TABLES FOR COMPUTING BIVARIATE NORMAL PROBABILITIES

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1. Introduction. Various tables have been published for obtaining probabilities over rectangles for correlated bivariate normal variables. Some of these tables give the probabilities as functions of three parameters (see [1], [2], and [3]). Others tabulate related two-parameter families from which these probabilities may be computed (see [3], [4], [5], [6], and [7]). The tables given here are of the latter type. They have been computed for use with a special two-dimensional interpolation scheme, which is described in Section 4. These new tabulations reduce considerably the amount of interpolation work required over that needed with previous tables. The function tabulated also eliminates an arctangent function from the formula for the bivariate normal over a region outside of a rectangle as compared with the formula for Nicholson's tabulation in [5]. Section 3 contains a derivation of the formulas given in Section 2 for using a two-parameter table to compute probabilities over rectangles. The tables given below should prove very useful, since examples where bivariate normal integrals over polygons are needed to solve practical problems abound in the literature. For example, see [6], [8], [9], and [10].

The usefulness of the $T(h, a)$ function tabulated below was also recognized by Professor Harry A. Bender, University of Rhode Island, who submitted, after this paper was received by the editor, a somewhat shorter tabulation than given here. An abstract of Professor Bender's paper appears in [15].

For h and $a > 0$, $T(h, a)$, the function tabulated, gives the volume of an uncorrelated bivariate normal distribution with zero means and unit variances over the area between $y = ax$ and $y = 0$ and to the right of $x = h$, i.e., the area shaded in Fig. 1.

Cadwell in [11] gives a method for obtaining the volume of a bivariate normal over any polygon. In Fig. 2, if AB is a side of any polygon, then the volume over the shaded area for an uncorrelated bivariate normal with zero means and unit variances is given by

$$T(h, a_2) - T(h, a_1)$$

for $a_2 > a_1$, where h is the length of the perpendicular from the origin to the line through AB and a_1h is the distance from the foot of the perpendicular, C, to B and a_2h is CA. If C lies between A and B, then the T-functions are added instead of subtracted. By composition of volumes like this, it is possible to obtain the volume over the area outside of any polygon. Section 2 includes some useful formulas for doing this.

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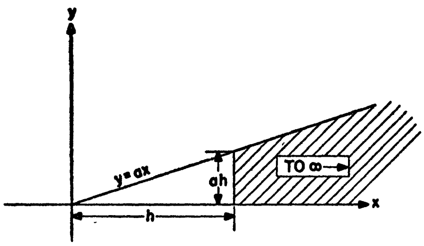

FIG. 1

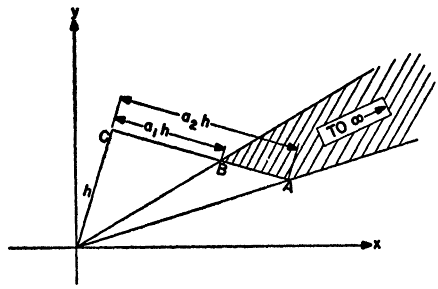

FIG. 2

FIG. 1. The area over which $T(h, a)$ gives the volume of a standardized bivariate normal with correlation zero.

FIG. 2. A typical area for computing the bivariate normal over a polygon.

2. Summary of formulas. The fundamental formula for finding volumes over rectangles is

$$B(h, k; \rho) = \tfrac{1}{2}G(h) + \tfrac{1}{2}G(k) - T(h, a_h) - T(k, a_k) - \begin{cases} 0 \\ \frac{1}{2} \end{cases}, \tag{2.1}$$

where the upper choice is made if $hk > 0$ or if $hk = 0$ but $h + k \geqq 0$, and the lower choice is made otherwise, where

$$a_h = \frac{k}{h\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}, \qquad a_k = \frac{h}{k\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}, \tag{2.2}$$

and where $B(h, k; \rho)$ is the volume of a bivariate normal with zero means and unit variances and correlation ρ over the lower left-hand quadrant of the xy-plane when divided at $x = h$ and $y = k$, $G(h)$ is the univariate normal with zero mean and unit variance integral from minus infinity to h, and $T(h, a)$ is the function tabulated below.

The T-function is tabulated only for $0 < a \leqq 1$, and ∞, but it is possible to obtain values for $1 < a < \infty$ by use of the following formula:

$$T(h, a) = \tfrac{1}{2}G(h) + \tfrac{1}{2}G(ah) - G(h)G(ah) - T\left(ah, \frac{1}{a}\right). \tag{2.3}$$

Values for negative a or h may be obtained by using

$$T(h, -a) = -T(h, a) \tag{2.4}$$

and

$$T(-h, a) = T(h, a). \tag{2.5}$$

Note that (2.3) requires a to be positive and hence when a is negative, first apply (2.4) and then (2.3).

