THE BIAS IN CERTAIN ESTIMATES OF THE PARAMETERS OF THE EXTREME-VALUE DISTRIBUTION

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Summary. This paper is mostly concerned with a modification of the maximum-likelihood estimate of the scale parameter of the extreme-value distribution for which the bias can be explicitly obtained. A formula for computing this bias is derived, and bias factors are tabulated for sample sizes from $n = 2$ to $n = 112$. A brief comparison is made between this estimator and the optimum linear estimator for a sample of size $n = 6$. Attention is called to a bias which results from the maximum-likelihood estimate of the second parameter, and formulas for the bias and the variance of this estimate are obtained. In the concluding section, the significance of certain aspects of the maximum-likelihood estimate of the scale parameter in practical applications is briefly discussed.

1. The equations for the maximum-likelihood estimate of the parameters. Given the extreme-value distribution of Type I [2]

$$\Phi(y) = \exp(-e^{-y}), \qquad y = \alpha(x - u), \tag{1.1}$$

where Φ denotes the cdf, the equations for the maximum-likelihood estimate of the parameters may be reduced to [6]

$$\frac{1}{\alpha} = \bar{x} - \frac{\sum xe^{-\alpha x}}{\sum e^{-\alpha x}}, \tag{1.2}$$

$$e^{-\alpha u} = \sum e^{-\alpha x}/n, \tag{1.3}$$

where the summations are taken over the n values of x in the sample. The first of these equations, although involving only the one unknown, α, is somewhat intractable of solution. A variation on such a solution may be accomplished as follows:

2. Modification of the maximum-likelihood estimate of α. Substituting equation (1.3) in (1.2) one can write

$$\frac{1}{\alpha} = \bar{x} - \frac{\sum xe^{-\alpha(x-u)}}{n}. \tag{2.1}$$

One may further note that from (1.1), using "log" to represent natural logarithm,

$$e^{-\alpha(x-u)} = -\log \Phi.$$

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Introducing the subscript i to denote a specific sample value, (2.1) becomes

$$\frac{1}{\alpha} = \bar{x} + \frac{\sum x_i \log \Phi_i}{n}.$$

So far this represents a transformation of the maximum-likelihood relations and hence a form of the maximum-likelihood estimate of the parameters. At this point it will be of interest to investigate the possible bias that would be incurred if $1/\alpha$ were estimated from the right-hand side of the last equation under the assumption that the value of $\log \Phi_i$ were the true value for each x_i (based on the true values of the population parameters). We denote such an estimate by $(1/\alpha)_0$. Thus

$$(1/\alpha)_0 = \bar{x} + \frac{\sum x_i \log \Phi_i}{n} = \bar{x} + \frac{\sum xe^{-\alpha(x-u)}}{n}, \tag{2.2}$$

and we note that the values of α and u which appear explicitly or implicitly on the right are the population parameters and not maximum-likelihood estimates of those parameters.

One seeks to evaluate $E[(1/\alpha)_0]$. From the well-known relation

$$E[\bar{y}] = E[\alpha(\bar{x} - u)] = \gamma = \text{Euler's constant } (=0.577216),$$

one obtains

$$E[\bar{x}] = u + \gamma/\alpha. \tag{2.3}$$

It is also known that (see [6], p. 111)

$$E[\sum (x_i - u)e^{-\alpha(x_i-u)}] = -n(1 - \gamma)/\alpha \tag{2.4}$$

and that

$$E[\sum e^{-\alpha(x_i-u)}] = n.$$

It follows that

$$E[\sum (x_i e^{-\alpha(x_i-u)})/n] = -(1 - \gamma)/\alpha + u. \tag{2.5}$$

Combining (2.3) and (2.5) in (2.2), we have

$$E[(1/\alpha)_0] = 1/\alpha. \tag{2.6}$$

A modification of the relation (2.2) is proposed by substitution of $E[\log \Phi_i]$ for $\log \Phi_i$. For any well-behaved cdf it is known that

$$E[-\log \Phi_m] = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} = \psi(n + 1) - \psi(m), \tag{2.7}$$

where Φ_m denotes the cumulative distribution function corresponding to the mth-ordered sample value proceeding from the smallest $x = x_1$ to the largest $x = x_n$, and where $\psi(x)$ is the logarithmic derivative of the gamma function.[1]

[1] This is a tabulated function; e.g., H. T. Davis, *Tables of Higher Mathematical Functions*, Vol. 1. Principia Press, Bloomington, Indiana, 1933.

