SOME FURTHER METHODS OF CONSTRUCTING REGULAR GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS

By G. H. Freeman

East Malling Research Station, Maidstone, England

1. Summary. Some further methods are given for the construction of regular group divisible incomplete block designs, and designs derivable by these methods are tabulated. The methods are (i) designs containing complete and incomplete groups; (ii) designs with groups arranged in sets; and (iii) designs derivable by addition. In the first two of these methods, which are related, it may be possible to avoid having to take all the blocks that the full procedure would indicate.

2. Introduction. An incomplete block design with r replicates of v treatments on b blocks with k plots in each is said to be group divisible if it contains m groups of n treatments each, where $mn = v$, in such a way that treatments in the same group concur in λ_1 blocks and treatments in different groups concur in λ_2 blocks, where $\lambda_1 \neq \lambda_2$. For all group divisible designs the following relationships hold: $bk = vr$, $\lambda_1(n - 1) + \lambda_2 n(m - 1) = r(k - 1)$, $r \geqq \lambda_1$, $rk \geqq \lambda_2 v$. Further, the efficiency factors are as follows:

$$\text{within groups,} \quad E_1 = \frac{n\{\lambda_1 + (m-1)\lambda_2\}}{rk} = 1 - \frac{r - \lambda_1}{rk};$$

$$\text{between groups,} \quad E_2 = \frac{mn^2\lambda_2\{\lambda_1 + (m-1)\lambda_2\}}{rk\{\lambda_1 + (mn-1)\lambda_2\}} = \frac{\lambda_2 v}{\lambda_1 + \lambda_2(v-1)} E_1.$$

Group divisible designs have been classified into three types by Bose and Connor [2]: (i) singular designs for which $r = \lambda_1$; (ii) semi-regular designs for which $r > \lambda_1$, $rk = \lambda_2 v$; (iii) regular designs for which $r > \lambda_1$, $rk > \lambda_2 v$. It is the purpose of this paper to present some unpublished methods for the construction of regular group divisible designs, and to give examples of the designs obtained by these methods.

Methods for the construction of group divisible incomplete block designs have been given by Bose, Shrikhande, and Bhattacharya [3], and tables of such designs, *inter alia*, have been prepared by Bose, Clatworthy, and Shrikhande [1]. The designs derived here are of the following kinds: (i) designs containing complete and incomplete groups; (ii) designs with groups arranged in sets; (iii) designs derivable by addition. The first two of these are related in that designs of a very general nature which are generalizations of the first kind are also, in one sense, generalizations of the second. As will be seen, these very general designs tend to require extremely large numbers of replicates and plots per block, and so they are not considered in any great detail below. An-

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other feature common to the first two kinds of design is that in both cases designs are possible which have only a fraction of the number of blocks and replicates necessary for the complete design. The third method of deriving designs, by addition, is unrelated to the other two; it is a slight generalization of a method given by Bose, *et al.* [3]. Apart from a few isolated examples, none of the designs given here appears in the tables of Bose, *et al.* [1].

3. Complete and incomplete groups. The first method, that of complete and incomplete groups, arises in the following manner. Consider a design with m groups of n members each, where $mn = v$. Let each block of the design contain u complete groups ($1 \leqq u \leqq m - 1$) and h members from one other group ($1 \leqq h \leqq n - 1$), the design containing sufficient blocks that the h extra treatments shall be selected in all prossible ways from each of the $(m - u)$ groups not wholly represented. This necessitates ${}^nC_h(m - u)$ blocks with the same u groups, and thus ${}^mC_u{}^nC_h(m - u)$ blocks in all. The complete design thus has ${}^{m-1}C_u{}^nC_h[(nu + h)/n]$ replicates of mn treatments on ${}^mC_u{}^nC_h(m - u)$ blocks with $(nu + h)$ plots per block.

