PROOF. Theorem 2 is an immediate consequence of Theorem 1 and Lemma 1 of [2] which established that if, for some $i, n_{ij}^{10} = 0$ for all $j \geqq 5$, then such an array cannot be extended to an eleven-rowed orthogonal array.

REMARK. It was also shown in Lemma 1 of [2] that if $k = 10$, then the array satisfies a unique set of solutions. Namely, $n_{i4}^{10} = 60$, $n_{i3}^{10} = n_{i2}^{10} = 0$, $n_{i1}^{10} = 20$, $n_{i0}^{10} = 0$, for all $i = 1, 2, \cdots, 81$. Hence any array constructed by the geometrical method developed by Bose and Bush [1] will satisfy this set of solutions. The problem of obtaining the totality of orthogonal arrays was investigated neither in the considered case nor in related cases.

In conclusion, we wish to remark that this paper restores the validity of the abstract published in *Ann. Math. Stat.*, Vol. 25 (1954), p. 177, which was unduly corrected in [2].

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A THEOREM ON CONVEX SETS WITH APPLICATIONS[1]

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1. Summary and introduction. T. W. Anderson [1] has proved the following theorem and has given applications to probability and statistics.

THEOREM 1. *Let E be a convex set in n-space, symmetric about the origin. Let $f(x) \geqq 0$ be a function such that* i) $f(x) = f(-x)$, ii) $\{x \mid f(x) \geqq u\} = K_u$ *is convex for every u* $(0 \leqq u \leqq \infty)$ *and* iii) $\int_E f(x)\,dx < \infty$, *then*

$$(1) \qquad \int_E f(x + ky)\,dx \geqq \int_E f(x + y)\,dx \qquad \text{for} \quad 0 \leqq k \leqq 1.$$

The purpose of this paper is to prove what can be considered a generalization of Anderson's Theorem and to give different statistical applications.

Functions in L_1 satisfying the hypothesis were called unimodal by Anderson and he noted in [1] that if we let $\varphi(y)$ be equal to the right hand side of (1) then φ is not unimodal in his sense insofar as it does not necessarily satisfy *ii* (i.e., there exist f, E, and u such that $\{x \mid \varphi(x) \geqq u\}$ is not convex). His example is

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the case where $n = 2$ and

$$f(x) = \begin{cases} 3, & |x_1| \leqq 1, \quad |x_2| \leqq 1, \\ 2, & |x_1| \leqq 1, \quad 1 < |x_2| \leqq 5, \\ 0, & \text{other } x, \end{cases}$$

where x_1, x_2 are the components of x relative to rectangular cartesian coordinate system.[2] Let E be the set of vectors where $|x_1| \leqq 1$, $|x_2| \leqq 1$. The set $\{x \mid \varphi(x) \geqq 6\}$ is not convex since for $x = (.5,4)$ and $x = (1,0)$, $\varphi(x) = 6$, while for $x = (.75,2)$, $\varphi(x) < 6$. The point of departure of this paper is to see what can be said about φ. This is achieved in Theorem 2, giving a stronger and more symmetrical statement than Anderson makes (but one which does not yield more information for his applications). The main Lemma, presented below, proceeds along the line of his argument but squeezes out additional information (convexity of level lines) under a weaker hypothesis (no symmetry assumptions) than Anderson uses at the corresponding stage of his argument.

2. Main result. Let E and K be convex sets in n-space, $\mathcal{E}$ (with no symmetry assumptions made at this time concerning E and K). For lebesgue measurable $A \subset \mathcal{E}$, let $V(A)$ be the lebesgue measure of A.

LEMMA 1. *If $\Phi(y) = V\{(E + y) \cap K\}$, then $\{y \mid \Phi(y) \geqq u\}$ is convex for each real u and convex E and K.*

PROOF. For $\alpha_1, \alpha_2 \geqq 0$, $\alpha_1 + \alpha_2 = 1$, we show

$$\alpha_1[(E + y_1) \cap K] + \alpha_2[(E + y_2) \cap K] \subset (E + \alpha_1 y_1 + \alpha_2 y_2) \cap K. \tag{2}$$

A typical element of the vector sum on the left hand side of the inclusion is $\alpha_1(x_1 + y_1) + \alpha_2(x_2 + y_2)$ where $x_1, x_2 \in E$, $x_1 + y_1 \in K$, $x_2 + y_2 \in K$. Since E and K are convex $\alpha_1(x_1 + y_1) + \alpha_2(x_2 + y_2) \in K$ and $\alpha_1 x_1 + \alpha_2 x_2 \in E$. These imply that $\alpha_1(x_1 + y_1) + \alpha_2(x_2 + y_2) \in (E + \alpha_1 y_1 + \alpha_2 y_2) \cap K$ which establishes relation (2).

