MULTISTAGE STATISTICAL DECISION PROCEDURES[1]

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1. Introduction. A class of problems which arise in a variety of forms can be formulated as follows: We are requested to make periodic decisions of the same type but based on an increasing amount of information. Suppose we have a collection D_k of decision procedures for the kth stage, given the amount of information available to us at that stage, and suppose that the procedures of D_k are admissible under the assumption that the kth-stage decision is all that is required of us. Is it then true that when we have to prescribe decision procedures $d_1, d_2, \cdots, d_n$ for each stage, we obtain an admissible class by taking an arbitrary procedure from each D_k? The answer turns out to be no in a large class of such decision problems which we consider here. This means that by planning our whole sequence of decision procedures in advance we are able to do better on an average than if we were to make each decision as it arises. The present paper is devoted to the problem of prescribing rules which tell how the single-stage decision procedures d_k should interlock with one another so as to give a minimal complete class of decision procedures for the multistage statistical decision problem. This problem is similar to the classical sequential decision problems which do not fix in advance the number of stages. In many respects the problem formulated here is simpler than the classical sequential decision problem, and sharper results are obtained—e.g., minimal complete classes of statistical decision procedures are determined.

In order to illustrate the nature of this type of decision problem and its analysis, we might look at the following simple example: A biased coin is tossed, and the player is required to call heads or tails, being paid one unit for a correct call and nothing for an incorrect one. The first call of the player is made in complete ignorance, but for the nth play he has the evidence of the first $(n - 1)$ tosses on which to make his call. To prescribe the classes D_n of strategies admissible for a single stage is to consider the problem of making the nth call on the basis of the first $(n - 1)$ outcomes, the first $(n - 1)$ calls having been forgotten. We then have the game in which nature chooses the bias p on the coin, the player observes a random variable z binomially distributed with parameter p and must choose between two actions with loss function $-p$ and $(p - 1)$. Here a decision procedure or strategy for the player consists of a function $\phi(z)$ which gives the probability[2] of calling heads if z is the number of heads that have previously been observed. Problems of this sort have been considered in [1]

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[2] We are thus allowing the player its use of a randomized strategy.

and [2], and it is known that each decision procedure is dominated by a "monotone" procedure—i.e., one for which $\phi(z) = 0$ for $z < t$, and $\phi(z) = 1$ for $z > t$.

The global approach to our coin-tossing endeavor is to look at the whole series of tosses and calls as a single decision problem. Suppose for convenience that we agree at the beginning that our decision problem is to terminate after n tosses. Then a decision procedure consists of n functions $\{\phi_i\}$, the function ϕ_i giving the probability of calling heads on the ith toss if z_i heads have appeared in the first $(i - 1)$ tosses. It is clear that we get a complete class of decision procedures if we allow only those functions ϕ_i which provide a monotone procedure for the ith toss, but it seems possible (and is in fact the case) that we might get a smaller complete class if we restricted ourselves to procedures ϕ_i where the separate components ϕ_i are related to one another in some fashion. We shall see that a complete class is formed by those strategies for which $\phi_i(z_i) = 0$ for $z_i < t_i$ and $\phi_i(z_i) = 1$ for $z_i > t_i$, with the numbers t_i having the property that both t_i and $(i - t_i)$ are increasing functions of i. If we impose suitable restrictions on $\phi_i(t_i)$, then we get not only a complete but also an admissible class.

Aside from the many straightforward statistical problems involving multistage decisions, there arises in inventory analysis an important class of examples where repeated decisions must be made in the face of uncertainty. Suppose the distribution of demand of a commodity has a known parametric form with unknown parameter. This is a very common assumption. Decisions must be made periodically (the first of each month, for example) for ordering a certain amount of the commodity to have in stock in order to meet demand. Certain costs are incurred for storage, for purchase of the commodity, and for not having enough on hand to satisfy demand. At the same time, the exact distribution is not known, and more information accumulates about these distributions as the various periods roll by. Here again we are faced with a multistage statistical decision problem of considerable importance, and it is of value to know how the decisions over the periods should be related to form an admissible procedure.

Many other examples of the above type can be cited which involve making decisions over several stages where more information of the uncertainty evolves with time. This investigation represents a first attempt in analyzing the relationship of the various decisions in the several stages from the point of view of statistical decision theory.

In the next section we describe a general class of games of this type and give theorems regarding complete and admissible classes.

2. Description of a class of games and decision procedures for them. We consider games of the following sort: Nature chooses a point ω in an interval Ω of the real line (or is in an unknown state specified by ω). Each play of the game consists of the player choosing one of two actions and observing a random variable x whose cumulative distribution is of the exponential class, i.e., is given by

$$P(x \mid \omega) = \beta(\omega) \int_{-\infty}^{x} e^{t\omega} \, d\mu(t), \tag{1}$$

where $\beta(\omega) > 0$ for $\omega \varepsilon \Omega$ and $\mu(t)$ in a σ-finite measure defined on the real line. This family of distributions includes many well-known examples such as the Normal with known variance, Poisson, Gamma, and Binomial.

In most games of this sort which arise in practice the loss on a play is given by a function $l_\nu(x)$ of the action ν taken by the player and the outcome x of the observation of the random variable. However, we shall only be interested in the expected loss

$$L_\nu(\omega) = \int l_\nu(x) \, dP(x \mid \omega),$$

and it is on this that we place our restrictions.

In the present article we consider only the case of two actions and require that there be a point ω_0 such that $L_1(\omega) < L_2(\omega)$ for $\omega < \omega_0$ and $L_1(\omega) > L_2(\omega)$ for $\omega > \omega_0$. This corresponds essentially to the usual one-sided statistical testing hypothesis. The loss functions $L_\nu(\omega)$ are henceforth assumed to be sufficiently regular to insure the existence of all the integrals involving them that we will have occasion to consider. Furthermore, we assume that $L_1 - L_2$ has at most a countable number of discontinuities of the first kind. This last assumption is useful in connection with Theorem 2.

