $p = 0.457$, and $p = 0.46$, respectively. The power and expected number of observations for this test for $\alpha = 0.05$ and $p = \frac{1}{2}$ are tabulated in Table 4.1.

4. Third (J) test procedure. This test is primarily an alternative procedure in case τ cannot be obtained for the procedure in Section 3. At the first stage go through the same steps outlined in Sections 2 and 3 and at the second stage take $n_2 = n - n_1$ additional observations and for n observations in all compute $\bar{x}_2' = \sum_{i=n_1+1}^{n} x_i/n_2$, $u_2' = \sqrt{n_2}(\bar{x}_2' - m_0)/\sigma$, $j_1 = G(u_1)$, $j_2 = G(u_2')$, $a = G(-\sqrt{p}h - \theta)$, $b = G(-\sqrt{p}h + \theta)$, and $q = j_1 j_2$, where the random variable Q has the distribution given by Theorem 4.1 below. Define k by $Pr(Q \leqq k) = \alpha$

and reject H if $q \leqq k$ and accept H if $q > k$. If the size of the double sample test is to be α as in Sections 2 and 3, Theorem 2.1 applies once more.

THEOREM 4.1. *The cumulative distribution function for the random variable, Q, when $a \leqq J_1 \leqq b$ is given by*

$$(2) \qquad Pr(Q \leqq q) = \begin{cases} \dfrac{q(\log b - \log a)}{b - a} & \text{for } 0 \leqq q \leqq a \\[2ex] \dfrac{q(1 + \log b - \log q) - a}{b - a} & \text{for } a \leqq q \leqq b. \end{cases}$$

PROOF. The random variable, J_2, has the rectangular distribution over the interval 0 to 1 and J_1 given J_2 has the rectangular distribution over the interval a to b. If $Q = J_1 J_2$, the joint distribution of Q and J_1 is given by $1/(b - a)J_1$ where the region of definition is the trapezoid bounded by $Q = 0$, $Q = J_1$, $J_1 = a$, and $J_1 = b$. Integrate out J_1 and then integrate again to obtain the cumulative distribution function (2).

The power of this double sample test is given by

$$G(-\sqrt{p}h - \theta + \sqrt{p}w) \\ + q^*[1 - G(-\sqrt{p}h - \theta + \sqrt{p}w) - G(\sqrt{p}h - \theta - \sqrt{p}w)]$$

where $q^* = Pr(Q \leqq k$ given that the mean is m and $a \leqq J_1 \leqq b)$. An exact formula for q^* can be found using the same process as that used in Theorem 4.1, but the result is complicated and is not practical to use in computations. A lower bound for q^* can be found as follows.

Let $J_3 = G(U_1 + \sqrt{p}w)$, $J_4 = G(U_2 + \sqrt{1 - p}w)$,

$$a' = G(-\sqrt{p}h - \theta + \sqrt{p}w),\ b' = G(-\sqrt{p}h + \theta + \sqrt{p}w),$$

and $Q' = J_3 J_4$. When the mean is m, Q' has the same distribution as Q, given in Theorem 4.1, but with a and b replaced by a' and b', respectively. Determine the pair of values of (U_1, U_2) that minimize the product $J_3 J_4$ such that $J_1 J_2 = k$ and $-\sqrt{p}h - \theta \leqq U_1 \leqq -\sqrt{p}h + \theta$ (not necessarily the same pair for each w). Denote the minimum value of $J_3 J_4$ by k'_w. Then $q^* \geqq Pr(Q' \leqq k'_w)$, since the hypothesis is always rejected if $Q' \leqq k'_w$.

The expected number of observations for this test is the same as for the test in Section 3 with the same maximum and minimum values.

EXAMPLE 4.1. If $\alpha = 0.05$ and $p = \frac{1}{2}$, $\theta = 0.6216$, $\tau = -1.96$, and $k = 0.006210$. For purposes of comparison in this case the powers of various single sample tests and the expected number of observations for the τ and J double sample tests are listed in Table 4.1. The column headed L. B. Power J-Test refers to the lower bound of the power of the double sample (J) test as outlined in this section. $G_1 = G(-h + w)$ is the power of a single sample test based on n observations. $G_5 = G(-h + 0.8561w)$ is the power of a single sample test based on $0.7329n$ observations, that is, on the maximum expected number of observations of the double sample (J and τ) tests. $G_6 = G(-h + 0.7927w)$ is the power of a single sample test based on $0.6284n$ observations, that is, on the expected number of observations of the double sample tests when the hypothesis H is true. $G_7 = G[-h + \gamma(w)]$ gives the values of the various single sample power curves based on the expected number of observations for the J and τ double sample tests for the particular alternative at hand, that is, $\gamma(w) = w\sqrt{E_w(N)/n}$, and $E_w(N)$ is the expected number of observations for the J and τ double sample tests. Note that the power of both of the double sample tests is everywhere better than $G[-h + \gamma(w)]$. Hence, if power is lost using either of the double sample tests for any alternative where w is positive, it is not lost in greater measure than is caused by taking fewer observations on the average.

TABLE 4.1

w	$E_w(N)$	Power τ-Test	L. B. Power J-Test	G_1	G_5	G_6	G_7
-1	$0.5498n$	—	0.0072	0.0041	0.0062	0.0074	0.0085
0	$0.6284n$	0.050	0.0500	0.0500	0.0500	0.0500	0.0500
0.5	$0.6746n$	0.112	0.1104	0.1261	0.1118	0.1059	0.1086
1	$0.7126n$	0.222	0.2136	0.2595	0.2151	0.1971	0.2117
1.6449	$0.7329n$	0.423	0.4096	0.5000	0.4064	0.3666	0.4064
2	$0.7265n$	0.555	0.5399	0.6387	0.5268	0.4763	0.5238
3	$0.6556n$	0.857	0.8341	0.9123	0.8221	0.7683	0.7835
4	$0.5686n$	0.977	0.9680	0.9907	0.9624	0.9365	0.9149
5	$0.5193n$	—	0.9968	0.9996	0.9958	0.9898	0.9749

5. Test of the Student hypothesis. A test of the hypothesis considered in Section 2 when the standard deviation is unknown can be constructed by simply making the probabilities of accepting and rejecting equal to the corresponding probabilities given in Section 2. That is, let

$$S_n(t) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_{-\infty}^{t} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} dt,$$

and let λ and η be defined by $S_{n_1-1}(-\lambda) = G(-\sqrt{p}h - \theta)$, $S_{n_1-1}(-\eta) = G(-\sqrt{p}h + \theta)$. Replace σ^2 in the test procedure of Section 2 by the corresponding unbiased estimates $s_1^2 = \sum_{i=1}^{n_1} (x_i - \bar{x}_1)^2/(n_1 - 1)$ and $s_2^2 = \sum_{i=n_1+1}^{n_1+n} (x_i - \bar{x}_2)^2/(n - 1)$. For n_1 observations reject H if $u_1 < -\lambda$ and accept H if $u_1 > -\eta$. Take n additional observations if $-\lambda \leqq u_1 \leqq -\eta$. For $n_1 + n$ observations make the usual test of size α using the last n observations only. For given n_1, α and p, λ and η can be obtained from [4]. The power of the test and the expected number of observations can be easily computed from the tables given in [5].

Tests similar to those given in Sections 3 and 4 can obviously be constructed for the Student hypothesis. Extension to any other hypothesis can easily be effected by assigning the probability of rejecting and the probability of accepting equal to the corresponding probabilities obtained using the normal test, as was done for the Student hypothesis above.

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