Suppose $\Phi(y_1) \geqq u$ and $\Phi(y_2) \geqq u$. It is desired to show that

$$\Phi(\alpha_1 y_1 + \alpha_2 y_2) \geqq u. \tag{3}$$

By (2),

$$\Phi(\alpha_1 y_1 + \alpha_2 y_2) \geqq V\{\alpha_1[(E + y_1) \cap K] + \alpha_2[(E + y_2) \cap K]\}. \tag{4}$$

By the Brunn-Minkowski inequality,
$V^{1/n}(\alpha_1[(E + y_1) \cap K] + \alpha_2[(E + y_2) \cap K]) \geqq \alpha_1\Phi^{1/n}(y_1) + \alpha_2\Phi^{1/n}(y_2) \geqq u^{1/n}$. The last inequality with (4) yields (3), which proves the Lemma.

Let $\mathcal{C}_0$ be the closed convex cone generated in the uniform norm (i.e., $\|f\| = \sup\{|f(x)|\}$) by the characteristic functions of symmetric, compact, convex sets. Let $\mathcal{C}_1$ be the closed convex cone generated in the L_1 norm (i.e., $\|f\|_1 = \int |f(x)|\, dx$) by the characteristic functions of symmetric, compact, convex sets.

[2] The subscript notation is to have this denotation only for this example and one at the end of the paper.

Consider the norm $\|\quad\|_3$ given by $\|f\|_3 = \max\{\|f\|, \|f\|_1\}$. Let $\mathcal{C}_3$ be the closed convex cone generated in the $\|\quad\|_3$ norm by the characteristic functions of symmetric, compact, convex sets. Note that for E, K compact, convex and symmetric, the convolution of χ_K and χ_E, the characteristic functions of K and E respectively, evaluated at y becomes

$$\chi_K * \chi_E(y) = \int \chi_K(x)\chi_E(y - x)\, dx = \Phi(y)$$

Furthermore $\Phi(y)$ is continuous and by Lemma 1 has the property that $\{y \mid \Phi(y) \geqq u\}$ is a symmetric, compact, convex set for each real $u > 0$. If we let $\Phi(x, u)$ be the characteristic function of $\{y \mid \Phi(y) \geqq u\}$ then $\|\epsilon^{-1}\sum_{j=1}^{\infty}\Phi(x; j\epsilon) - \Phi(x)\| \to 0$ as $\epsilon \to 0^+$. Thus $\Phi \varepsilon \mathcal{C}_0$. The same argument shows that $\Phi \varepsilon \mathcal{C}_1$ and $\Phi \varepsilon \mathcal{C}_3$. The continuity of convolution in the $\|\quad\|_1$ and $\|\quad\|_3$ norms implies

THEOREM 2. *$\mathcal{C}_1 * \mathcal{C}_1 \subset \mathcal{C}_1$ and $\mathcal{C}_3 * \mathcal{C}_3 \subset \mathcal{C}_3$.*

By observing that $V(K)^{-1}\chi_K \varepsilon \mathcal{C}_1$ for K compact, symmetric, and convex and using the continuity properties of translation in the L_1 norm one can extend the conclusion of Theorem 2 to read $\mathcal{C}_1 * \mathcal{C}_1 = \mathcal{C}_1$.

It should be observed that for each $\Phi \varepsilon \mathcal{C}_3$ and each vector $y \varepsilon \mathcal{E}$, $\Phi(ky) \geqq \Phi(y)$ for $0 \leqq k \leqq 1$. This is true for Φ, the characteristic function of a symmetric convex set, and therefore for convex combinations of these. Since this property is preserved by taking uniform limits it follows that this property is true for each $\Phi \varepsilon \mathcal{C}_3$. Thus Theorem 2 implies Anderson's Theorem, since $\int_E f(x + y)\, dx \varepsilon \mathcal{C}_3$.