There is undoubtedly some loss in efficiency in making the above substitution. This at present we have not found it possible to measure. The *bias* introduced by this substitution can be measured, and knowledge of it is essential.

3. General equation for measuring the bias. Denoting $E[\log \Phi_m]$ by $\langle \log \Phi_m \rangle$ and replacing $1/\alpha$ by β, the proposed equation for the estimate of $1/\alpha$ is

$$\hat{\beta} = \bar{x} + \sum x_m \langle \log \Phi_m \rangle / n. \tag{3.1}$$

Since (2.2) does not produce a bias, the bias incurred by the estimate $\hat{\beta}$ is given by

$$E[\hat{\beta} - 1/\alpha] = E[\sum x_m(\langle \log \Phi_m \rangle - \log \Phi_m)/n]. \tag{3.2}$$

Note that

$$E[u \sum \langle \log \Phi_m \rangle] - E[u \sum \log \Phi_m] = 0. \tag{3.3}$$

Subtracting this from (3.2) and noting definition of y in (1.1),

$$\alpha E[\hat{\beta} - 1/\alpha] = E[\sum y_m \langle \log \Phi_m \rangle / n] - E[\sum y_m \log \Phi_m / n]. \tag{3.4}$$

From (2.4)

$$E[\sum y_m(-\log \Phi_m)/n] = -(1 - \gamma),$$

and with

$$E[y_m] = \bar{y}_m ,$$

the general formula for the bias of $\hat{\beta}$ *is given by*

$$\alpha E[\hat{\beta} - 1/\alpha] = -(1 - \gamma) - \sum \bar{y}_m \langle -\log \Phi_m \rangle / \text{n}, \tag{3.5}$$

where

$$\langle -\log \Phi_m \rangle = \frac{1}{m} + \frac{1}{m + 1} + \cdots + \frac{1}{n}, \quad m = 1, 2, \cdots, n,$$

and index $m = 1$ *corresponds to smallest extreme and* $m = n$ *to the largest, with* γ *representing Euler's constant* 0.577216.

4. Reduction of general formula for bias. Two formulas for the computation of $\bar{y}_m$ exist in the literature. One was published by the author in 1947 [5]:

$$\bar{y}_m = \gamma + \sum_{i=0}^{r} (-1)^i C_i^n \Delta^i \log (n - i), \qquad r = n - m, \tag{4.1}$$

where Δ^i represents forward difference of ith order. Another formula for $\bar{y}_m$ was derived by Lieblein and published in 1953 [7]:

$$\bar{y}_m = mC_m^n \sum_{i=0}^{r} (-1)^i C_i^r g(m + i), \qquad g(z) = (\gamma + \log z)/z, \quad r = n - m. \tag{4.2}$$

Although for computational purposes there is little difference between the two formulas, the author found that the reduction in question proceeds more directly from the substitution of (4.2) into (3.5). After considerable maneuvering,[2] one arrives at

$$-\frac{\sum \bar{y}_m\langle \log \Phi_m \rangle}{n} = \gamma - \sum_{r=2}^{n} \Delta_{-1}^{r-2}\left(\frac{\log r}{r(r-1)}\right), \tag{4.3}$$

where Δ_{-1}^{k} refers to the kth-order difference with unit interval, proceeding in the negative sense. This, in turn, can be expressed as

$$\frac{\sum \bar{y}_m\langle -\log \Phi_m \rangle}{n} = \gamma - \sum_{r=2}^{n} \frac{1}{r(r-1)} \sum_{i=1}^{r-1} (-1)^{i+1}\Delta^{i} \log 1. \tag{4.4}$$