If we take n observations of a random variable with distribution (1), then their sum z_n is a sufficient statistic [1] and has the cumulative distribution

$$P_n(z_n \mid \omega) = [\beta(\omega)]^n \int_{-\infty}^{z_n} e^{t\omega} \, d\mu_n(t), \tag{2}$$

where $\mu_n(t)$ is the convolution of μ with itself n times. Thus, if we look at the $(n + 1)$-st decision by itself on the basis of the first n observations, a decision procedure consists of specifying the probability $\varphi_n(z_n)$ with which we take action 1, having observed z_n. The risk on this play becomes

$$\rho_n(\varphi_n \mid \omega) = [\beta(\omega)]^n \int_{-\infty}^{\infty} e^{\omega z_n}[\varphi_n(z_n)L_1(\omega) + (1 - \varphi_n(z_n))(L_2(\omega)] \, d\mu_n(z_n). \tag{3}$$

Various aspects of this fixed sample size game have been considered in [2] and a minimal complete class of decision procedures for this game is the class of monotone procedures: those of the form $\varphi_n(z) = 0$ for $t < t_n$ and $\varphi_n(z) = 1$ for $z > t_n$. We speak of t_n as the "critical number" for φ_n. If μ_n has a discontinuity at t_n, then $z_n = t_n$ occurs with positive probability, and the value $\varphi_n(t_n)$ becomes important. In this case we shall refer to $\varphi_n(t_n)$ as the randomization at t_n.

If we now take a multistage view of the first n plays of the game, a decision procedure becomes a set $\varphi = \{\varphi_i\}_{i=0}^{n}$ consisting of the procedures for each play, and the risk becomes

$$\rho(\varphi \mid \omega) = \sum_{i=0}^{n} \rho_i(\varphi_i \mid \omega), \tag{4}$$

where ρ_i is given by (3).

Thus all of the decision procedures which we shall consider from now on are completely specified by the critical numbers t_j and the randomizations $\lambda_j = \varphi_j(t_j)$ at them.

A complete class of decision procedures is obtained if we restrict each φ_j to be a monotone procedure. Two strategies are said to be equivalent if they give the same value to the risk function (4) for each value of ω. We shall not regard equivalent strategies as distinct, although the functions $\{\varphi_j\}$ which specify them may be different. For example, in the coin-tossing game, a strategy which tells us to call heads on the fourth try if more than $(1\frac{1}{3})$ heads have occurred on the first three plays is equivalent to one which tells us to call heads on the fourth if two or more heads have occurred in the first three plays. In case the measure $d\mu$ in (1) is atomless, two strategies with the same critical numbers are equivalent, and the randomizations are irrelevant. The specification becomes unique if we restrict t_j to the spectrum of $d\mu_j$ and require $\varphi_j(t_j) = 1$ unless t_j is an atom of $d\mu_j$.

3. The Principal theorems. We recall that the spectrum of a random variable (or of its distribution) is the set of those points x with the property that every open interval containing x is assigned positive probability. We define the range Γ of a random variable (or of its distribution) as the convex hull of its spectrum. Γ is automatically a closed interval, which we shall denote by $[a, b]$ where a may be $-\infty$ and b may be $+\infty$.

THEOREM 1. *Let F be an a priori probability measure on Ω with sets of positive measure above and below ω_0. If $s = (\varphi_1, \cdots, \varphi_n)$ is a Bayes procedure against F with critical numbers $(t_1, \cdots, t_n)$, then $t_i - t_{i-1}$ must be an interior point of Γ for $i = 2, \cdots, n$.*

In this theorem $\infty - \infty$ and $(-\infty) - (-\infty)$ are always to be taken as interior points of Γ. The necessary restrictions placed on the relation between t_i and t_{i-1} when a strategy is Bayes are given below for the four principal members of the exponential family.

1) Binomial $\quad t_{i-1} + 1 > t_i > t_{i-1}$,
2) Poisson $\quad t_i > t_{i-1}$,
3) Gamma $\quad t_i > t_{i-1}$,
4) Normal with known σ $\quad t_i$ can be anything.

We defer the proof of this theorem until Section 5.

As a corollary of Theorem 1 we have the following description of a complete class of strategies.

THEOREM 2. *Let $\mathcal{S}$ be the class of decision procedures whose critical numbers satisfy $t_i - t_{i-1} \in \Gamma$ and whose randomizations have the property that if $t_i - t_{i-1}$ is an end-*

point of Γ, *and* $\varphi_{i-1}(t_{i-1}) > 0$, *then* $\varphi_i(t_i) = 1$. *Then* $\mathcal{S}$ *is an essentially complete class.*

The question which arises at this point is whether or not the decision procedures of class $\mathcal{S}$ are all admissible. For an important class of distributions of the exponential family, we are able to say that they are. These are the distributions for which the natural range Ω of ω is open, where by the natural range of ω we mean the set of ω for which

$$\int_{-\infty}^{\infty} e^{\omega t} \, d\mu(t)$$

is finite. We shall use c and d to denote the endpoints of the interval.

THEOREM 5. *If the natural range of* ω *is open, then all the strategies of class* $\mathcal{S}$ *are admissible.*

On the basis of these theorems we can now describe a complete and admissible class of procedures for some simple examples. In the example of the biased coin, the natural range of ω is[3] $(-\infty, \infty)$, and so the class $\mathcal{S}$ is both complete and admissible. Thus one's strategy for such a game should depend only on the number of heads and tails that have occurred; and if λ_{ij} is the probability of taking action 1 (calling heads) when i heads and j tails have been observed, then this strategy is in $\mathcal{S}$ if and only if $\lambda_{ij} > 0$ implies that λ'_{ij} and λ'_{ij} are all equal to one for $i' > i$ and $j' < j$. This criterion can be expressed loosely as follows:

If on a given play the player calls heads with a nonzero probability, and if a head occurs on that play, then the player must call heads on the following play. Thus this criterion seems to be one of consistency.