Other useful formulas are:

$$T(h, 0) = 0,$$

$$T(0, a) = \frac{1}{2\pi} \arctan a,$$

$$T(h, 1) = \tfrac{1}{2}G(h)[1 - G(h)],$$

and

$$T(h, \infty) = \begin{cases} \tfrac{1}{2}[1 - G(h)] & \text{if } h \geqq 0, \\ \tfrac{1}{2}G(h) & \text{if } h \leqq 0. \end{cases}$$

For finding volumes of the general correlated bivariate normal over polygons, the first step is to make a rotation and stretching of the axes to reduce the function under the integral to the form of the T-function. A transformation that will do this is

$$u = \frac{1}{\sqrt{2 + 2\rho}} \left[\frac{x - \mu_X}{\sigma_X} + \frac{y - \mu_Y}{\sigma_Y} \right],$$

$$v = \frac{-1}{\sqrt{2 - 2\rho}} \left[\frac{x - \mu_X}{\sigma_X} - \frac{y - \mu_Y}{\sigma_Y} \right],$$

for $\rho^2 < 1$, where μ_X, μ_Y are the means of the X and Y variables and σ_X, σ_Y are the standard deviations of the X and Y variables, respectively. This will take the original polygon into another polygon in the uv plane. The vertices of the new polygon should be computed and a graph drawn. For each side of the polygon the volume over a region like that shown in Fig. 2 may be computed with the aid of these formulas:

$$h = \frac{| h_1 k_2 - h_2 k_1 |}{\sqrt{(h_2 - h_1)^2 + (k_2 - k_1)^2}},$$

$$a_1 = \frac{| h_1(h_2 - h_1) + k_1(k_2 - k_1) |}{| h_1 k_2 - h_2 k_1 |},$$

$$a_2 = \frac{| h_2(h_2 - h_1) + k_2(k_2 - k_1) |}{| h_1 k_2 - h_2 k_1 |},$$

where the vertical bars indicate absolute value and where (h_1, k_1) and (h_2, k_2) are the coordinates of two adjacent vertices on the polygon. With the aid of the graph, these volumes are then easily combined to give the volume over the outside (or inside) of the polygon.

3. Derivation of the relationship between the bivariate normal and the tabulated function.

Let

$$B(h, k; \rho) \tag{3.1}$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{h} \int_{-\infty}^{k} \exp\left[-\tfrac{1}{2}(x^2 - 2\rho xy + y^2)/(1-\rho^2)\right] dx\, dy,$$

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}t^2\right) dt, \tag{3.2}$$

and

$$T(h, a) = \frac{1}{2\pi} \int_{0}^{a} \frac{\exp\left[-\tfrac{1}{2}h^2(1+x^2)\right]}{1+x^2}\, dx. \tag{3.3}$$

It is also convenient to have a second form of (3.3), which is the function in Tables A, B and C. It may be obtained by differentiating with respect to h and then reintegrating. The result is

$$T(h, a) = \frac{-1}{2\pi} \int_{0}^{h} \int_{0}^{ax} \exp\left[-\tfrac{1}{2}(x^2 + y^2)\right] dy\, dx + \frac{\arctan a}{2\pi}. \tag{3.4}$$

The T-function is related to the V-function tabulated by Nicholson in [5] as follows:

$$T(h, a) = \frac{1}{2\pi} \arctan a - V(h, ah).$$

If (3.4) is integrated by parts,

$$T(h, a) = \begin{cases} \tfrac{1}{2}G(h) + \tfrac{1}{2}G(ah) - G(h)G(ah) - T(ah, 1/a) & \text{if } a \geqq 0, \\ \tfrac{1}{2}G(h) + \tfrac{1}{2}G(ah) - G(h)G(ah) - T(ah, 1/a) - \tfrac{1}{2} & \text{if } a < 0. \end{cases} \tag{3.5}$$

It will be shown that (3.1) can be expressed as a function of expressions like (3.2) and (3.3). If (3.1) is differentiated with respect to ρ, then integration with respect to x and y can be effected. Integrating that result with respect to ρ yields

$$B(h, k; \rho) \tag{3.6}$$
$$= \frac{1}{2\pi} \int_{0}^{\rho} (1 - z^2)^{-1/2} \exp\left[-\tfrac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2)\right] dz + G(h)G(k).$$

From this $B(0, 0; \rho) = 1/(2\pi) \arcsin \rho + \frac{1}{4}$, a well-known result (see [12], [13], and [14]). Now (3.6) may be rewritten as

$$B(h, k; \rho)$$
$$= \frac{1}{2\pi} \int_{0}^{\rho} (1 - z^2)^{-1/2} \frac{h(h - kz)}{h^2 - 2hkz + k^2} \exp\left[-\tfrac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2)\right] dz$$
$$+ \frac{1}{2\pi} \int_{0}^{\rho} (1 - z^2)^{-1/2} \frac{k(k - hz)}{h^2 - 2hkz + k^2} \exp\left[-\tfrac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2)\right] dz$$
$$+ G(h)G(k).$$

In the integrals above, making the substitutions

$$u = \frac{k - hz}{h\sqrt{1 - z^2}} \quad \text{and} \quad v = \frac{h - kz}{k\sqrt{1 - z^2}},$$

respectively, produces

$$(3.7) \quad \begin{aligned} B(h, k; \rho) = T\left(h, \frac{k}{h}\right) + T\left(k, \frac{h}{k}\right) - T\left(h, \frac{k - \rho h}{h\sqrt{1 - \rho^2}}\right) \\ - T\left(k, \frac{h - \rho k}{k\sqrt{1 - \rho^2}}\right) + G(h)G(k). \end{aligned}$$

Applying (3.5) to (3.7), gives

$$(3.8) \quad B(h, k; \rho) = \begin{cases} \tfrac{1}{2}G(h) - T\left(h, \dfrac{k - \rho h}{h\sqrt{1 - \rho^2}}\right) + \tfrac{1}{2}G(k) - T\left(k, \dfrac{h - \rho k}{k\sqrt{1 - \rho^2}}\right), \\ \qquad \text{if } hk > 0 \text{ or if } hk = 0,\ h \text{ or } k \geqq 0 \\ \tfrac{1}{2}G(h) - T\left(h, \dfrac{k - \rho h}{h\sqrt{1 - \rho^2}}\right) + \tfrac{1}{2}G(k) \\ - T\left(k, \dfrac{h - \rho k}{k\sqrt{1 - \rho^2}}\right) - \tfrac{1}{2}, \ \text{if } hk < 0 \text{ or if } hk = 0, h \text{ or } k < 0 \end{cases}$$

which expresses the bivariate normal in terms of the G- and T-functions in a compact form.