Here the differences are taken in the positive sense. This formula is better suited to computation than the preceding one when differences of the logarithm are already available. A further reduction gives a form which is better suited to theoretical evaluation, as will be seen in the next section. This is

$$\frac{\sum \bar{y}_i\langle -\log \Phi_i \rangle}{n} = \gamma - \sum_{r=1}^{n-1} (-1)^{r+1} \frac{\Delta^{r} \log 1}{r} + \frac{1}{n}\left[\sum_{r=1}^{n-1} (-1)^{r+1}\Delta^{r} \log 1\right]. \tag{4.5}$$

5. On the convergence of the above series. From the general theory of a distribution function of an ordered sample value, it can be proved that as $n \to \infty$

$$E[\sum y_i\langle \log \Phi_i\rangle/n] - E[\sum y_i \log \Phi_i/n] \to 0$$

if Φ is continuous. Accordingly, from (3.5), the expressions on the right of equations (4.3) to (4.5) should approach $\gamma - 1$ as $n \to \infty$. Set

$$R_n(x) = \sum_{1}^{n-1} \frac{(-1)^{r+1}\Delta^{r} \log x}{r}. \tag{5.1}$$

By expanding a function $f(x + t)$ in a series of differences about $t = 0$ and differentiating as to t, it is easily verified that

$$f'(x) + \omega(x) = \sum_{1}^{\infty} (-1)^{r+1} \frac{\Delta^{r} f(x)}{r}, \tag{5.2}$$

where $\omega(x)$ is periodic with period unity. Thus

$$1/x + \omega(x) = R_\infty(x), \qquad x > 0. \tag{5.3}$$

[2] An alternative reduction has been suggested by a referee. This involves writing out the differences $\Delta^i \log (n - i)$ in (4.1) as linear functions of $\log (n - i)$ and multiplying by the expression for $E[-\log \Phi_m]$ given in (2.7). He obtains

$$\frac{1}{n} \sum_{m=1}^{n} \bar{y}_m\langle -\log \Phi_m\rangle = \gamma + \frac{1}{n} \sum_{i=1}^{n-1} \frac{(-1)^i}{i} C_{i+1}^{n} \log(i + 1).$$

This becomes identical with (4.4) when expressed in terms of differences of the logarithm. It is believed that the form (4.4) is more convenient for purposes of computation because of the relation (6.6).

It is known that [11]

$$\lim_{n\to\infty} n^x \log n \mid \Delta^n \log x \mid = \Gamma(x), \qquad x > 0, \tag{5.4}$$

and that for finite n [3]

$$\mid \Delta^n \log x \mid < \frac{n!}{x(x+1)\cdots(x+n)}, \qquad n > 2, \quad x > 0. \tag{5.5}$$

Hence the infinite series $R_\infty(x)$ converges for positive values of x uniformly for $x \geqq x_0 > 0$. Accordingly,

$$\lim_{x\to\infty} R_\infty(x) = 0.$$

It follows that

$$\omega(x) \equiv 0, \qquad x \geqq x_0 .$$

We have then

$$R_\infty(x) = 1/x, \qquad x > 0,$$

and

$$R_\infty(1) = 1.$$

It is easily proved from (5.4) and (5.5) that the last series in (4.5) approaches zero as $n \to \infty$. This completes the proof of the convergence of the sum of the two series on the right of (4.5) to value unity. It follows that the series on the right of (4.3) and (4.4) also converge to value unity as n becomes infinite.