In the case of a Poisson distribution, if λ_{in} is the probability of betting on a "successful" outcome on the nth play with i successes having been observed, then a procedure is in $\mathcal{S}$ if and only if $\lambda_{in} < 1$ implies that $\lambda_{in'} = 0$ for all $n' > n$.

For a normally distributed variable, however, we have $\Gamma = (-\infty, \infty)$, and so the single-stage procedures can be related to one another in a completely arbitrary fashion, and the resulting multistage procedure will still be admissible. This special result was obtained independently by H. Rubin.

4. Preliminary lemmas. This section is devoted to establishing the fundamental lemmas needed throughout the sequel.

LEMMA 1. *If* $h(\omega)$ *changes sign at most once and* ω_0 *is a change point, then*

$$g(x) = \int e^{x\omega} h(\omega) \, dH(\omega) \qquad \text{with } dH(\omega) \geqq 0$$

has at most one zero, counting multiplicity, provided $H(\omega)$ *does not concentrate its measure fully in the set of zeros of* $h(\omega)$ *or* ω_0.

REMARK. A point ω_0 is called a change point of $h(\omega)$ if $h(\omega')h(\omega) \leqq 0$ for $\omega' \leqq \omega_0 \leqq \omega$, $\omega' \neq \omega$, with inequality for at least one choice of ω' and ω.

[3] If p is the probability of heads, then $\omega = \log [p/(1 - p)]$.

PROOF. Consider the relation

$$\frac{d}{dx}[g(x)e^{-x\omega_0}] = \int e^{x(\omega-\omega_0)}(\omega - \omega_0)h(\omega)\, dH(\omega).$$

As $(\omega - \omega_0)h(\omega)$ has only one sign since both $(\omega - \omega_0)$ and $h(\omega)$ change signs at the same point, we deduce by virtue of the hypothesis that $d/dx)[g(x)e^{-x\omega_0}]$ is strictly of one sign, and hence $g(x)e^{-x\omega_0}$ is strictly monotone. This implies that $g(x)$ can vanish at most once, counting multiplicities. If H has positive measure in both intervals $(-\infty, \omega_0)$ and $(\omega_0, +\infty)$, then there exists precisely one zero. This is easy to see by letting x tend to infinity and analyzing the rate of growth of the integrand.

On further careful examination of the proof of Lemma 1, we notice that if $g(x)$ possesses one change of sign, then both $g(x)$ and $h(\omega)$ change signs in the same directions as their respective arguments increase.

COROLLARY. *If*

$$\lambda(x) = \int e^{x\omega}[L_1(\omega) - L_2(\omega)]\, dF(\omega),$$

where F does not concentrate fully at ω_0, then $\lambda(x)$ vanishes at most once, and if F has measure in both $(-\infty, \omega_0)$ and (ω_0, ∞), then $\lambda(x)$ vanishes precisely once.

LEMMA 2. *If $s_1 - s_0$ does not belong to Γ, then $\beta(\omega)e^{\omega(s_1-s_0)}$ is monotonic. (In particular, provided μ has at least two points in its spectrum if $s_1 - s_0 \leqq a$, then $\beta(\omega)e^{\omega(s_1-s_0)}$ is strictly decreasing; if $s_1 - s_0 \geqq b$, then $\beta(\omega)e^{\omega(s_1-s_0)}$ is strictly increasing.)*

PROOF. Consider the function

$$\beta(\omega)e^{\omega(s_1-s_0)} = \beta(\omega)e^{\omega y} = \frac{1}{\int e^{\omega(x-y)}\, d\mu(x)} = \frac{1}{m(\omega)}.$$

We obtain

$$m'(\omega) = \int e^{\omega(x-y)}(x - y)\, d\mu(x) \geqq 0 \quad \text{if} \quad y \leqq a,$$

and thus $m(\omega)$ is monotonic increasing, whence $\beta(\omega)e^{\omega(s_1-s_0)}$ is decreasing. A similar argument applies to the case where $y \geqq b$.

The next lemma will be useful in determining the admissible strategies of the multistage decision problem.

LEMMA 3. *Let the natural range Ω be open. If x is interior to Γ, then $\beta(\omega)e^{\omega x} \to 0$ as $\omega \to$ the end points of Ω.*

Suppose first that $\Omega = (-\infty, \infty)$. Let $A = \{\xi \mid \xi > x + \epsilon\}$. Since $x \epsilon$ int Γ for ϵ sufficiently small, $\mu A > 0$.

$$\beta(\omega)e^{\omega x} = \frac{1}{\int_\Omega e^{\omega(\xi-x)}\, d\mu(\xi)} \leqq \frac{1}{\int_A e^{\omega(\xi-x)}\, d\mu(\xi)} \leqq \frac{1}{\int_A e^{\omega\epsilon}\, d\mu(\xi)} = \frac{1}{e^{\omega\epsilon}\mu A}.$$

As $\omega \to +\infty$, the last quantity $\to 0$. A similar argument proves the assertion for $\omega \to -\infty$.

Suppose $\Omega = (a, b)$ where, say, b is finite. Form the ratio

$$\frac{\int e^{\omega\xi}\, d\mu(\xi)}{\int e^{\omega(\xi - x)}\, d\mu(\xi)} = e^{\omega x},$$

for $\omega \varepsilon \Omega$. As $\omega \to b$, the ratio tends to e^{bx}. But as $\omega \to b$, $\int e^{\omega\xi}\, d\mu(\xi) \to \infty$ by virtue of the fact that Ω is open. Hence $\int e^{\omega(\xi - x)}\, d\mu(\xi)$ must tend to ∞ as $\omega \to b$ if the ratio is to tend to the finite limit e^{bx} at b. A similar argument at the point a proves the assertion, whenever a is finite.

It should be noticed that if an endpoint of Ω is infinite, then the requirement that $x \varepsilon \operatorname{int} \Gamma$ is essential, whereas if the endpoint is finite, this is not needed. If Ω is not open then for every x, $e^{\omega x}\beta(\omega)$ need not converge to zero as ω tends to the endpoints of Ω.

