ON THE UNIQUENESS OF WALD SEQUENTIAL TESTS[1]

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1. Summary. Under certain mild restrictions on the distributions involved, it is shown that the probabilities of the two types of error uniquely determine the two bounds characterizing the Wald sequential probability ratio test.

2. Introduction. $X_1, X_2, \cdots$ is an infinite sequence of independent and identically distributed chance variables. The density of X_1 is $f_i(x)$ under H_i, where $i = 1, 2$. We assume that under either H_1 or H_2 the chance variable $f_2(X_1)/f_1(X_1)$ has a distribution which assigns a positive probability to any nondegenerate interval in the interval $[0, \infty]$, and zero probability to any point in that interval.

B, A shall denote the stopping bounds characterizing the usual Wald sequential probability ratio test. As usual, $B < A$. $Q_i[R; T]$ shall denote the probability under H_i that the value of the final probability ratio is in the region R, when the sequential stopping rule is to stop the first time the value of the probability ratio is in T, and not before; $u(z)$ shall denote the set of numbers less than or equal to z; $v(z)$ shall denote the set of numbers greater than or equal to z. The union of any two sets R and T shall be denoted by $R + T$. We note the following easily proved inequality for future reference: if b, a are any two finite positive numbers with $b < a$, then

$$Q_2[u(b); u(b) + v(a)] < b \cdot Q_1[u(b); u(b) + v(a)].$$

In what follows, $\theta_1, \theta_2, \cdots$ shall be numbers between zero and one.

For any given B, A, we denote by $\alpha(B, A)$ the probability of accepting H_2 when H_1 is true when using the Wald test with bounds B, A; while $\beta(B, A)$ denotes the probability of accepting H_1 when H_2 is true and the Wald test with bounds B, A is used.

3. Proof of uniqueness. Let α, β be two given numbers between zero and one, such that the equalities $\alpha(B, A) = \alpha$ and $\beta(B, A) = \beta$ imply the strict inequalities $0 < B < A < \infty$. Then we have:

Theorem. *There is at most one solution to the equations* $\alpha(B, A) = \alpha$, $\beta(B, A) = \beta$, *the unknowns being* B, A.

Proof. We assume that there is at least one solution to these equations. Let B be any number for which it is possible to find an A greater than B with

$$\alpha(B, A) = \alpha.$$

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Fixing B, we shall show that the equation $\alpha(B, A) = \alpha$ is satisfied for exactly one value of A. This is so because for a fixed B, $\alpha(B, A)$ is a strictly decreasing function of A under the assumptions made above. We denote the value of A satisfying $\alpha(B, A) = \alpha$ by $A(B)$. It is easily seen that $A(B)$ is a strictly decreasing and continuous function of B, and the set of all B for which $A(B)$ exists is an interval.

From now on, we shall denote $\beta(B, A(B))$ by $\beta(B)$. Our next step is to show that $\beta(B)$ is a strictly increasing function of B, and this will complete the proof of the theorem. For a given B, we can find a positive ΔB so small that $B + \Delta B < A(B + \Delta B)$. We denote $A(B) - A(B + \Delta B)$ by ΔA. We denote by R the set of numbers no greater than B, by S the set of numbers between B and $B + \Delta B$, by T the set of numbers between $A(B) - \Delta A$ and $A(B)$, by U the set of numbers greater than $A(B)$, and finally we denote the set $R + S + T + U$ by V. We have the following relationships, where z is the variable of integration:

$$\begin{aligned}\beta(B) = Q_2[R; V] &+ \int_B^{B+\Delta B} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2[u(z); V] \\ &+ \int_{A(B)-\Delta A}^{A(B)} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2\,[u(z); V];\end{aligned} \tag{3.1}$$

and

$$\beta(B + \Delta B) = Q_2[R; V] + Q_2[S; V]. \tag{3.2}$$

Then we get

$$\begin{aligned}\beta(B + \Delta B) - \beta(B) = Q_2[S; V] &\\ &- \int_B^{B+\Delta B} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2[u(z); V] \\ &- \int_{A(B)-\Delta A}^{A(B)} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2[u(z); V].\end{aligned} \tag{3.3}$$

Also, we have that the expression we get by replacing the subscripts 2 in the right-hand side of (3.1) by the subscripts 1 is equal to $1 - \alpha$, as is the expression on the right-hand side of (3.2) when the same change of subscripts is made. Then we get by subtraction

$$
\begin{aligned}
Q_1[S;V] = &\int_B^{B+\Delta B} Q_1\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_1[u(z); V] \\
&+ \int_{A(B)-\Delta A}^{A(B)} Q_1\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_1[u(z); V].
\end{aligned} \tag{3.4}
$$

Using some obvious continuity properties of the function Q_1, we get

$$
Q_2[S; V] = (B + \theta_1 \Delta B) \cdot Q_1[S; V], \tag{3.5}
$$

and combining (3.3), (3.4), and (3.5), we get

$$
\begin{aligned}
&\beta(B + \Delta B) - \beta(B) \\
&\quad = (B + \theta_1 \Delta B) \int_B^{B+\Delta B} Q_1\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_1[u(z); V] \\
&\quad + (B + \theta_1 \Delta B) \int_{A(B)-\Delta A}^{A(B)} Q_1\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_1[u(z); V] \\
&\quad - \int_B^{B+\Delta B} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2[u(z); V] \\
&\quad - \int_{A(B)-\Delta A}^{A(B)} Q_2\left[u\left(\frac{B}{z}\right); u\left(\frac{B}{z}\right) + v\left(\frac{A}{z}\right)\right] dQ_2[u(z); V].
\end{aligned} \tag{3.6}
$$