6. Tabulation of the bias factor. For purposes of computing a table of bias factors, the formula (4.4) has been used, since it was possible to obtain from the National Bureau of Standards a table of $\Delta^i \log 1$ from $i = 1$ to $i = 111$. For purposes of reference we refer to the series on the right of (4.4) as $S(n)$. Substituting (4.4) into (3.5), the general equation for the bias reduces to

$$E[\hat{\beta} - 1/\alpha] = -1/\alpha + S(n)/\alpha \tag{6.1}$$

or

$$E[\hat{\beta}] = S(n)/\alpha. \tag{6.2}$$

In tabulating the bias for ready use it seems preferable to tabulate a factor b_n such that

$$1/\alpha = b_n E[\hat{\beta}]; \tag{6.3}$$

hence *the bias factor* b_n *is given by*

$$b_n = 1/S(n), \tag{6.4}$$

where

$$S(n) = \sum_{r=2}^{n} \frac{1}{r(r-1)} \sum_{i=1}^{r-1} (-1)^{i+1} \Delta^i \log 1. \tag{6.5}$$

As a computational matter it is to be noted that

$$S(n) - S(n-1) = \frac{1}{n(n-1)} \sum_{1}^{n-1} (-1)^{i+1} \Delta^i \log 1. \tag{6.6}$$

The author has computed the series to seven decimal places, using the tabulated values of $\Delta^i \log 1$ furnished by the National Bureau of Standards. Through

$n = 21$, values of $S(n)$ were checked by alternate computation, using the series on the right of (4.5) Values of $S(n)$ agreed to within two units in the seventh decimal place. Beyond that point computations of $S(n)$ were checked by repetition and by comparing cumulative adding-machine tapes. In this way $S(n)$ was tabulated to $n = 112$, accuracy being guaranteed to the sixth decimal place, subject, of course, to the accuracy of values of $\Delta^i \log 1$ furnished by the Bureau of Standards.

A table of the reciprocals of $S(n)$ carried to the nearest fourth decimal place, and giving the bias factor b_n, is shown below.

TABLE OF BIAS FACTOR

Estimate of $1/\alpha = b_n\hat{\beta}$, n = *size of sample*

n	b_n	n	b_n	n	b_n	n	b_n
1	—	31	1.0743	61	1.0388	91	1.0265
2	2.8854	32	1.0720	62	1.0382	92	1.0263
3	1.9606	33	1.0699	63	1.0376	93	1.0260
4	1.6503	34	1.0679	64	1.0371	94	1.0257
5	1.4941	35	1.0661	65	1.0365	95	1.0255
6	1.3997	36	1.0643	66	1.0360	96	1.0252
7	1.3363	37	1.0626	67	1.0355	97	1.0250
8	1.2907	38	1.0610	68	1.0350	98	1.0247
9	1.2563	39	1.0595	69	1.0345	99	1.0245
10	1.2294	40	1.0581	70	1.0340	100	1.0243
11	1.2078	41	1.0567	71	1.0336	101	1.0240
12	1.1900	42	1.0555	72	1.0331	102	1.0238
13	1.1751	43	1.0542	73	1.0327	103	1.0236
14	1.1625	44	1.0530	74	1.0323	104	1.0234
15	1.1516	45	1.0519	75	1.0319	105	1.0232
16	1.1421	46	1.0508	76	1.0315	106	1.0229
17	1.1337	47	1.0498	77	1.0311	107	1.0227
18	1.1264	48	1.0488	78	1.0307	108	1.0225
19	1.1198	49	1.0478	79	1.0303	109	1.0223
20	1.1139	50	1.0469	80	1.0300	110	1.0222
21	1.1085	51	1.0460	81	1.0296	111	1.0220
22	1.1037	52	1.0452	82	1.0293	112	1.0218
23	1.0993	53	1.0444	83	1.0289		
24	1.0952	54	1.0436	84	1.0286		
25	1.0915	55	1.0428	85	1.0283		
26	1.0881	56	1.0421	86	1.0280		
27	1.0849	57	1.0414	87	1.0277		
28	1.0820	58	1.0407	88	1.0274		
29	1.0792	59	1.0401	89	1.0271		
30	1.0767	60	1.0394	90	1.0268		