The following examples will illustrate these last facts:

(i) If $d\mu(x) = e^{(-x^2/2)}$, then $\Omega = (-\infty, \infty)$ is open.

(ii) If $d\mu(x) = \begin{cases} x^\alpha\, dx & x > 0, \quad \alpha \geqq 0, \\ 0 & x \leqq 0, \end{cases}$

then $\Omega = (-\infty, 0)$ is open. Note that for $x > 0$ interior to Γ, then $e^{-x\omega}\beta(\omega) = C\,|\,\omega\,|^\alpha e^{+x\omega} \to 0$ as $\omega \to -\infty$, while for $x = 0$, $e^{x\omega}\beta(\omega) = C\,|\,\omega\,|^\alpha \to +\infty$.

(iii) If $d\mu(x) = e^{-|x|}\, dx$, then $\Omega = (-1, 1)$.

(iv) If $d\mu(x) = [e^{-|x|}/(1 + x^2)]$, then $\Omega = (-1, 1)$ is not open.

(v) Any exponential distribution where the spectrum of μ is a compact set has $\Omega = (-\infty, \infty)$ and thus falls in the domain of validity of Lemma 3.

5. Bayes procedures and an essentially complete class. A complete class of procedures generally too large is easy to determine. In fact, the class of procedures $s = (\phi_1, \cdots, \phi_n)$, where ϕ_i is a monotone procedure for the ith stage, is complete. A monotone procedure is determined by a critical number t_i such that $\phi_i(z_i) = 1$ for $z_i < t_i$ and $\phi_i(z_1) = 0$ for $z_i > t_i$ (with possible randomization relevant on t_i if $\mu\{z_i\} > 0$).

At each stage the risk function is

$$\rho_i(\omega, \phi_i) = \int_{-\infty}^{\infty} \{L(\omega)\phi_i(z_i) + L_2(\omega)[1 - \phi_i(z_i)]\}\, [\beta(\omega)]^i e^{z_i\omega}\, d\mu_i.$$

This is the usual risk function for a single-stage, one-sided decision problem with random variable z_i, so any procedure which is not monotone for the ith stage can be improved upon by inserting a monotone procedure at the ith stage. (See [2].) Our object here for the multistage decision problem is to obtain an essentially minimal complete class of procedures. To do this, we first characterize the Bayes solutions.

A procedure s is Bayes against an a priori distribution $F(\omega)$ on Ω if

$$\rho(F, s) = \min_s \rho(F, s), \quad \text{where} \quad \rho(F, s) = \textstyle\sum_1^n \rho_i(\omega, \phi_i)\, dF(\omega).$$

Interchanging the order of integration

$$\rho(F, s) = \sum_1^n \int \phi_i(z_i)\, d\mu_i \int e^{z_i\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^i\, dF(\omega) + C(F),$$

where $C(F)$ is a function of F which does not involve s. It is clear now what s must be in order to minimize this expression. The optimal procedure $s = (\phi_1, \cdots, \phi_n)$ is

$$\phi_i(z_i) = \begin{cases} 1, & \text{if } \int e^{z_i\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^i\, dF(\omega) < 0, \\ 0, & \text{if } \int e^{z_i\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^i\, dF(\omega) > 0. \end{cases}$$

But

$$g(z_i) = \int e^{z_i\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^i\, dF(\omega)$$

changes sign once, since $L_1(\omega) - L_2(\omega)$ changes sign once, and it changes in the same order as $L_1(\omega) - L_2(\omega)$. (This assumes that $F(\omega)$ has sets of positive measure both above and below ω_0. The reader can easily complete the elementary analysis necessary when this is not satisfied.)

Thus there exists a t_i such that

$$\phi_i(z_i) = \begin{cases} 1, & \text{for } z_i < t_i, \\ 0, & \text{for } z_i > t_i. \end{cases}$$

THEOREM 1. *Let F be an a priori probability measure on Ω with sets of positive measure above and below ω_0. If $s = (\phi_1, \cdots, \phi_n)$ is a Bayes procedure against F with critical numbers $(t_1, \cdots, t_n)$, then $t_i - t_{i-1}$ must be an interior point of Γ for $i = 2, \cdots, n$.*

PROOF OF THEOREM 1:

Let $\Omega = (c, d)$, where d may be $+\infty$ and c may be $-\infty$. For $i = 2, \cdots, n$,

$$\int_c^d e^{t_i\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^i\, dF(\omega) = 0, \tag{5}$$

and

$$\int_c^d e^{t_{i-1}\omega}[L_1(\omega) - L_2(\omega)][\beta(\omega)]^{i-1}\, dF(\omega) = 0. \tag{6}$$

(5) and (6) can be written as

$$\int_{\omega_0}^d e^{t_i\omega} L_1(\omega)\beta^i(\omega)\, dF(\omega) = \int_c^{\omega_0} e^{t_i\omega} L_2(\omega)\beta^i(\omega)\, dF(\omega), \tag{7}$$

and

$$(8) \qquad \int_{\omega_0}^{d} e^{t_{i-1}\omega} L_1(\omega)\beta^{i-1}(\omega)\, d f(\omega) = \int_{c}^{\omega_0} e^{t_{i-1}\omega} L_2(\omega)\beta^{i-1}(\omega)\, dF(\omega).$$

Suppose $t_i - t_{i-1}$ is not interior to Γ, say, $\geqq b$. Then by Lemma 2,

$$\int_{\omega_0}^{d} e^{t_i\omega} L_1(\omega)\beta^{i}(\omega)\, dF(\omega) = \int_{\omega_0}^{d} e^{(t_i - t_{i-1})\omega}\beta(\omega) e^{t_{i-1}\omega} L_1(\omega)\beta^{i-1}(\omega)\, dF(\omega)$$

$$> e^{(t_i - t_{i-1})\omega_0}\beta(\omega_0) \int_{\omega_0}^{d} e^{t_{i-1}\omega} L_1(\omega)\beta^{i-1}(\omega)\, dF(\omega)$$

$$= e^{(t_i - t_{i-1})\omega_0}\beta(\omega_0) \int_{c}^{\omega_0} e^{t_{i-1}\omega} L_2(\omega)\beta^{i-1}(\omega)\, dF(\omega)$$

$$> \int_{c}^{\omega_0} e^{(t_i - t_{i-1})\omega}\beta(\omega) e^{t_{i-1}\omega} L_2(\omega)\beta^{i-1}(\omega)\, dF(\omega)$$

$$= \int_{c}^{\omega_0} e^{t_i\omega} L_2(\omega)\, \beta^{i}(\omega) dF(\omega),$$

which is impossible by virtue of (7). Thus the theorem is proved.