A series expression for $T(h, a)$ may be obtained by expanding the numerator of the integrand of (3.3) in the usual exponential series, dividing by the denominator, and integrating term by term. Rearrangement of the terms of this series gives

$$(3.9) \qquad T(h, a) = \frac{\arctan a}{2\pi} - \frac{1}{2\pi}\sum_{j=0}^{\infty} c_j a^{2j+1},$$

where

$$c_j = (-1)^j \frac{1}{2j + 1}\left[1 - \exp\left(-\tfrac{1}{2}h^2\right)\sum_{i=0}^{j}\frac{h^{2i}}{2^i i!}\right],$$

which converges rapidly for small values of a and h.

The values of $T(h, a)$ given in Tables A, B, and C were computed using the series (3.9). They were checked by using Gauss' seven-point integration formula on (3.3). The tables were also checked by taking differences. These checks show that at the points of tabulation the table is accurate to as many places as given, i.e., to six decimal places.

4. Interpolation in the tables. Table A has a coarse interval in the parameter a and an interval fine enough for ordinary linear interpolation in the parameter h. Table B has intervals in parameter a fine enough for ordinary linear interpolation and has parameter h at a coarse interval. Ordinary linear interpolation

TABLE A OF T(h, a)

Note that T(h, 0) = 0

h \ a	.25	.50	.75	1.00
0.00	.038990	.073792	.102416	.125000
0.01	988	788	410	.124992
0.02	982	776	392	968
0.03	972	756	363	928
0.04	958	728	321	873
0.05	940	692	267	801
0.06	918	649	202	714
0.07	892	597	124	611
0.08	862	538	035+	492
0.09	829	470	.101934	357
0.10	791	395-	821	207
0.11	750-	312	697	041
0.12	704	221	561	.123860
0.13	655-	122	413	663
0.14	602	016	253	450+
0.15	545-	.072902	082	223
0.16	484	780	.100900	.122980
0.17	419	651	706	722
0.18	350+	514	501	449
0.19	278	369	285-	162
0.20	202	217	057	.121859
0.21	122	058	.099818	542
0.22	038	.071891	569	210
0.23	.037951	717	308	.120864
0.24	860	535+	037	503
0.25	766	347	.098755-	129
0.26	668	151	462	.119740
0.27	566	.070948	158	337
0.28	461	738	.097844	.118921
0.29	352	521	520	492
0.30	240	297	186	048

h \ a	.25	.50	.75	1.00
0.30	.037240	.070297	.097186	.118048
0.31	124	066	.096841	.117592
0.32	005-	.069828	487	123
0.33	.036882	584	122	.116641
0.34	756	333	.095748	146
0.35	627	076	365-	.115639
0.36	495-	.068812	.094971	119
0.37	359	542	569	.114587
0.38	220	265+	157	044
0.39	078	.067983	.093736	.113489
0.40	.035933	694	306	.112922
0.41	785-	399	.092868	344
0.42	634	098	420	.111755+
0.43	479	.066791	.091965-	155+
0.44	322	479	501	.110545-
0.45	162	161	028	.109924
0.46	.034999	.065837	.090548	293
0.47	834	508	060	.108652
0.48	665+	173	.089564	001
0.49	494	.064834	060	.107341
0.50	320	489	.088549	.106671
0.51	144	139	031	.105993
0.52	.033965-	.063784	.087506	305+
0.53	783	424	.086973	.104609
0.54	599	059	435-	.103905+
0.55	413	.062690	.085889	193
0.56	224	316	337	.102473
0.57	033	.061938	.084779	.101745+
0.58	.032840	555+	215-	010
0.59	645-	168	.083645-	.100268
0.60	447	.060778	069	.099519

TABLE A OF T(h,a)

h \ a	.25	.50	.75	1.00
0.60	.032447	.060778	.083069	.099519
0.61	247	383	.082487	.098764
0.62	046	.059984	.081901	002
0.63	.031842	581	309	.097234
0.64	636	175+	.080712	.096460
0.65	429	.058765+	110	.095681
0.66	219	352	.079504	.094896
0.67	008	.057936	.078893	106
0.68	.030795+	516	278	.093312
0.69	581	093	.077658	.092512
0.70	365-	.056667	035+	.091709
0.71	147	239	.076408	.090901
0.72	.029928	.055807	.075777	089
0.73	707	373	143	.089274
0.74	485+	.054937	.074506	.088455+
0.75	262	498	.073866	.087634
0.76	038	056	223	.086809
0.77	.028812	.053613	.072577	.085982
0.78	585-	167	.071928	152
0.79	357	.052720	278	.084320
0.80	128	270	.070625-	.083486
0.81	.027898	.051819	.069970	.082651
0.82	667	367	313	.081814
0.83	435-	.050912	.068655-	.080975+
0.84	202	457	.067995-	136
0.85	.026968	000-	333	.079296
0.86	734	.049542	.066671	.078455+
0.87	499	083	007	.077614
0.88	264	.048622	.065343	.076773
0.89	027	161	.064678	.075932
0.90	.025791	.047700	013	091