By the method of its construction the design is group divisible, save where $u = m - 1$ and $h = n - 1$ simultaneously, in which case it degenerates into a totally balanced incomplete block design. Ignoring this case, it is seen that the other parameters of the first kind are $\lambda_1 = {}^{m-1}C_u\{u{}^nC_h + {}^{n-2}C_{h-2}\}$ and $\lambda_2 = {}^{m-2}C_{u-1}\{(u - 1){}^nC_h + 2{}^{n-1}C_{h-1}\}$. Thus the design satisfies the conditions $r > \lambda_1$, $rk > \lambda_2 v$, where r is the number of replicates and k the number of plots per block, and so is regular.

As an illustration of the construction of the design consider the case $m = 2$, $u = 1$, $n = 4$, $h = 2$, giving a design with 9 replicates of 8 treatments on 12 blocks with 6 plots in each, with $\lambda_1 = 7$, $\lambda_2 = 6$. The design is as follows. The two groups are ABCD and EFGH and the blocks are given by columns.

A	A	A	A	A	A	E	E	E	E	E	E
B	B	B	B	B	B	F	F	F	F	F	F
C	C	C	C	C	C	G	G	G	G	G	G
D	D	D	D	D	D	H	H	H	H	H	H
E	E	E	F	F	G	A	A	A	B	B	C
F	G	H	G	H	H	B	C	D	C	D	D

All designs of this type with $r \leqq 10$ are shown in Table I.

In certain circumstances it is possible to obtain a regular design with the same m, u, n and h without taking as many blocks as would be implied by the designs just described. For example, when $m = 3$, $u = 1$, designs with just half the blocks are possible by taking blocks with the first group of treatments complete and the second incomplete, the second complete and the third incomplete, and the third complete and the first incomplete. There are then ${}^nC_h[(n + h)/n]$ replicates of $3n$ treatments on $3{}^nC_h$ blocks with $(n + h)$ plots per block, $\lambda_1 = {}^nC_h + {}^{n-2}C_{h-2}$, $\lambda_2 = {}^{n-1}C_{h-1}$. As an illustration, consider $n = 4$, $h = 1$. The design is as follows, and here $r = k = 5$, $v = b = 12$, $\lambda_1 = 4$,

$\lambda_2 = 1$. The groups are ABCD, EFGH, and IJKL, and the blocks are given by columns.

A	A	A	A	E	E	E	E	I	I	I	I
B	B	B	B	F	F	F	F	J	J	J	J
C	C	C	C	G	G	G	G	K	K	K	K
D	D	D	D	H	H	H	H	L	L	L	L
E	F	G	H	I	J	K	L	A	B	C	D

All designs of this type with $r \leqq 10$ are given in Table II.

Half designs are possible on this same principle for any odd value of m when $u = 1$; in such designs those blocks are taken in which one group is complete and another incomplete, but not conversely for the same pair of groups. As an illustration, $m = 5$, $n = 3$, $h = 1$ gives the following design with 8 replicates of 15 treatments on 30 blocks with 4 plots each, $\lambda_1 = 6$, $\lambda_2 = 1$:

A	A	A	D	D	D	G	G	G	J	J	J	M	M	M	A	A	A	G	G	G	M	M	M	D	D	D	J	J	J
B	B	B	E	E	E	H	H	H	K	K	K	N	N	N	B	B	B	H	H	H	N	N	N	E	E	E	K	K	K
C	C	C	F	F	F	I	I	I	L	L	L	O	O	O	C	C	C	I	I	I	O	O	O	F	F	F	L	L	L
D	E	F	G	H	I	J	K	L	M	N	O	A	B	C	G	H	I	M	N	O	D	E	F	J	K	L	A	B	C

The groups are ABC, DEF, GHI, JKL, and MNO, and the blocks are given by columns. For $u = 1$ all the possible half designs with 10 or fewer replicates are given below:

m	n	h	r	v	b	k	λ_1	λ_2	E_1	E_2
5	2	1	6	10	20	3	4	1	0.89	0.68
5	3	1	8	15	30	4	6	1	0.94	0.70
5	4	1	10	20	40	5	8	1	0.96	0.71
5	3	2	10	15	30	5	8	2	0.96	0.80
7	2	1	9	14	42	3	6	1	0.89	0.65