THEOREM 2. Let $\mathcal{S}$ be the class of decision procedures whose critical numbers satisfy $t_i - t_{i-1} \varepsilon \Gamma$ and whose randomizations have the property that if $t_i - t_{i-1}$ is an endpoint of Γ, and $\varphi(t_{i-1}) > 0$, then $\varphi(t_i) = 1$. Then $\mathcal{S}$ is an essentially complete class.

PROOF. Let $\omega_1, \omega_2, \omega_3 \cdots$ represent a dense set of ω_i, not including ω_0, which includes all other discontinuities of $L_1(\omega) - L_2(\omega)$. Considering only $\omega_1, \omega_2, \cdots, \omega_m$ as the pure states of nature, and in view of Theorem 1, we know that all Bayes procedures have the form as described in the theorem with $t_j - t_{j-1}$ in the interior of Γ. Thus these procedures constitute an essentially complete class when the states of nature are $\omega = \omega_k (i = 1, \cdots, m)$. (See [1].) Hence if s is any procedure, there exists a procedure $s^{(m)}$ of the type indicated in the theorem where

$$\rho(\omega_i, \varphi) \geqq \rho(\omega, \varphi^m) \qquad i = 1, \cdots, m,$$

and hence

$$\sum_{j=1}^{n} [\beta(\omega)]^j [L_1(\omega_i) - L_2(\omega_i)] \int e^{x\omega_i}(\varphi_j - \varphi_j^m)\, d\mu_j(x) \geqq 0.$$

By the usual diagonal process we can select a limit strategy s^0 from s^m, that is $\varphi_i^r(x)$ converges to $\varphi_1^0(x)$ for every x, where r is a subsequence of m. It is easy to see that for each i,

$$\rho(\omega_i, s) \geqq \rho(\omega_i, s^0).$$

Since ω_i are dense in Ω, it follows that the above inequality can be extended to hold for all ω. Moreover, each s^m is in $\mathcal{S}$, and $\mathcal{S}$ is clearly closed with respect to pointwise convergence. Thus the limit procedure s^0 must also be in $\mathcal{S}$.

6. Admissibility. In the preceding section an essentially complete class of procedures was determined. It is now our purpose to investigate the admissibility of such procedures. Throughout this section we assume that Ω is open. The analysis will be divided into two cases: (a) where $d\mu(x)$ is atomless, and (b) where $d\mu(x)$ can possess atoms.

Case a. The σ-finite measure $d\mu(x)$ is atomless. For this situation the collection of procedures in $\mathcal{S}$ are uniquely characterized by the critical values $(t_1, \cdots, t_n)$, where

$$\varphi_i(t) = \begin{cases} 1, & z < t_i \\ 0, & z > t_i, \end{cases}$$

and any randomization at t_i is of no consequence in terms of the expected risk. Consequently, a strategy s and the collection of critical numbers $(t_1, t_2, \cdots, t_n)$ shall be referred to interchangeably.

THEOREM 3. *All monotone strategies* $s = (t_1, t_2, \cdots, t_n)$ *with* $t_{k+1} - t_k$ *in* Γ *for* $k = 1, 2, \cdots, n-1$ *are admissable.*

PROOF. Suppose not. Then for some $s = (t_1, \cdots, t_n)$ with $t_{k+1} - t_k \varepsilon \Gamma$, there exists a strategy s' which is better. If s' is not a monotone strategy with $t_{k+1} - t_k \varepsilon \Gamma$, then by the completeness of this class of strategies, there exists a monotone strategy $s^* = (t_1^*, \cdots, t_n^*)$ with $t_{k+1}^* - t_k^* \varepsilon \Gamma$ better than s'.

By definition,

$$\rho(\omega, s) = \sum_1^n [\beta(\omega)]^k e^{z_k\omega} \int \{\varphi_k(z_k)L_1(\omega) + (1 - \varphi_k(z_k))L_2(\omega)\}\, d\mu_k(z_k),$$

$$\rho(\omega, s) - \rho(\omega, s^*) = \sum_1^n [\beta(\omega)]^k \int e^{z_k\omega}(\varphi_k - \varphi_k^*)[L_1(\omega) - L_2(\omega)]\, d\mu_k(z_k)$$

$$= [L_1(\omega) - L_2(\omega)] \sum_1^n [\beta(\omega)]^k \int_{t_k^*}^{t_k} e^{z_k\omega}\, d\mu_k \geqq 0.$$

The theorem will be proved if it can be shown that this holds true only if $t_k = t_k^*$ for all k, or equivalently, $\rho(s, \omega) = \rho(s^*, \omega)$ for all ω.

For the sake of definiteness suppose that $t_1 \leqq t_1^*$. Let $i_1, i_2, i_3, \cdots \leqq n$ be defined by the following relation.

$$\begin{aligned} &t_1 \leqq t_1^*, t_2 \leqq t_2^*, \cdots, t_{i_1-1} \leqq t_{i_1-1}^*, t_{i_1} < t_{i_1}^* \\ &t_{i_1+1} = t_{i_1+1}^*, \quad \cdots, t_{i_2-1} = t_{i_2-1}^*, t_{i_2} > t_{i_2}^* \\ &t_{i_2+1} \geqq t_{i_2+1}^*, \quad \cdots, t_{i_3-1} \geqq t_{i_3-1}^*, t_{i_3} > t_{i_3}^* \\ &t_{i_3+1} = t_{i_3+1}^*, \quad \cdots, t_{i_4-1} = t_{i_4-1}^*, t_{i_4} < t_{i4}^* \\ &t_{i_4+1} \leqq t_{i_4+1}^*, \quad \cdots, t_{i_4-1} \leqq t_{i_4-1}^*, t_{i_5} < t_{i_5}^*, \end{aligned}$$

and so on.