h \ a	.25	.50	.75	1.00
0.90	.025791	.047700	.064013	.075091
0.91	554	237	.063347	.074251
0.92	316	.046775-	.062681	.073411
0.93	079	311	015+	.072572
0.94	.024840	.045848	.061349	.071734
0.95	602	384	.060684	.070898
0.96	363	.044920	018	063
0.97	125-	456	.059354	.069230
0.98	.023886	.043992	.058690	.068398
0.99	647	528	027	.067569
1.00	408	065-	.057365-	.066742
1.01	169	.042602	.056704	.065917
1.02	.022931	139	044	095+
1.03	692	.041677	.055386	.064276
1.04	454	216	.054729	.063459
1.05	216	.040755-	075-	.062646
1.06	.021978	295+	.053421	.061836
1.07	740	.039836	.052770	029
1.08	503	378	121	.060226
1.09	266	.038922	.051474	.059426
1.10	030	466	.050830	.058630
1.11	.020794	012	188	.057839
1.12	559	.037559	.049548	051
1.13	325-	107	.048911	.056268
1.14	091	.036657	277	.055489
1.15	.019857	209	.047646	.054714
1.16	625-	.035762	018	.053945-
1.17	393	317	.046393	180
1.18	162	.034874	.045771	.052420
1.19	.018931	432	152	.051664
1.20	702	.033993	.044537	.050914

TABLE A OF T(h, a)

h \ a	.25	.50	.75	1.00
1.20	.018702	.033993	.044537	.050914
1.21	473	556	.043925+	169
1.22	246	120	317	.049430
1.23	019	.032687	.042712	.048696
1.24	.017794	256	112	.047967
1.25	569	.031828	.041515-	244
1.26	345+	402	.040922	.046527
1.27	123	.030978	333	.045815-
1.28	.016902	556	.039748	109
1.29	682	138	167	.044409
1.30	463	.029721	.038590	.043715+
1.31	245-	308	018	027
1.32	028	.028897	.037450+	.042345+
1.33	.015813	488	.036886	.041670
1.34	599	083	327	000
1.35	387	.027680	.035773	.040337
1.36	176	281	223	.039680
1.37	.014966	.026884	.034678	030
1.38	757	490	137	.038386
1.39	550+	099	.033601	.037749
1.40	345-	.025711	070	118
1.41	140	326	.032543	.036493
1.42	.013938	.024944	022	.035875+
1.43	737	566	.031505+	264
1.44	537	190	.030994	.034659
1.45	339	.023818	487	061
1.46	142	449	.029985+	.033470
1.47	.012947	084	489	.032885+
1.48	754	.022721	.028997	308
1.49	562	362	511	.031736
1.50	372	006	029	172

h \ a	.25	.50	.75	1.00
1.50	.012372	.022006	.028029	.031172
1.51	184	.021654	.027553	.030614
1.52	.011997	305+	082	063
1.53	812	.020959	.026616	.029519
1.54	628	617	155+	.028982
1.55	446	279	.025700	451
1.56	266	.019943	249	.027927
1.57	088	611	.024804	410
1.58	.010911	283	364	.026899
1.59	736	.018958	.023929	395+
1.60	563	637	500-	.025898
1.61	391	319	075+	408
1.62	221	005-	.022656	.024924
1.63	053	.017694	242	447
1.64	.009887	387	.021833	.023976
1.65	723	083	430	512
1.66	560	.016783	032	055-
1.67	399	486	.020638	.022604
1.68	240	193	250+	159
1.69	082	.015903	.019868	.021721
1.70	.008927	617	490	290
1.71	773	334	117	.020865-
1.72	621	055+	.018750-	446
1.73	470	.014780	388	033
1.74	322	508	030	.019627
1.75	175-	239	.017678	227
1.76	030	.013974	331	.018833
1.77	.007887	713	.016989	446
1.78	745+	454	651	064
1.79	605+	200	319	.017689
1.80	467	.012949	.015992	320

TABLE A OF T(h, a)

h \ a	.25	.50	.75	1.00	h \ a	.25	.50	.75	1.00
1.80	.007467	.012949	.015992	.017320	2.30	.002625+	.004334	.005079	.005305-
1.81	331	701	669	.016956	2.32	505-	126	.004825+	034
1.82	197	457	352	599	2.34	388	.003926	582	.004774
1.83	064	216	039	247	2.36	277	735-	350-	527
1.84	.006933	.011978	.014731	.015901	2.38	169	551	127	291
1.85	804	744	428	561	2.40	066	375-	.003915-	065+
1.86	676	514	130	227	2.42	.001967	206	711	.003850+
1.87	550+	286	.013836	.014898	2.44	872	044	517	645-
1.88	426	062	547	575+	2.46	781	.002890	332	449
1.89	304	.010841	263	258	2.48	693	742	155+	263
1.90	183	624	.012983	.013946	2.50	609	600	.002987	086
1.91	064	410	708	639	2.52	529	465+	826	.002917
1.92	.005947	199	437	338	2.54	452	336	673	756
1.93	831	.009991	171	042	2.56	379	213	527	603
1.94	717	786	.011909	.012752	2.58	308	095+	388	458
1.95	605+	585-	652	467	2.60	241	.001983	256	320
1.96	495-	387	399	187	2.62	177	876	130	189
1.97	386	192	150-	.011911	2.64	115+	774	011	064
1.98	278	000	.010905+	641	2.66	057	677	.001897	.001946
1.99	172	.008811	665+	376	2.68	001	585-	789	834
2.00	068	625+	429	116	2.70	.000947	497	687	727
2.02	.004865-	263	.009970	.010611	2.72	896	413	590	627
2.04	667	.007912	527	124	2.74	848	334	498	531
2.06	476	573	099	.009656	2.76	802	258	410	441
2.08	291	245+	.008687	205+	2.78	758	186	327	355+
2.10	112	.006929	291	.008773	2.80	716	118	249	274
2.12	.003939	624	.007909	357	2.82	676	054	175-	198
2.14	771	330	542	.007958	2.84	638	.000992	104	125+
2.16	610	046	188	575-	2.86	602	934	038	057
2.18	453	.005772	.006849	207	2.88	568	879	.000975-	.000992
2.20	303	509	523	.006855+	2.90	535+	827	916	931
2.22	157	255+	209	517	2.92	504	777	859	874
2.24	017	011	.005909	194	2.94	475+	731	806	819
2.26	.002881	.004777	620	.005884	2.96	448	686	756	768
2.28	751	551	344	588	2.98	421	645-	709	720
2.30	625+	334	079	305-	3.00	396	605+	665-	674