Designs of this nature with $u \neq 1$ are also sometimes possible, but there is only one with 10 or fewer replicates. This is given by $m = 5$, $u = 2$, $n = 2$, $h = 1$, and gives rise to the following design with 10 replicates of 10 treatments on 20 blocks with 5 plots each, $\lambda_1 = 8$, $\lambda_2 = 4$, $E_1 = 0.96$, $E_2 = 0.87$:

A	A	C	C	E	E	G	G	I	I	A	A	C	C	E	E	G	G	I	I
B	B	D	D	F	F	H	H	J	J	B	B	D	D	F	F	H	H	J	J
E	E	G	G	I	I	A	A	C	C	C	C	E	E	G	G	I	I	A	A
F	F	H	H	J	J	B	B	D	D	D	D	F	F	H	H	J	J	B	B
C	D	E	F	G	H	I	J	A	B	G	H	I	J	A	B	C	D	E	F

The groups are AB, CD, EF, GH, and IJ, and the blocks are given by columns.

A valid design is obtained by taking half the blocks, the first ten as the design is written down. The design has 5 replicates of 10 treatments on 10 blocks of 5 plots each, $\lambda_1 = 4$, $\lambda_2 = 2$, $E_1 = 0.96$, $E_2 = 0.87$.

Half designs are also possible for other cases where $n = 2$, $h = 1$, $m = 2u +$

TABLE I

Regular group divisible designs formed by the method of complete and incomplete groups

m	u	n	h	r	v	b	k	λ_1	λ_2	E_1	E_2
2	1	3	1	4	6	6	4	3	2	0.94	0.87
2	1	4	1	5	8	8	5	4	2	0.96	0.85
2	1	4	2	9	8	12	6	7	6	0.96	0.94
2	1	5	1	6	10	10	6	5	2	0.97	0.85
2	1	6	1	7	12	12	7	6	2	0.98	0.84
2	1	7	1	8	14	14	8	7	2	0.98	0.84
2	1	8	1	9	16	16	9	8	2	0.99	0.83
2	1	9	1	10	18	18	10	9	2	0.99	0.83
3	1	2	1	6	6	12	3	4	2	0.89	0.76
3	1	3	1	8	9	18	4	6	2	0.94	0.77
3	1	3	2	10	9	18	5	8	4	0.96	0.86
3	1	4	1	10	12	24	5	8	2	0.98	0.78
3	2	3	1	7	9	9	7	6	5	0.98	0.96
3	2	4	1	9	12	12	9	8	6	0.99	0.96
4	1	2	1	9	8	24	3	6	2	0.89	0.71

TABLE II

Regular group divisible designs with half the blocks for complete and incomplete groups with $m = 3$, $u = 1$

n	h	r	v	b	k	λ_1	λ_2	E_1	E_2
2	1	3	6	6	3	2	1	0.89	0.76
3	1	4	9	9	4	3	1	0.94	0.77
3	2	5	9	9	5	4	2	0.96	0.86
4	1	5	12	12	5	4	1	0.96	0.77
4	2	9	12	18	6	7	3	0.96	0.87
4	3	7	12	12	7	6	3	0.98	0.90
5	1	6	15	15	6	5	1	0.97	0.77
5	4	9	15	15	9	8	4	0.99	0.93
6	1	7	18	18	7	6	1	0.98	0.77
7	1	8	21	21	8	7	1	0.98	0.77
8	1	9	24	24	9	8	1	0.99	0.76
9	1	10	27	27	10	9	1	0.99	0.76

1. Putting $u = 3$ and 4 respectively gives designs with 7 replicates of 14 treatments on 14 blocks of 7 plots each, $\lambda_1 = 6$, $\lambda_2 = 3$, $E_1 = 0.98$, $E_2 = 0.91$ and 9 replicates of 18 treatments on 18 blocks of 9 plots each, $\lambda_1 = 8$, $\lambda_2 = 4$, $E_1 = 0.99$, $E_2 = 0.94$.