The strict inequality signs above mean that the measure μ_h has positive measure in the half-open interval between t_h and t_h^*. Otherwise t_h and t_h^* define the same test and are to be taken equal.

Let $\Omega = (c, d)$, where c or d or both may be infinite. The first step is to show that as $\omega \to d$,

$$R_{i_1}^{j} = \frac{\beta(\omega)^{j}\left|\int_{t_j^*}^{t_j} e^{t\omega}\, d\mu_j(t)\right|}{\beta(\omega)^{i_1}\left|\int_{t_{i_1}^*}^{t_{i_1}} e^{t\omega}\, d\mu_{i_1}(t)\right|} \to 0,$$

for $i_1 + 1 \leqq j \leqq i_3$. If $d > 0$, then for ω near d,

$$\left|\int_{t^*}^{t_j} e^{t\omega}\, d\mu_j(t)\right| \leqq c_1 e^{t_j\omega}; \tag{9}$$

and as $t_{i_1}^* > t_{i_1}$, we get

$$\int_{t_{i_1}}^{t_{i_1}^*} e^{t\omega}\, d\mu_{i_1}(t) \geqq c_2(\epsilon) \exp{(t_{i_1} + \epsilon)\omega} > 0 \tag{10}$$

where $\epsilon > 0$, and $c_2(\epsilon)$ is a constant which depends only on ϵ and not on ω. This is certainly true for ϵ sufficiently small, since the half-open interval $(t_{i_1}, t_{i_1}^*)$ has positive measure by assumption.

Since $t_{i_1} < t_{i_1}^*$ and $t_{i_1+1} \geqq t_{i_1+1}^*$, $t_{i_1+1} - t_{i_1} > t_{i_1+1}^* - t_{i_1}^*$. But both $t_{i_1+1} - t_{i_1}$ and $t_{i_1+1}^* - t_{i_1}^*$ are in Γ. Thus for $\epsilon > 0$ sufficiently small, $t_{i_1+1} - t_{i_1} - \epsilon$ will be in the interior of Γ. In fact, two sufficiently small numbers $\epsilon > 0$, $\eta > 0$ can be found such that $t_{i_1+1} - t_{i_1} - \epsilon'$ is in the interior of Γ for $\epsilon - \eta \leqq \epsilon' \leqq \epsilon + \eta$. In the bound (10), above, choose $\epsilon > 0$ sufficiently small so that not only is (10) satisfied, but ϵ also has the property just indicated. Combining (9) and (10), we have

$$R_{i_1} \leqq c_3(\epsilon)\beta(\omega)^{j-i_1} \exp{\{(t_j - t_{i_1} - \epsilon)\omega\}}; \tag{11}$$

$c_3(\epsilon)$ is independent of ω, $\epsilon > 0$, and $t_{i_1+1} - t_{i_1} - \epsilon'$ is in int Γ for $\epsilon - \eta \leqq \epsilon' \leqq \epsilon + \eta$.

Since $t_k - t_{k-1} \epsilon \Gamma$, $k = i_1 + 2, \cdots, j$, for η_k near zero and of the correct sign, $t_k - t_{k-1} - \eta_k$ will be in the interior of Γ. Choose each η_k sufficiently small in absolute value so that $\sum_{i_1+2}^{j} |\eta_k| < \eta$. Then the ϵ' defined by $\epsilon' = \epsilon - \sum_{i_1+2}^{j} \eta_k$ satisfies $\epsilon - \eta \leqq \epsilon' \leqq \epsilon + \eta$. Splitting ϵ up into $\epsilon' + \sum \eta_k$, the upper bound (11) can be rewritten as $R_{i_1}^{j} \leqq c_3(\epsilon)\beta(\omega) \exp{\{(t_{i_1+1} - t_{i_1} - \epsilon')\omega\}} \prod_{i_1+2}^{j} \beta(\omega) \exp{\{(t_k - t_{k-1} - \eta_k)\}}$. Since $t_{i_1+1} - t_{i_1} - \epsilon'$ is int Γ and $t_k - t_{k-1} - \eta_k$ are int Γ for $k = i_1 + 2, \cdots, j$, each term involving ω in the above bound $\to 0$ as $\omega \to d$ by Lemma 3. Thus $R_{i_1} \to 0$ as $\omega \to d > 0$ for $j = i_1 + 1, \cdots, i_3$. When $d \leqq 0$, a similar proof using t_j^*, $t_{i_1}^*$ instead of t_j, t_{i_1} and $t_{i_1+1}^* - t_{i_1}^* + \epsilon$ for $\epsilon > 0$ instead of $t_{i_1+1} - t_{i_1} - \epsilon$ in the upper bound gives the same result.

An analogous proof shows that $R_{i_1}^{j} \to 0$ for $i_1 + 1 \leqq j \leqq i_3$ as $\omega \to c$. In a similar manner, if $R_{i_3}^{j}$ is defined by

$$R_{i_3}^{j} = \frac{\beta(\omega)^{j}\left|\int_{t_j^*}^{t_j} e^{t\omega}\, d\mu_j(t)\right|}{\beta(\omega)^{i_3}\left|\int_{t_{i_3}^*}^{t_{i_3}} e^{t\omega}\, d\mu_{i_3}(t)\right|},$$

then $R^j_{i_3} \to 0$ as $\omega \to c, d$ for $i_3 + 1 \leqq j \leqq i_5$. Similar results are obtained for $R^j_{i_5}$, $R^j_{i_7}$, etc.