7. Application to Type II extreme-value distribution. In certain problems where a finite lower limit bounds the observed variable, the Type II or Type III extreme-value distribution is found more descriptive of the practical situation (e.g., application to breaking strength of materials [1], maximum wind speeds [10], etc.). Since the treatment is the same except for sign, we examine only the case of the Type II distribution. This distribution is usually given in the following form (see formula (3.19) of [2]):

$$\Psi = \exp\left[-(z/b)^{-\alpha}\right], \qquad z > 0, \tag{7.1}$$

where Ψ is the cdf of the variable z, and α and b are parameters. Taking logarithms, this is the same as

$$-\log(-\log\Psi) = \alpha(\log z - \log b). \tag{7.2}$$

Thus if we set

$$x = \log z \qquad \text{and} \qquad u = \log b, \tag{7.3}$$

the distribution (7.1) becomes identical with (1.1), with

$$y = \alpha(x - u) = \alpha(\log z - \log b), \tag{7.4}$$

and the parameter α is the same in the two distributions.

Hence the estimation of the parameter α, or its reciprocal β, may proceed by setting

$$x_i = \log z_i \tag{7.5}$$

and applying equation (3.1) for estimate of β followed by use of the tabulated bias factor.

8. Comparison of estimator $\hat{\beta}$ with that of optimum weighting for sample of size six. The estimator $\hat{\beta}$ defined by (3.1) is a linear function of the sample values x_m, with coefficients given by

$$c_m = (1 + \langle\log\Phi_m\rangle)/n, \qquad \sum c_m = 0.$$

In a recent monograph, Lieblein [8] has developed a linear unbiased estimator of β in which the coefficients are determined so that the variance of the estimate is a minimum. The "optimum" coefficients or "weights" have been determined explicitly for samples as large as six. For larger samples, a grouping procedure is recommended. Specific optimum weights for each ordered value of larger samples are not available.

It will be of some interest to compare the series of weights of the two linear estimators for a sample of size $n = 6$. For the estimator $\hat{\beta}$ of (3.1), the bias factor for $n = 6$ is found from the table to be $b_6 = 1.3997$. Thus the unbiased estimate of $1/\alpha$ is $b_6\hat{\beta}$ with coefficients $b_6 c_m$, which turn out to be

$$w_1 = -.3383, \quad w_2 = -.1050, \quad w_3 = .0117,$$
$$w_4 = .0894, \quad w_5 = .1477, \quad w_6 = .1944.$$

The optimum weights found by Lieblein for a sample of size six are

$$w_1 = -.4593, \quad w_2 = -.0360, \quad w_3 = .0732,$$
$$w_4 = .1267, \quad w_5 = .1495, \quad w_6 = .1458.$$

It is perhaps of some significance that in each case the sum of the weights is zero.

9. Bias of estimate of parameter u from the maximum-likelihood equations. A curious fact about the relationship of the parameter u to the maximum-likelihood equations is that *if u be estimated from the maximum-likelihood equations under the assumption that the other parameter α be known, a bias results.* The author discovered this some ten years ago while working on a related paper [4]. With the present interest in the extreme-value distribution, it seems worth while to bring this out. Evaluation of the bias proceeds as follows:

Define variable ξ by

$$\xi_i = e^{-\alpha x_i} \tag{9.1}$$

and take

$$\bar{\xi} = \sum e^{-\alpha x_i}/n, \qquad \xi_0 = e^{-\alpha u}, \tag{9.2}$$

where x_i is distributed as in (1.1).