Further, if there is a balanced incomplete block design of n treatments with h plots per block which is not unreduced, or if the unreduced design with these parameters is resolvable, a regular design may be possible without taking all possible blocks. The only design of this type of practicable size appears to be that with $m = 2$, $u = 1$, $n = 7$, $h = 3$, which gives rise to a design with 10 replicates of 14 treatments on 14 blocks with 10 plots in each as follows:

A − G with HIJ, HKL, HMN, IKM, ILN, JKN, JLM;

H − N with ABC, ADE, AFG, BDF, BEG, CDG, CEF.

$\lambda_1 = 8$, $\lambda_2 = 6$, $E_1 = 0.98$, $E_2 = 0.96$

Other designs, of a similar type but not necessarily containing any complete groups, can be obtained as follows. Let each block contain h_j members from the jth group of treatments in all possible ways, there being m_j groups of treatments. If j goes from 1 to s we have $k = \sum_{j=1}^{s} h_j m_j$, and the blocks can be divided into sets such that each set contains $\prod_{j=1}^{s} {}^nC_{h_j}$ blocks, where h_j occurs m_j times. The number of such sets of blocks is thus $m!/\prod_{j=1}^{s} m_j!$, where

$$\sum_{j=1}^{s} m_j = m.$$

The design contains r replicates of mn treatments on $(m!\prod^n C_{h_j})/\prod m_j$ blocks with $\sum h_j m_j$ plots per block, where summations and products run from 1 to s and the term $\prod {}^nC_{h_j}$ contains h_j m_j times. Further,

$$\lambda_1 = \sum_{j=1}^{s} {}^{n-2}C_{h_j-2} \left(\prod_{j' \neq j} {}^nC_{h_{j'}}\right)$$

and

$$\lambda_2 = \sum_{j \neq j'} {}^{n-1}C_{h_j-1}\, {}^{n-1}C_{h_{j'}-1} \left(\prod_{j'' \neq j, j'} {}^nC_{h_{j''}}\right),$$

where h_j occurs m_j times.

Thus, if h_j is a constant, the design is semi-regular; otherwise it is regular. The only case with $r \leqq 10$, $k > 2$ is given by $m = 2$, $n = 3$, $s = 2$, $m_1 = m_2 = 1$, $h_1 = 2$, $h_2 = 1$, giving 9 replicates of 6 treatments on 18 blocks with 3 plots per block, $\lambda_1 = 3$, $\lambda_2 = 4$, $E_1 = 0.78$, $E_2 = 0.81$. The design is:

A	A	A	A	A	A	B	B	B	D	D	D	D	D	D	E	E	E
B	B	B	C	C	C	C	C	C	E	E	E	F	F	F	F	F	F
D	E	F	D	E	F	D	E	F	A	B	C	A	B	C	A	B	C

The groups are ABC and DEF, and the blocks are given by columns.

As before, only a fraction of the total number of blocks may be needed and two designs derived in this fashion have respectively $m = 3$, $n = 3$, $s = 3$, $m_1 = m_2 = m_3 = 1$, $h_1 = 2$, $h_2 = 1$, $h_3 = 0$, giving a half design with 9 replicates of 9 treatments on 27 blocks and 3 plots per block, $\lambda_1 = 3$, $\lambda_2 = 2$, and $m = 4$, $n = 4$, $s = 3$, $m_1 = m_2 = 1$, $m_3 = 2$, $h_1 = 2$, $h_2 = 1$, $h_3 = 0$, giving a 1/6 design with 9 replicates of 16 treatments on 48 blocks with 3 plots per block, $\lambda_1 = 2$, $\lambda_2 = 1$. Designs with these last two sets of parameters are given by Bose, *et al.* [1], but the blocks comprising the designs are different in each case, even though the efficiency factors are unaltered.

4. Designs with groups arranged in sets. In certain designs the groups of treatments may be arranged in sets in such a fashion that treatments from groups within a set concur in one way while treatments from groups in different sets concur in another way. At first sight this would appear to lead to designs with three associate-classes, and in general it does, but in many particular cases the treatments from different groups concur the same number of times whether or not the groups are in the same set In such a case the property of group divisibility ensures that there are only two associate-classes. The simplest of these designs, those with $2n$ plots per block, are derived below.