3. Applications. In the succeeding paragraphs we generalize to n dimensions and strengthen slightly in the case of 1 dimension some results of Z. W. Birnbaum [2] on random variables with comparable peakedness. It should be noted that the statistical problems with which Anderson concerned himself could be formulated in terms of peakedness and so the succeeding remarks, e.g., Lemma 2, apply to his applications also. Since in the applications Birnbaum is concerned with continuous random variables [3] and peakedness about the origin we will formulate our definitions for this case.

If Y and Z are continuous random variables whose values lie in $\mathcal{E}$ and whose probability densities are $\varphi(Y)$ and $f(Z)$ respectively, then Y is said to be more peaked (about the origin) than Z if

$$(\varphi - f) * \chi_E(0) \geqq 0 \tag{5}$$

for each compact, symmetric, convex E of $\mathcal{E}$. This coinincides with Birnbaum's definition when $n = 1$.

LEMMA 2. *If* 1) $(\varphi - f) * \chi_E(0) \geqq 0$ *for each compact, symmetric, convex* $E \subset \mathcal{E}$ *and* 2) $h \varepsilon \mathcal{C}_3$, *then* $(\varphi - f) * h * \chi_E(0) \geqq 0$.

PROOF. It suffices to show that $(\varphi - f) * X_F * \chi_E(0) \geqq 0$ for compact, symmetric, convex F since the closed (in $\|\quad\|_3$ norm) convex cone generated by these functions is dense in $\mathcal{C}_3$. However $\chi_F * \chi_E \in \mathcal{C}_2$ and so Lemma 2 follows from 1).

Using the lemma above we establish an n-dimensional variant of Birnbaum's Lemma (with slight reduction of symmetry hypotheses).

LEMMA 3. *Let Y_1, Y_2, Z_1, Z_2, be continuous random variables with values in $\mathcal{E}$ and probability densities $\varphi_1(Y_1)$, $\varphi_2(Y_2)$, $f_1(Z_1)$, and $f_2(Z_2)$ such that*

1) *Y_1 and Y_2 are independent, Z_1 and Z_2 are independent,*
2) *$\varphi_2 \, \varepsilon \, \mathcal{C}_3$ and $f_1 \, \varepsilon \, \mathcal{C}_3$*
3) *Y_i is more peaked (about 0) than Z_i for $i = 1, 2$.*

Then

4) *$Y_1 + Y_2$ is more peaked (about 0) than $Z_1 + Z_2$.*

PROOF. For each compact, symmetric, convex set, E,

$$
\begin{aligned}
\varphi_1*\varphi_2*\chi_E(0) - f_1*f_2*\chi_E(0) \\
&= \varphi_1*\varphi_2*\chi_E(0) - f_1*\varphi_2*\chi_E(0) + f_1*\varphi_2*\chi_E(0) - f_1*f_2*\chi_E(0) \\
&= (\varphi_1 - f_1)*\varphi_2*\chi_E(0) + (\varphi_2 - f_2)*f_1*\chi_E(0) \geqq 0.
\end{aligned}
$$

NOTE. The algebraic manipulations in the proof are already implicit at one stage of Birnbaum's argument but after that he uses devices whose direct analogue did not go through for $n > 1$ and whose weight is carried here by previous Lemma. The slight reduction in symmetry assumptions in the case $n = 1$ can be established without all the machinery used here. The later parts of his paper also go through for the multivariate case.

Here it may be wondered whether the requirement that $\varphi_2 \, \varepsilon \, \mathcal{C}_3$ and $f_1 \, \varepsilon \, \mathcal{C}_3$ can be changed to $\varphi_2 \, \varepsilon \, \mathcal{C}_3$ and $f_2 \, \varepsilon \, \mathcal{C}_3$. In the following example (constructed by T. W. Anderson and the author) for $n = 1$ not only $\varphi_2 \, \varepsilon \, \mathcal{C}_3$, $f_2 \, \varepsilon \, \mathcal{C}_3$ but also Y_1 and Z_1 are symmetrically distributed and the other hypotheses of the Lemma are satisfied (with the exception of the random variables being continuously distributed, but that can be taken care of by considering nearby distributions). Nevertheless $Y_1 + Y_2$ is not more peaked (about 0) than $Z_1 + Z_2$.