TABLE B OF T(h, a)

a \ h	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
.00	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
.01	.001591	.001543	.001404	.001201	965+	729	517	344	215+	127	070	036	018
.02	.003183	.003085-	.002809	.002402	.001930	.001457	.001033	688	431	253	140	073	035+
.03	.004773	.004626	.004212	.003603	.002895-	.002185-	549	.001032	646	379	210	109	053
.04	.006363	.006167	.005615-	.004802	.003858	912	.002064	375-	860	506	279	145-	071
.05	.007951	.007706	.007016	.006000	.004821	.003638	579	717	.001074	631	348	181	088
.06	.009538	.009244	.008416	.007197	.005782	.004363	.003092	.002059	288	757	417	216	105+
.07	.011123	.010780	.009814	.008392	.006741	.005086	604	399	500+	881	486	252	123
.08	.012705+	.012314	.011209	.009585-	.007698	807	.004115-	739	712	.001005+	555-	287	140
.09	.014285+	.013845-	.012603	.010775+	.008653	.006527	624	.003077	923	129	622	322	157
.10	.015863	.015373	.013993	.011963	.009605+	.007244	.005131	413	.002133	251	690	357	174
.11	.017437	.016898	.015380	.013147	.010555-	958	635+	748	341	373	757	392	190
.12	.019008	.018420	.016764	.014328	.011501	.008670	.006138	.004081	548	494	823	426	207
.13	.020575-	.019938	.018144	.015506	.012444	.009379	638	412	754	614	888	459	223
.14	.022138	.021452	.019521	.016680	.013384	.010084	.007135-	740	958	733	953	492	239
.15	.023697	.022962	.020893	.017850-	.014319	786	629	.005067	.003160	850+	.001017	525+	255-
.16	.025251	.024467	.022260	.019015-	.015251	.011485-	.008120	391	360	967	081	558	270
.17	.026800	.025968	.023623	.020176	.016178	.012179	608	712	559	.002082	143	590	285+
.18	.028344	.027463	.024980	.021331	.017100	869	.009092	.006031	755+	195+	205-	621	300
.19	.029883	.028953	.026333	.022482	.018018	.013555+	573	347	950-	308	266	652	315-
.20	.031416	.030437	.027679	.023627	930	.014237	.010050+	659	.004142	418	325+	682	329
.21	.032944	.031916	.029020	.024766	.019837	913	523	969	332	528	384	712	343
.22	.034465+	.033388	.030355-	.025899	.020739	.015585+	992	.007276	520	635+	442	741	357
.23	.035980	.034854	.031683	.027027	.021635-	.016252	.011456	579	705+	741	499	769	371
.24	.037488	.036313	.033005+	.028148	.022525-	913	917	879	888	846	555-	797	384
.25	.038990	.037766	.034320	.029262	.023408	.017569	.012372	.008175-	.005068	948	609	824	396
.26	.040484	.039211	.035628	.030370	.024286	.018219	823	467	246	.003049	663	851	409
.27	.041971	.040649	.036929	.031470	.025156	864	.013269	756	421	148	716	877	421
.28	.043451	.042080	.038223	.032564	.026021	.019502	710	.009041	593	245+	767	902	432
.29	.044923	.043503	.039509	.033650-	878	.020135-	.014147	322	762	341	817	927	444
.30	.046387	.044918	.040787	.034728	.027728	761	578	599	929	434	866	951	455-

TABLE B OF T(h, a)