Consider a design with $2n^2$ treatments in $2n$ groups of n members each, the design having $2n$ plots per block. Divide the groups of treatments into two sets, set 1 containing the first n groups and set 2 the remainder. The design then has blocks of the following kinds:

(i) Blocks containing two complete groups from set 1 or set 2, there being $[n(n - 1)]/2$ from each set and thus $n(n - 1)$ in all. There are thus $(n - 1)$ replicates of each treatment in these blocks.

(ii) Blocks containing one complete group from set 1 and one member from each group in set 2, or conversely. These blocks are sufficient in number that each group from one set occurs once with every treatment from the other and that the groups of a set occur equally frequently. This necessitates n^2 blocks with complete groups from one set and one member from each group of the other, and thus $2n^2$ blocks in all. Further, in order that each treatment shall occur once and once only with all treatments in the same set but different groups, n must be such that there are $(n - 1)$ orthogonal Latin squares of side n. Each treatment is replicated $2n$ times in these blocks.

The complete design thus has $(3n - 1)$ replicates and $n(3n - 1)$ blocks.

In blocks of the first kind, each treatment concurs $(n - 1)$ times with treatments of its own group, once with treatments of the other groups of its own set, and not at all with treatments of the other set. In blocks of the second kind, each treatment concurs n times with treatments of its own group, once with treatments of the other groups in its own set, and twice with treatments of the other set. Thus the design is group divisible with $\lambda_1 = 2n - 1$, $\lambda_2 = 2$, and so, further, is regular.

$n = 2$ and $n = 3$ give the only examples with 10 or fewer replicates, these having respectively 5 replicates of 8 treatments on 10 blocks with 4 plots each, $\lambda_1 = 3$, $E_1 = 0.90$, $E_2 = 0.85$, and 8 replicates of 18 treatments on 24 blocks with 6 plots each, $\lambda_1 = 5$, $E_1 = 0.94$, $E_2 = 0.87$. The design for $n = 3$ is:

A	A	D	J	J	M
B	B	E	K	K	N
C	C	F	L	L	O
D	G	G	M	P	P
E	H	H	N	Q	Q
F	I	I	O	R	R

Blocks of the first kind

A	A	A	D	D	D	G	G	G	J	J	J	M	M	M	P	P	P
B	B	B	E	E	E	H	H	H	K	K	K	N	N	N	Q	Q	Q
C	C	C	F	F	F	I	I	I	L	L	L	O	O	O	R	R	R
J	K	L	J	K	L	J	K	L	A	B	C	A	B	C	A	B	C
M	N	O	N	O	M	O	M	N	D	E	F	E	F	D	F	D	E
P	Q	R	R	P	Q	Q	R	P	G	H	I	I	G	H	H	I	G

Blocks of the second kind

The groups are ABC, DEF, GHI, JKL, MNO, and PQR, and the blocks are given by columns.

If there are $3n$ groups of n members each instead of $2n$ groups, a design with $2n$ plots per block is possible in a similar fashion. Here, however, only the second kind of block described above is used, and so the design is, in a sense, a complete and incomplete group design as understood in the last section. The three possible pairs of sets of groups all have to be considered, thus giving rise to a design with $6n^2$ blocks and $4n$ replicates. If two treatments belong to different groups, whether of the same set or not, they concur twice, i.e. $\lambda_2 = 2$; further, $\lambda_1 = 2n$, and the design is thus regular group divisible. The design with $n = 2$ is the only one with 10 replicates or fewer, and has in fact 8 replicates of 12 treatments on 24 blocks with 4 plots each, $\lambda_1 = 4$, $\lambda_2 = 2$, $E_1 = 0.88$, $E_2 = 0.81$. The design is:

A	A	C	C	E	E	G	G	A	A	C	C	I	I	K	K	E	E	G	G	I	I	K	K
B	B	D	D	F	F	H	H	B	B	D	D	J	J	L	L	F	F	H	H	J	J	L	L
E	F	E	F	A	B	A	B	I	J	I	J	A	B	A	B	I	J	I	J	E	F	E	F
G	H	H	G	C	D	D	C	K	L	L	K	C	D	D	C	K	L	L	K	G	H	H	G

The groups are AB, CD, EF, GH, IJ, and KL, and the blocks are given by columns.