What this shows is that as ω tends to the end-points of Ω, the i_1 term is of a larger order of magnitude than the later terms up to and including i_3. By the same reasoning, the i_3 term is of larger order of magnitude than the later terms up to and including i_5, and so on. Thus as $\omega \to c$ or d, the i_1 term dominates the remaining finite number of terms. Since the first i_1 terms have the same sign, the sign of

$$\sum_1^n [\beta(\omega)]^k \int_{t_k^*}^{t_k} e^{z_2\omega} \, d\mu_k$$

is the same as the sign of

$$\int_{t_{i_1}^*}^{t_{i_1}} e^{z_{i_1}\omega} \, d\mu_{i_1},$$

which is one sign when either $\omega \to c$ or when $\omega \to d$. But the sign of $L_1(\omega) - L_2(\omega)$ for ω near c is negative while for ω near d it is positive. Thus for ω near c, $\rho(\omega, s) - \rho(\omega, s^*)$ is of one sign, and for ω near d it is of the opposite sign. But this contradicts the fact that $\rho(\omega, s) - \rho(\omega, s^*) \geqq 0$ for all ω. Thus the assumption that there exists an i_1 such that $t_{i_1} < t_{i_1}^*$ is incorrect, and the following must hold:

$$t_1 = t_1^*, \qquad t_2 = t_2^*, \cdots, t_{i_1} = t_{i_1}^*.$$

The first i_1 terms in $\rho(\omega, s) - \rho(\omega, s^*)$ are zero. The argument can be repeated again on i_3, i_5, etc. The conclusion is that $t_k = t_k^*$ for all k. Thus $s \equiv s^*$ and $\rho(\omega, s) \equiv \rho(\omega, s^*)$. This completes the proof of the theorem.

Case b. The restriction that $d\mu(x)$ is atomless is removed.

THEOREM 4. *Without any assumptions on* μ, *if* s *is a procedure in* $\mathcal{S}$ *where* $t_j - t_{j-1}$ *is interior to* Γ *for every* j, *then* s *is admissible.*

PROOF: Let s^* denote a strategy of $\mathcal{S}$ which improves s. The same domination argument used in Theorem 3 when $t_j - t_{j-1}$ is interior to Γ shows that $\mu_j(t_j)(\lambda_j - \lambda_j^*) = 0$, and thus $\rho(s, \omega) \equiv \rho(s^*, \omega)$. This therefore implies that s^* is admissible.

THEOREM 5. *If the natural range* Ω *of* ω *is open, then each decision procedure in* $\mathcal{S}$ *is admissible.*

PROOF. Theorem 4 already disposes of the case in which the differences of all critical numbers are interior to Γ. For the sake of simplicity of exposition, we shall assume that the right-hand endpoint b of Γ is infinite. The case in which both endpoints are finite has a completely analogous but tedious proof. Let s be a procedure in $\mathcal{S}$, and let s^* be a procedure which betters s. Since $\mathcal{S}$ is complete, we may assume that s^* is in $\mathcal{S}$.

The critical numbers of s are denoted as previously by t_i, and the respective randomizations at t_i by λ_i. The notation $t_i > t_i^*$ for this case shall mean either $t_i > t_i^*$ or $t_i = t_i^*$ and $\lambda_i > \lambda_i^*$ where the last possibility only has relevance when $\mu_i\{t_i\} > 0$. Let the indices $i_1, i_2, i_3, \cdots$ be defined as in the proof of Theorem 3, subject to the new interpretation of $t_j \geqq t_j^*$.

Pursuing a similar analysis as in Theorem 3, it remains only to show that

$$R_{i_1}^j = \frac{\beta(\omega)^j \left| \int e^{t\omega}(\varphi_j - \varphi_j^*)\, d\mu_j \right|}{\beta(\omega)^{i_1} \left| \int e^{t\omega}(\varphi_{i_1} - \varphi_{i_1}^*)\, d\mu_{i_1} \right|}$$

tends to zero as ω tends to the end points of Ω for all $j = i_1 + 1, \cdots, i_3$. The need to represent the expression for $R_{i_1}^j$ in terms of $\varphi_j, \varphi_j^*, \varphi_{i_1}$ and $\varphi_{i_1}^*$ is that the randomization of φ_j, φ_j^*, etc., at the critical numbers may contribute to the integral.

The only situation necessitating an additional argument distinct from that used in the proof of Theorem 3 arises when $t_{i_1} = t_{i_1}^*$ with $\lambda_{i_1} < \lambda_{i_1}^*$. The expression $R_{i_1}^j$ becomes

$$R_{i_1}^j = \frac{\beta(\omega)^j \left| \int e^{t\omega}(\varphi_j - \varphi_j^*)\, d\mu_j \right|}{C\beta(\omega)^{i_1}\mu_{i_1}\{t_{i_1}\}\, e^{\omega t_{i_1}}}.$$

For the term $j(i_1 + 1 \leqq j \leqq i_3)$ if $t_j = t_{i_1} + (j - i_1)a$, then the same must hold for t_j^*, since $t_j \geqq t_j^*$; and $\lambda_j = \lambda_j^* = 1$, since s and s^* both belong to $\mathcal{S}$. Consequently $t_j = t_j^*$. Therefore we may assume that $t_j > t_j^* \geqq t_{i_1} + (j - i_1)a$. If also $t_j^* > t_{i_1} + (j - i_1)a = t_{i_1}^* + (j - i)a$, then the argument can be carried through as in Theorem 5, so let us further suppose that $t_j^* = t_{i_1} + (j - i_1)a$. This requires that $\lambda_j^* = 1$. It is now asserted that

$$\beta(\omega)^{j-i_1}e^{-t_{i_1}\omega}\int e^{t\omega}(\varphi_j - \varphi_j^*)\, d\mu_j \tag{12}$$

tends to zero as $\omega \to$ the endpoints of Ω, where the range of integration of t is a region which is included in the half-open interval $t_j \geqq t > t_{i_1} + (j - i_1)a$.

To establish this last fact, we distinguish two cases according as $\mu\{a\} > 0$ or $\mu\{a\} = 0$.