Again using continuity properties of Q_1, we can write (3.6) as follows:

$$
\begin{aligned}
&\beta(B + \Delta B) - \beta(B) \\
&\quad = (B + \theta_1 \Delta B) \cdot Q_1\left[u\left(\frac{B}{B + \theta_2 \Delta B}\right); u\left(\frac{B}{B + \theta_2 \Delta B}\right)\right. \\
&\qquad\qquad \left. + v\left(\frac{A}{B + \theta_2 \Delta B}\right)\right] \cdot Q_1[S; V] \\
&\quad + (B + \theta_1 \Delta B) \cdot Q_1\left[u\left(\frac{B}{A(B) - \theta_3 \Delta A}\right); u\left(\frac{B}{A(B) - \theta_3 \Delta A}\right)\right. \\
&\qquad\qquad \left. + v\left(\frac{A}{A(B) - \theta_3 \cdot \Delta A}\right)\right] \cdot Q_1[T; V] \\
&\quad - Q_2\left[u\left(\frac{B}{B + \theta_4 \Delta B}\right); u\left(\frac{B}{B + \theta_4 \Delta B}\right) + v\left(\frac{A}{B + \theta_4 \Delta B}\right)\right] \cdot Q_2[S; V] \\
&\quad - Q_2\left[u\left(\frac{B}{A(B) - \theta_5 \Delta A}\right); u\left(\frac{B}{A(B) - \theta_5 \Delta A}\right)\right. \\
&\qquad\qquad \left. + v\left(\frac{A}{A(B) - \theta_5 \Delta A}\right)\right] \cdot Q_2[T; V].
\end{aligned} \tag{3.7}
$$

But $Q_2[S; V] = (B + \theta_1 \Delta B) \cdot Q_1[S; V]$, while $Q_2[T; V] = (A(B) - \theta_6 \cdot \Delta A) \cdot Q_1[T; V]$, and using these relationships in (3.7) we get:

$$
\begin{aligned}
&\beta(B + \Delta B) - \beta(B) \\
&= (B + \theta_1 \Delta B) \cdot Q_1[S; V] \cdot \left\{ Q_1 \left[u\left(\frac{B}{B + \theta_2 \Delta B}\right); u\left(\frac{B}{B + \theta_2 \Delta B}\right) \right. \right. \\
&\qquad\qquad \left. + v\left(\frac{A}{B + \theta_2 \Delta B}\right) \right] \\
&\quad \left. - Q_2\left[u\left(\frac{B}{B + \theta_4 \Delta B}\right); u\left(\frac{B}{B + \theta_4 \Delta B}\right) + v\left(\frac{A}{B + \theta_4 \Delta B}\right) \right] \right\} \\
&\quad + Q_1[T; V] \cdot \left\{ (B + \theta_1 \Delta B) \cdot Q_1 \left[u\left(\frac{B}{A(B) - \theta_3 \Delta A}\right); \right. \right. \\
&\qquad\qquad \left. u\left(\frac{B}{A(B) - \theta_3 \Delta A}\right) + v\left(\frac{A}{A(B) - \theta_5 \Delta A}\right) \right] \\
&\quad - (A(B) - \theta_6 \Delta A) \cdot Q_2 \left[u\left(\frac{B}{A(B) - \theta_5 \Delta A}\right); \right. \\
&\qquad\qquad \left. \left. u\left(\frac{B}{A(B) - \theta_5 \Delta A}\right) + v\left(\frac{A}{A(B) - \theta_5 \Delta A}\right) \right] \right\}.
\end{aligned} \tag{3.8}
$$

Recalling that

$$
Q_2\left[u(1); u(1) + v\left(\frac{A}{B}\right)\right] < Q_1\left[u(1); u(1) + v\left(\frac{A}{B}\right)\right],
$$

$$
\begin{aligned}
&Q_2\left[u\left(\frac{B}{A(B)}\right); u\left(\frac{B}{A(B)}\right) + v\left(\frac{A}{A(B)}\right)\right] \\
&\qquad < \frac{B}{A(B)} \cdot Q_1\left[u\left(\frac{B}{A(B)}\right); u\left(\frac{B}{A(B)}\right) + v\left(\frac{A}{A(B)}\right)\right],
\end{aligned}
$$

and from continuity considerations on Q_1 and Q_2, it follows that each of the two expressions in braces in (3.8) becomes positive for small enough ΔB. This proves that $\beta(B)$ is strictly increasing in B, and completes the proof of the theorem.

4. Extensions. All the results above go through in the same way under the following somewhat less restrictive conditions: Under either H_1 or H_2, the chance variable $f_2(X_1)/f_1(X_1)$ has a continuous distribution which assigns a positive probability to any nondegenerate subinterval of $[C, D]$, where $0 \leqq C < 1 < D$; and the equalities $\alpha(B, A) = \alpha$ and $\beta(B, A) = \beta$ imply the strict inequalities $C < B < A < D$.