In the same way that, for complete and incomplete groups half designs are possible with $m = 3$, $u = 1$, so also are half designs possible here by means of the same device, i.e., with complete groups from the first set and incomplete from the second, and so on in a cyclic fashion only. $n = 2$ gives a design with 4 replicates of 12 treatments on 12 blocks with 4 plots each, and this design is given by Bose, *et al.* [1]. In general the design has $2n$ replicates of $3n^2$ treatments on $3n^2$ blocks with $2n$ plots each, $\lambda_1 = n$, $\lambda_2 = 1$. For $n = 3$, 4, or 5 respectively, the design thus has 6, 8, or 10 replicates and plots per block, and it has 27, 48, or 75 treatments and blocks; $\lambda_1 = 3$, 4, or 5; $E_1 = 0.92$, 0.94, or 0.95; $E_2 = 0.85$, 0.88, or 0.90.

Designs of this kind are possible with more than $2n$ plots per block, but the numbers of replicates and plots per block very soon give designs which are beyond the bounds of practicality. Even the smallest designs with $3n^2$ treatments and $3n$ plots per block have 12 replicates. However, the smallest design with $2n^2$ treatments and $3n$ plots per block, that with $n = 3$, gives a more practical design. This design, which has 9 replicates of 18 treatments on 18 blocks with 9 plots each, $\lambda_1 = 6$, $\lambda_2 = 3$, $E_1 = 0.96$, $E_2 = 0.91$, arises in each of the following forms:

A	A	A	A	A	A	D	D	D	J	J	J	J	J	J	M	M	M
B	B	B	B	B	B	E	E	E	K	K	K	K	K	K	N	N	N
C	C	C	C	C	C	F	F	F	L	L	L	L	L	L	O	O	O
D	D	D	G	G	G	G	G	G	M	M	M	P	P	P	P	P	P
E	E	E	H	H	H	H	H	H	N	N	N	Q	Q	Q	Q	Q	Q
F	F	F	I	I	I	I	I	I	O	O	O	R	R	R	R	R	R
J	K	L	J	K	L	J	K	L	A	B	C	A	B	C	A	B	C
M	N	O	N	O	M	O	M	N	D	E	F	E	F	D	F	D	E
P	Q	R	R	P	Q	Q	R	P	G	H	I	I	G	H	H	I	G

or

G	G	G	D	D	D	A	A	A	P	P	P	M	M	M	J	J	J
H	H	H	E	E	E	B	B	B	Q	Q	Q	N	N	N	K	K	K
I	I	I	F	F	F	C	C	C	R	R	R	O	O	O	L	L	L
K	J	J	K	J	J	K	J	J	B	A	A	B	A	A	B	A	A
L	L	K	L	L	K	L	L	K	C	C	B	C	C	B	C	C	B
N	M	M	M	M	N	M	N	M	E	D	D	D	D	E	D	E	D
O	O	N	O	N	O	N	O	O	F	F	E	F	E	F	E	F	F
Q	P	P	P	Q	P	P	P	Q	H	G	G	G	H	G	G	G	H
R	R	Q	Q	R	R	R	Q	R	I	I	H	H	I	I	I	H	I

The groups are ABC, DEF, GHI, JKL, MNO, and PQR, and the blocks are given by columns.

These two designs illustrate the principles on which the general designs of this type are derived, *viz.*, either two complete groups from one set and one member from each group of the other set or one complete group from one set and two members from each group of the other set.