a \ h	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
.30	.046387	.044918	.040787	.034728	.027728	.020761	.014578	.009599	.005929	.003434	.001866	.000951	.000455-
.31	.047843	.046326	.042057	.035799	.028571	.021381	.015003	873	.006092	526	914	974	465+
.32	.049291	.047725-	.043319	.036862	.029407	994	424	.010142	253	615+	961	997	476
.33	.050730	.049115+	.044573	.037916	.030235-	.022601	839	406	411	703	.002006	.001019	486
.34	.052161	.050497	.045818	.038963	.031056	.023201	.016248	667	565+	789	050+	040	495+
.35	.053583	.051871	.047054	.040001	.031868	795-	652	923	717	872	093	061	505-
.36	.054997	.053235+	.048282	.041031	.032673	.024381	.017050+	.011175+	866	954	135+	081	513
.37	.056401	.054591	.049501	.042052	.033470	961	443	423	.007011	.004034	176	101	522
.38	.057797	.055937	.050711	.043065-	.034259	.025533	829	666	154	112	215+	119	530
.39	.059183	.057275-	.051912	.044068	.035040	.026098	.018210	905+	293	188	254	137	538
.40	.060559	.058602	.053103	.045063	812	656	585+	.012140	430	261	291	155-	546
.41	.061927	.059921	.054285+	.046048	.036576	.027207	954	370	563	333	327	171	553
.42	.063284	.061230	.055458	.047025-	.037332	750+	.019318	595+	693	403	362	188	560
.43	.064633	.062529	.056621	992	.038079	.028286	675-	816	820	471	395+	203	567
.44	.065971	.063818	.057775-	.048950-	817	815-	.020026	.013033	944	536	428	218	573
.45	.067299	.065098	.058918	.049898	.039547	.029336	371	245+	.008065+	601	459	232	579
.46	.068618	.066368	.060052	.050837	.040268	849	710	453	183	663	490	246	585-
.47	.069926	.067628	.061176	.051767	980	.030355+	.021043	656	298	723	519	259	590
.48	.071225+	.068877	.062290	.052687	.041684	854	370	855-	410	782	547	272	595+
.49	.072513	.070117	.063395-	.053597	.042379	.031345-	691	.014049	519	838	574	284	600
.50	.073792	.071347	.064489	.054498	.043065-	828	.022006	239	625+	893	600	295+	605+
.51	.075060	.072566	.065573	.055389	742	.032304	315+	425-	728	946	626	306	610
.52	.076318	.073775+	.066647	.056270	.044410	772	619	606	828	997	650-	317	614
.53	.077566	.074974	.067710	.057141	.045069	.033232	916	783	926	.005046	673	327	618
.54	.078803	.076163	.068764	.058003	720	685+	.023207	956	.009020	094	695+	337	622
.55	.080030	.077341	.069807	855+	.046361	.034131	492	.015124	112	140	716	346	625+
.56	.081247	.078509	.070841	.059697	994	569	771	288	201	184	737	354	629
.57	.082453	.079667	.071864	.060530	.047618	999	.024045-	448	287	227	756	363	632
.58	.083649	.080814	.072876	.061353	.048233	.035422	312	604	370	268	775+	370	635-
.59	.084835+	.081951	.073879	.062165+	839	838	574	756	451	308	793	378	638
.60	.086010	.083078	.074871	969	.049436	.036246	831	904	530	346	810	385-	640
.61	.087176	.084194	.075854	.063762	.050024	646	.025081	.016048	606	383	826	391	643
.62	.088330	.085300	.076826	.064546	604	.037040	326	188	679	418	842	398	645+
.63	.089475-	.086396	.077788	.065320	.051175+	426	566	324	750-	452	857	404	647
.64	.090609	.087482	.078739	.066084	738	805+	800	456	818	485-	871	409	649
.65	.091733	.088557	.079681	839	.052291	.038177	.026028	585-	885-	516	884	415-	651

TABLE B OF T(h, a)

a \ h	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
.65	.091733	.088557	.079681	.066839	.052291	.038177	.026028	.016585-	.009885-	.005516	.002884	.001415-	.000651
.66	.092847	.089622	.080612	.067584	836	542	251	709	949	546	897	420	653
.67	.093950+	.090677	.081534	.068320	.053373	899	469	831	.010010	574	909	424	655-
.68	.095044	.091722	.082445+	.069046	901	.039250-	682	948	070	602	921	429	656
.69	.096127	.092756	.083347	762	.054421	594	889	.017063	127	628	932	433	658
.70	.097200	.093781	.084239	.070469	932	930	.027091	173	182	653	942	437	659
.71	.098263	.094796	.085120	.071167	.055435-	.040261	289	281	236	677	952	441	660
.72	.099316	.095800	992	856	929	584	481	385-	287	700	961	444	662
.73	.100360	.096795-	.086854	.072535-	.056416	901	669	486	336	722	970	447	663
.74	.101393	.097780	.087707	.073205-	894	.041211	851	583	384	743	979	450+	664
.75	.102416	.098755-	.088549	866	.057365-	515-	.028029	678	429	763	987	453	665-
.76	.103430	.099720	.089382	.074518	827	812	203	770	473	782	994	456	665+
.77	.104434	.100675+	.090206	.075161	.058282	.042103	371	858	515+	800	.003001	458	666
.78	.105428	.101621	.091020	794	728	388	536	944	556	818	008	461	667
.79	.106413	.102557	824	.076420	.059167	666	695+	.018027	595-	834	014	463	668
.80	.107388	.103484	.092620	.077036	598	939	851	107	632	850-	020	465+	668
.81	.108354	.104401	.093406	643	.060022	.043205+	.029002	184	668	865-	026	467	669
.82	.109310	.105309	.094182	.078242	438	466	149	259	702	879	031	469	669
.83	.110257	.106208	950-	832	847	721	292	331	735-	892	036	470	670
.84	.111195+	.107097	.095708	.079414	.061248	970	431	401	766	905+	041	472	670
.85	.112124	977	.096458	988	642	.044213	566	469	796	917	045+	473	671
.86	.113043	.108848	.097198	.080553	.062029	451	697	533	825+	929	049	475-	671
.87	954	.109710	930	.081109	409	683	825-	596	853	940	053	476	672
.88	.114855-	.110563	.098653	658	782	910	948	656	879	950-	057	477	672
.89	.115747	.111407	.099367	.082199	.063148	.045132	.030068	715-	905-	960	060	478	672
.90	.116631	.112243	.100073	731	507	348	185-	771	929	969	063	479	672
.91	.117506	.113069	770	.083256	859	559	298	825-	952	978	066	480	673
.92	.118372	887	.101458	772	.064205-	766	408	877	974	986	069	481	673
.93	.119230	.114696	.102138	.084281	544	967	514	927	995-	994	072	482	673
.94	.120079	.115497	810	783	877	.046163	617	975-	.011015-	.006001	074	482	673
.95	920	.116290	.103474	.085276	.065203	355-	717	.019021	034	008	076	483	673
.96	.121752	.117074	.104129	762	523	542	814	066	052	015-	078	483	674
.97	.122577	850-	777	.086241	837	724	908	109	069	021	080	484	674
.98	.123392	.118617	.105416	713	.066145-	902	999	150-	086	027	082	485-	674
.99	.124601	.119377	.106047	.087177	446	.047075-	.031087	189	101	032	084	485+	674
1.00	.125000	.120129	671	634	742	244	172	227	116	038	086	485+	674
∞	.250000	.200647	.154269	.113314	.079328	.052825-	.033404	.020030	375+	112	105-	490	675-