EXAMPLE. Let Y_1, Y_2 be independent random variables, Z_1, Z_2 independent random variables such that $Y_1 = Z_1$ with $\Pr\{Y_1 = 5\} = \Pr\{Y_1 = -5\} = \Pr\{Z_1 = 5\} = \Pr\{Z_1 = -5\} = \frac{1}{2}$ and

$$
\varphi_2(Y_2) = \begin{cases} \frac{1}{2}, & |Y_2| \leqq 1, \\ 0, & |Y_2| > 1, \end{cases}
$$

$$
f_2(Z_2) = \begin{cases} \frac{1}{4}, & |Z_2| \leqq 2, \\ 0, & |Z_2| > 2. \end{cases}
$$

Here it is not true that $Y_1 + Y_2$ is more peaked (about 0) than $Z_1 + Z_2$.

We close this note with the following

CONJECTURE. Let $f \, \varepsilon \, L_1(\mathcal{E})$ and let $\int_E f(x + y)\, dx = \Phi(y)$, then $\Phi_E(ky) \geqq \Phi_E(y)$ for each $y \, \varepsilon \, \mathcal{E}$, $0 \leqq k \leqq 1$, and for each compact, symmetric, convex $E \subset \mathcal{E}$ implies that $f \, \varepsilon \, \mathcal{C}_1$.

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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Berkeley meeting of the Institute, July 14–16, 1955*)

1. Nonparametric Mean Estimation of Percentage Points and Density Function Values. JOHN E. WALSH, Lockheed Aircraft Corporation.

Consider a sample of size n from a statistical population with probability density function $f(x)$ and $100p$ per cent point θ_p. The function $f(x)$ is of an analytic nature. Some methods are presented for approximate nonparametric expected value estimation of θ_p and of $1/f(\theta_p)$. A nonparametric estimate whose expected value differs from θ_p by terms of order $n^{-7/2}$ can be obtained. For $1/f(\theta_p)$, an estimate whose expected value is accurate to terms of order n^{-3} can be obtained. The estimates developed consist of linear functions of specified order statistics of the sample. The order statistics used are sample percentage points with percentage values which are near $100p$. Let m be the number of order statistics appearing in an estimate ($m \leqq 7$). Coefficients for the linear estimation function are obtained by solving a specified set of m linear equations in m unknowns. All estimates derived for θ_p have variances of the form $p(1 - p)/nf(\theta_p)^2 + 0(n^{-3/2})$. Without additional information, all that can be determined about the variances of the estimates derived for $1/f(\theta_p)$ is that they are $0(n^{-1/2})$. Thus both types of estimates are consistent but the estimates for θ_p are more efficient than those for $1/f(\theta_p)$.

2. On the Concept of Probability in Quantum Mechanics. A. O. BARUT, Stanford.

Some mathematical consequences of the following particular probability measure are discussed: Consider the one to one correspondence between the elements of the sample space Ω and the linearly independent elements of a unitary space $\bigvee$ (in general a Hilbert space). The probability measure of sets in Ω is defined by $p(S) = (P_S x, x) = \| P_S x \|^2$, where $(x, x) = 1$ and P_S is the projection operator on the manifold spanned by vectors corresponding to the points in S. The vector x characterizes the system or the experiment. It follows from $p(S)$ that random variables are represented by linear Hermitian operators. These random variables may have an intrinsic correlation coefficient even though they are independent in the ordinary sense; they apply to a larger class of phenomena.

3. Two-Sample Estimates of Prescribed Precision. (Preliminary Report.) ALLAN BIRNBAUM, Columbia University and Stanford University.

Let $x_1, x_2, \cdots$ be independent observations on a random variable X with density (or discrete probability) function $f(x, \theta)$, with θ unknown, $\theta \in \Omega$, $E(X) = \mu = \mu(\theta)$, Var $(X) = \sigma^2(\theta)$. Suppose an unbiased estimate of μ is required, with variance not exceeding a given