With α known, u is to be estimated from (1.3) above. Denoting this estimate by $\hat{u}$, we have as the equation of estimate

$$e^{-\alpha \hat{u}} = \sum e^{-\alpha x_i}/n = \bar{\xi} \tag{9.3}$$

and note from (9.2) that

$$e^{-\alpha(\hat{u}-u)} = \bar{\xi}/\xi_0 \tag{9.4}$$

and hence that

$$\alpha(\hat{u} - u) = -\log(\bar{\xi}/\xi_0). \tag{9.5}$$

Thus taking the moment generating function as (see [6])

$$G(\theta) = E[e^{-\theta \log(\bar{\xi}/\xi_0)}] = E[(\bar{\xi}/\xi_0)^{-\theta}],$$
$$G'(0) = \alpha E[\hat{u} - u].$$

In a previous paper (see formula (4.3) of [4]), the author showed that the pdf of $\bar{\xi}$ is given by

$$P(\bar{\xi})\, d\bar{\xi} = [1/\Gamma(n)] e^{-n\bar{\xi}/\xi_0} (n\bar{\xi}/\xi_0)^{n-1} n \, d\bar{\xi}/\xi_0 .$$

Hence

$$G(\theta) = n^{\theta}\Gamma(n - \theta)/\Gamma(n) \tag{9.6}$$

and

$$G'(0) = -\Gamma'(n)/\Gamma(n) + \log n = \gamma + \log n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right).$$

Accordingly, *the bias in the maximum-likelihood estimate of u, with α known, is given by*

$$(9.7) \qquad E[\hat{u} - u] = (1/\alpha)\left[\gamma + \log n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)\right].$$

In this connection, note from (9.6) that

$$E[\bar{\xi}/\xi_0] = G(-1) = 1$$

and hence that

$$(9.8) \qquad E[e^{-\alpha\hat{u}}] = e^{-\alpha u}.$$

Thus, if, in the Type II distribution, *one took the parameters as α and $b^{-\alpha}$, and if $b^{-\alpha}$ were estimated from*

$$(9.9) \qquad \text{Estimate of } b^{-\alpha} = e^{-\alpha\hat{u}} = \bar{\xi}$$

with α known, no bias would result from such estimate.

It may be of interest in this connection to evaluate the variance of the estimate $\hat{u}$, with parameter α known. From the definition of $G(\theta)$, it follows that

$$G''(0) = E[\alpha^2(\hat{u} - u)^2].$$

From (9.6) we have

$$G''(0) = (\log n)G'(0) - (\log n)\Gamma'(n)/\Gamma(n) + \Gamma''(n)/\Gamma(n).$$

This can be reduced to

$$(9.10) \qquad G''(0) = [G'(0)]^2 + \psi'(n),$$

where

$$(9.11) \qquad \psi(x) = \Gamma'(x)/\Gamma(x).$$

This latter function is a well-known mathematical function [9]. Its derivative when x is an integer is an infinite series of the reciprocals of the squares of the natural numbers beginning with x, and $\psi'(1) = \pi^2/6$. Thus we have the result that *the variance of the estimate $\hat{u}$, with parameter α known, is given by*

$$(9.12) \qquad E[(\hat{u} - u)^2] - (\text{Bias})^2 = (1/\alpha^2)[\pi^2/6 - (1 + 1/2^2 + 1/3^2 + \cdots + 1/(n-1)^2].$$

10. General remarks. A fact about the maximum-likelihood estimate of the scale parameter $\beta = 1/\alpha$ as it relates to practical problems, which does not seem to have been brought out in the literature, is the following: Inspection of equation (1.2) or (3.1) shows that in this estimate values of the observed series x_i that are near the lower extreme are much more heavily weighted than those near the upper extreme. From a theoretical point of view this is entirely rational. In practice, however, the extreme-value distribution is often used to fit an observed series of extremes without any very sound theoretical basis—merely because it seems to describe fairly well the behavior of the *upper* part of the series of extremes. It should be pointed out that in such cases the maximum-likelihood estimate of β may very well be *worse* than an estimate which gives less weight to the lower part of the series.

For example, the distribution of Type I is sometimes used even when the lower limit of the series of observations is *fixed*. In such a case, theory is not satisfied at that end of the series and, accordingly, some distortion of the theoretical fit at that end of the series is to be expected. Hence in such a case preference for the maximum-likelihood estimate because of its greater theoretical efficiency is questionable.

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