TABLE C OF T(h, a)

h \ a	.10	.20	.25	.30	.35	.40	.45	.50	.60	.70	.80	1.00	∞
3.00	.000174	.000329	.000396	.000455-	.000505-	.000546	.000579	.000605+	.000640	.000659	.000668	.000674	.000675-
3.05	149	283	340	389	431	466	494	516	545-	560	567	571	572
3.10	128	242	290	333	368	397	420	438	462	474	480	483	484
3.15	109	206	248	283	313	338	357	372	391	401	405+	408	408
3.20	093	176	211	241	266	286	302	314	330	338	341	343	344
3.25	079	149	179	204	225-	242	255+	265+	278	284	287	288	289
3.30	067	127	151	172	190	204	215-	223	233	238	241	242	242
3.35	057	107	128	145+	160	172	181	187	196	199	201	202	202
3.40	048	090	108	122	134	144	151	157	163	167	168	168	168
3.45	040	076	090	103	113	121	127	131	136	139	140	140	140
3.50	034	064	076	086	094	101	105+	109	113	115+	116	116	116
3.60	024	044	053	060	065+	070	073	075+	078	079	079	080	080
3.70	017	031	036	041	045-	048	050-	051	053	054	054	054	054
3.80	011	021	025-	028	030	032	034	035-	036	036	036	036	036
3.90	008	014	017	019	020	022	023	023	024	024	024	024	024
4.00	005+	010	011	013	014	014	015-	015+	016	016	016	016	016
4.10	003	006	007	008	009	009	010	010	010	010	010	010	010
4.20	002	004	005-	005+	006	006	006	006	007	007	007	007	007
4.30	001	003	003	004	004	004	004	004	004	004	004	004	004
4.40	001	002	002	002	002	003	003	003	003	003	003	003	003
4.50	001	001	001	001	002	002	002	002	002	002	002	002	002
4.60	000	001	001	001	001	001	001	001	001	001	001	001	001
4.70	000	000	001	001	001	001	001	001	001	001	001	001	001
4.76	000	000	000	000	000	000	000	000	000	000	000	000	000

may be used throughout Table C. Tables A and B were designed for interpolation as follows: To interpolate for a value $T(h_2, a_2)$, say, a_1 and a_3 should be picked closest to a_2 from Table A so that $a_1 \leqq a_2 < a_3$, and h_1 and h_3 should be picked closest to h_2 from Table B so that $h_1 \leqq h_2 < h_3$. Then the interpolated value of $T(h_2, a_2)$ is obtained from

$$T(h_2, a_2) = \sum_{i=1}^{3} \sum_{j=1}^{3} w_{ij} T(h_i, a_j),$$

where the weights w_{ij} are given by

$$w_{ij} = \begin{pmatrix} -(1-b)(1-c) & 1-c & -b(1-c) \\ (1-b) & 0 & b \\ -(1-b)c & c & -bc \end{pmatrix},$$

where

$$b = \frac{a_2 - a_1}{a_3 - a_1} \quad \text{and} \quad c = \frac{h_2 - h_1}{h_3 - h_1}.$$

The weights were obtained by considering the result of ordinary linear interpolation where nearby values of the function are subtracted before interpolating, say, $T(h_2, a_1) - T(h_1, a_1)$ and $T(h_2, a_3) - T(h_1, a_3)$. These numbers are interpolated with respect to a to obtain $T(h_2, a_2) - T(h_1, a_2)$, and then $T(h_1, a_2)$ is added. This process may also be followed with (h_3, a_1), (h_3, a_3), and (h_3, a_2). If the two estimates of $T(h_2, a_2)$ are then combined as in linear interpolation with respect to h, i.e., $(1 - c)$ times the first estimate plus c times the second, the above weights w_{ij} follow. The interpolation on the differences could also have been first with respect to h to obtain the two estimates and then with respect to a between these two. The same weights w_{ij} are obtained by doing this.

This method of interpolation has resulted in approximately a 90 per cent reduction over the size of a table needed for linear interpolation. Quadratic interpolation using Bessel's formula would give comparable results to the new method with approximately an additional 80 per cent reduction in the number of entries, but the additional work involved more than outweighs that reduction in the number of entries, even though the table is used only a few times. The procedure given here may be termed a compromise between linear and quadratic interpolation.