I. $\mu\{a\} > 0$: Note that

$$\beta(\omega)^{j-i_1}e^{(t-t_{i_1})\omega} < \frac{e^{(t-t_{i_1})\omega}}{[\mu\{a\}e^{a\omega}]^r} \leqq C\, e^{(t-t_{i_1}-ar)\omega} \leqq C',$$

as ω tends to the left-hand endpoint of Ω for $t > t_{i_1} + ar(r = j - i_1)$. Of course, for each t in the same region by Lemma 3,

$$\beta(\omega)^{j-i_1}e^{(t-t_{i_1})\omega} \to 0.$$

As $d\mu$ is integrable on any closed interval contained in the spectrum of μ, which may include an endpoint if it belongs to the point spectrum, the expression in (11) on account of the Lebesgue convergence theorem tends to zero as $\omega \to c$ where $\Omega = (c, d)$. If $d < \infty$ the argument is the same as at c. If $d = \infty$, then

$\beta(\omega)$ vanishes faster than any exponential as $\omega \to d$, and the result in (12) is immediately clear.

II. $\mu\{a\} = 0$. The formula (12) is bounded by

$$[\beta(\omega)]^{j-i_1} \int_{[t_{i_1}+(j-i_1)a]+}^{\infty} e^{(t-t_{i_1})\omega}\, d\mu_j(t),$$

which on some simple calculation reduces to

$$\int_{a+}^{\infty} e^{t\omega}\, d\mu_{i_1}(t + t_{i_1}).$$

If $c = -\infty$ as $\omega \to -\infty$, then it is easy to see that

$$\int_{a+}^{\infty} e^{t\omega}\, d\mu(t + t_{i_1}) \to 0.$$

If c is finite, then the argument of showing that (12) tends to zero is similar to that of I above. The reasoning involved at the other endpoint d is similar and is omitted.

Having thus established the assertion of (12) the remainder of the reasoning proceeds as in the proof of Theorem 3.

7. Converse of Theorem 1.

THEOREM 6. *If a decision procedure s has critical numbers t_j with $t_j - t_{j-1}$ interior to Γ, then s is Bayes against an a priori distribution fully concentrated at $n + 1$ points, n being the total number of plays of the game.*

PROOF. Let us consider a distribution F whose spectrum is the $n + 1$ point ω_i. In order for s to be Bayes against F, it is necessary and sufficient that

$$\sum_{i=1}^{n+1} e^{\omega_i t_j}[\beta(\omega)]^j (L_1 - L_2)(\omega_i)\alpha_i = 0 \qquad j = 1, 2, \cdots, n. \tag{13}$$

The solutions $\gamma_i = [\alpha_i(L_1 - L_2)(\omega_i)]$ of these equations are proportional to the cofactors of the last row of

$$\begin{pmatrix} e^{\omega_1 t_1}\beta(\omega_1), & e^{\omega_2 t_1}\beta(\omega_2), & \cdots, & e^{\omega_{n+1} t_1}\beta(\omega_{n+1}) \\ e^{\omega_1 t_2}\beta^2(\omega_1), & e^{\omega_2 t_2}\beta^2(\omega_2) & \cdots, & e^{\omega_{n+1} t_2}\beta^2(\omega_{n+1}) \\ e^{\omega_1 t_n}\beta^n(\omega_1), & e^{\omega_2 t_n}\beta^n(\omega_2) & \cdots, & e^{\omega_{n+1} t_n}\beta^n(\omega_{n+1}) \\ a_1 & a_2 & \cdots, & a_{n+1}. \end{pmatrix}$$

The values ω_1 and ω_2 are chosen fixed subject only to the inequalities $\omega_1 < \omega_0 < \omega_2$.

The remaining ω_i, $i = 3, \cdots, n + 1$, are chosen near c and d appropriately so that $\beta(\omega_i) \exp\{\omega_i(t_j - t_{j-1})\}$ are all near zero but each of smaller order of magnitude. In fact, each ω_i is chosen successively, each closer to c or d so that $\beta(\omega_3) \exp\{\omega_3(t_j - t_{j-1})\}$ is of larger order of magnitude than $\beta(\omega_4) \exp\{\omega_4(t_j - t_{j-1})\}$, which in turn is of larger order of magnitude than $\beta(\omega_5) \exp\{\omega_5(t_j - t_{j-1})\}$, etc.

This careful selection of ω_i implies that the cofactor of a_1 is dominated by $(-1)\ \exp\{\omega_2 t_n\}\ \exp\{\omega_3 t_{n-1}\} \cdots \exp\{\omega_{n+1} t_1\}[\beta^n(\omega_3)\beta(\omega_3)^{n-1} \cdots \beta(\omega_{n+1})]$ and thus is negative, and in a similar manner we can infer that the cofactor of a_2 is positive. Thus according to our choice of ω_1 and ω_2, it follows that $\alpha_1 > 0$, $\alpha_2 > 0$.

The sign of the cofactors $a_r(r = 3, \cdots, n + 1)$ is independent of the choice of ω_i, provided only that $\beta(\omega)\exp\{\omega_r(t_j - t_{j-1})\}$ has the right order of magnitude as described above. The exact sign of a_r is

$$\text{sign of cofactor } a_r = (-1)^r \begin{vmatrix} e^{\omega_1 t_{n-1}} & e^{\omega_2 t_{n-1}} \\ e^{\omega_1 t_n}\beta(\omega) & e^{\omega_2 t_n}\beta(\omega) \end{vmatrix}$$

The ω_i are then selected near either c or d, so that they produce the correct magnitude and so that $\alpha_i > 0$. This entails choosing ω_r near c and d alternately. The Bayes distribution is of the form: place the mass $\lambda_i = \alpha_i / \sum \alpha_i$ at ω_i; then

$$\int e^{\omega t}[\beta(\omega)]^j(L_1 - L_2)(\omega)\, dF(\omega) = 0$$

reduces to the equations (13), and hence the Bayes strategy for this F is the given s. This completes the proof.

The above proof was suggested by J. Pratt and replaces a more cumbersome construction by the authors, which showed also that each Bayes strategy was Bayes against a strategy fully concentrated at only n points.

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