5. Designs derivable by addition. Bose, *et al.* [3] describe designs derivable by addition of further blocks to a balanced incomplete block design with

TABLE III

Regular group divisible designs derivable by addition

v	k	m	n	r_1	b_1	Type	Rep.	r_2	b_2	Type	Rep.	r	b	λ_1	λ_2	E_1	E_2
6	3	2	3	5	10	T	1	1	2	S	1	6	12	3	2	0.83	0.77
6	3	2	3	5	10	T	1	1	2	S	2	7	14	4	2	0.86	0.73
6	3	2	3	5	10	T	1	1	2	S	3	8	16	5	2	0.88	0.70
6	3	2	3	5	10	T	1	1	2	S	4	9	18	6	2	0.89	0.67
6	3	3	2	5	10	T	1	2	4	SR	1	7	14	2	3	0.76	0.81
6	3	3	2	5	10	T	1	2	4	SR	2	9	18	2	4	0.74	0.81
6	3	3	2	6	12	R	1	2	4	SR	1	8	16	4	3	0.83	0.79
8	4	2	4	7	14	T	1	1	2	S	1	8	16	4	3	0.88	0.84
8	4	2	4	7	14	T	1	1	2	S	2	9	18	5	3	0.89	0.82
8	4	2	4	7	14	T	1	1	2	S	3	10	20	6	3	0.90	0.80
8	4	4	2	7	14	T	1	3	6	S	1	10	20	6	4	0.90	0.85
8	4	4	2	5	10	R	1	3	6	S	1	8	16	6	3	0.94	0.83
8	4	4	2	6	12	SR	1	3	6	S	1	9	18	3	4	0.83	0.86
9	6	3	3	8	12	T	1	2	3	S	1	10	15	7	6	0.95	0.93
10	4	2	5	6	15	T	1	4	10	R	1	10	25	5	2	0.88	0.76
10	4	5	2	6	15	T	1	4	10	S	1	10	25	6	3	0.90	0.82
10	5	5	2	9	18	T	1	1	2	S	1	10	20	5	4	0.90	0.88
12	3	3	4	4	16	SR	1	3	12	R	2	10	40	4	1	0.80	0.64

r_1, b_1 and r_2, b_2 are the numbers of replicates and blocks in the two designs which are added to make the final design given here, the number of complete replications of these initial designs being given in the appropriate column. The type of initial design is also given, T for totally balanced incomplete block designs and S, SR, and R respectively for singular, semi-regular, and regular group divisible incomplete block designs.

$v = mk$, m blocks giving a complete replicate, and designs derivable by taking together the blocks of two group divisible designs with the same v and k. However, it is possible to add blocks to a balanced design even with $v \neq mk$, or $v = mk$ when m of the balanced design blocks do not give a complete replicate, or to add a group divisible design to a balanced design, all of which amount to the same thing. New designs ($r \leqq 10$) derived by these and other addition methods, are given in Table III. An example of this type with $v = mk$ is given by the following design with 7 replicates of 6 treatments in 2 groups of 3 on 14 blocks with 3 plots each, $\lambda_1 = 4$, $\lambda_2 = 2$. The groups are ABC, and DEF, and the blocks are given by columns.

A	A	A	A	A	B	B	B	C	C		A	D	A	D
B	B	C	D	E	C	D	E	D	D		B	E	B	E
C	D	F	E	F	E	F	F	E	F		C	F	C	F

Totally balanced incomplete block | Partially balanced (singular, disconnected)

An example with $v \neq mk$ is the design with 10 replicates of 10 treatments in 2 groups of 5 on 25 blocks with 4 plots each, $\lambda_1 = 5$, $\lambda_2 = 2$. The design is:

A	B	A	A	A	B	C	D	D	A	B	C	A	B	C	A	A	A	A	B	F	F	F	F	G
B	C	C	B	D	E	D	E	F	E	F	F	F	D	E	B	B	B	C	C	G	G	G	H	H
C	E	G	I	E	G	I	H	G	G	H	G	H	G	H	C	C	D	D	D	H	H	I	I	I
D	F	H	J	F	I	J	I	J	J	J	I	I	H	J	D	E	E	E	E	I	J	J	J	J

Totally balanced incomplete block | Partially balanced (disconnected)

The groups are ABCDE and FGHIJ, and the columns represent the blocks. This design illustrates the point made by Bose, *et al.* [3] that it does not matter if a design is disconnected if it is added to another design to form a new one.

REFERENCES

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