EXAMPLE. Find $T(.15, .625)$. From the tables, the following entries are extracted:

h	a		
	.50	.625	.75
0	.073792	.088903	.102416
.15	.072902		.101082
.25	.071347	.085848	.098755

The weights to be applied are

h	a		
	.50	.625	.75
0	−.2	.4	−.2
.15	.5		.5
.25	−.3	.6	−.3

The result is $T(.15, .625) = .0877898$. Calculation of this number from the series gives .0877919. The result of the interpolation therefore provides a difference of two in the sixth place. Further calculations show that this difference could be reduced to five in the seventh place if the linear interpolations for $T(0, .625)$ and $T(.25, .625)$ were eliminated and the exact values for these points were used. The value for $T(0, .625)$ was rounded up during the linear interpolation with respect to a in Table B, since second differences in the a direction for all h are negative. A similar working rule for rounding when interpolating in the h direction is to round up the interpolated value when $0 < h < .9$ and to round down for $h \geqq .9$ in Table A. The value obtained from the above interpolation scheme should be rounded up for $0 < h < 1.50$ and rounded down for $h \geqq 1.50$, for all values of $a > 0$.

Empirical examination of the errors in interpolation by this scheme shows that the maximum error that would occur anywhere in Tables A and B is seven in the sixth decimal place, and that this could be reduced to six in the sixth decimal place if the linear interpolations in Tables A and B were eliminated and the exact values were used. Linear interpolation in Table C gives errors less than four in the sixth decimal place. Table D gives the maximum error in the sixth decimal place, which will be committed when using the above

TABLE D

h	a			
	0.00–0.25	0.25–0.50	0.50–0.75	0.75–1.00
0.00–0.25	+0.9	+2.1	+2.8	+3.3
0.25–0.50	+1.3	+3.1	+4.4	+5.3
0.50–0.75	+1.7	+4.4	+6.2	+7.1
0.75–1.00	+2.0	+4.9	+6.5	+6.4
1.00–1.25	+1.8	+4.2	+4.6	+3.1
1.25–1.50	+1.3	+2.5	+1.5	−1.3
1.50–1.75	+0.5	+0.5	−1.7	−3.9
1.75–2.00	−0.3	−1.6	−3.6	−4.7
2.00–2.25	−0.8	−2.6	−4.0	−3.9
2.25–2.50	−1.0	−2.8	−3.4	−2.6
2.50–2.75	−1.0	−2.4	−2.4	−1.4
2.75–3.00	−0.8	−1.7	−1.4	−0.7

interpolation scheme over the ranges of h and a indicated. The sign preceding the entry is the sign of the exact value of $T(h, a)$ minus the interpolated value for that difference which is the largest in absolute value. These are empirical results obtained on the digital computer by using the interpolation process and the exact value for fifteen points systematically spaced in each block. A number in Tables A, B, and C whose last nonzero digit is five is followed by a plus or minus sign, respectively, to indicate that the number should be rounded up or down when dropping the digit with the five. Any entry having the first three digits the same as those of the entry directly above it has had these digits dropped.

REFERENCES

[1] KARL PEARSON, *Tables for Statisticians and Biometricians*, Part II, Cambridge University Press, 1931.

[2] *Tables of the Bivariate Normal Distribution*, University of California, Statistical Laboratory, Berkeley. (These tables are unpublished; however, a description of these tables is given in *Mathematics Tables and Other Aids to Computation*, Vol. II, 1948, pp. 372–373.)

[3] D. TEICHROEW, "Numerical Analysis Research unpublished statistical tables," *J. Amer. Stat. Assn.*, Vol. 50, 1955, pp. 550–556.

[4] W. F. SHEPPARD, "II. On the calculation of the double-integral expressing normal correlation," *Trans. Cambridge Phil. Soc.*, Vol. 19, 1900, pp. 23–66.

[5] C. NICHOLSON, "The probability integral for two variables," *Biometrika*, Vol. 33, 1943, pp. 59–72.

[6] R. C. T. SMITH, "Conduction of heat in the semi-infinite solid, with a short table of an important integral," *Australian Journal of Physics*, Vol. 6, 1953, pp. 127–130.

[7] C. HASTINGS AND M. PIEDEM, *Miscellaneous Probability Tables*, Applied Mathematics Panel, National Defense Research Committee, Note No. 14, New York, 1944.

[8] A. R. EAGLE, "A method for handling errors in testing and measuring," *Industrial Quality Control*, Vol. 10, 1954, pp. 10–15.

[9] A. H. BOWKER AND H. P. GOODE, *Sampling Inspection by Variables*, McGraw-Hill Book Company, 1952, pp. 127–132.

[10] A. R. KAMAT, "Incomplete and absolute moments of the multivariate normal distribution with some applications," *Biometrika*, Vol. 40, 1953, pp. 20–34.

[11] J. H. CADWELL, "The bivariate normal integral," *Biometrika*, Vol. 38, 1951, pp. 475–481.

[12] H. CRAMER, *Mathematical Methods of Statistics*, Princeton University Press, 1946, p. 290.

[13] F. N. DAVID, "A note on the evaluation of the multivariate normal integral," *Biometrika*, Vol. 40, 1953, pp. 458–459.

[14] R. L. PLACKETT, "A reduction formula for normal multivariate integrals," *Biometrika*, Vol. 41, 1954, pp. 351–360.

[15] H. A. BENDER, "Bivariate distribution," *Bull. Amer. Math. Soc.*, Vol. 61, abstract No. 813, pp. 561–562.