TESTING THE HYPOTHESIS THAT TWO POPULATIONS DIFFER ONLY IN LOCATION[1]

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0. Summary. Let $X_1, X_2, \cdots, X_n$ be n independent identically distributed random variables with cumulative distribution function $F(x - \xi)$. Let

$$\hat{\xi}(X_1, X_2, \cdots, X_n)$$

be an estimate of ξ such that $\sqrt{n}(\hat{\xi} - \xi)$ is bounded in probability. The first part of this paper (Secs. 2 through 4) is concerned with the asymptotic behavior of U-statistics modified by centering the observations at $\hat{\xi}$. A set of necessary and sufficient conditions are given under which the modified U-statistics have the same asymptotic normal distribution as the original U-statistics. These results are extended to generalized U-statistics and to functions of several generalized U-statistics. The second part gives an application of the asymptotic theory developed earlier to the problem of testing the hypothesis that two populations differ only in location.

1. Introduction. Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples of observations from populations with cumulative distribution functions $F(x - \xi)$ and $G(x - \eta) = F[(x - \eta)/\delta]$ respectively, ξ and η being the unknown location parameters and δ a scale parameter. No knowledge is assumed concerning the distribution functions F and G except that they are absolutely continuous. The problem considered in this paper is that of testing the hypothesis that the two populations differ only in location against the alternative that the Y's are more spread out than the X's and vice versa, or in symbols

$$H: \delta = 1, \qquad A: \delta \neq 1. \tag{1.1}$$

From intuitive considerations and the work of Fraser [1], it seems likely that there do not exist similar tests for testing the hypothesis H, which are very satisfactory. The following simplified problem was therefore considered by the author [2]. Let the location parameters ξ and η be known, say $\xi = \eta = 0$, so that the distribution functions of X and Y differ only in the scale parameter. Then the problem considered is that of testing the hypothesis

$$H': \delta = 1, \quad \text{i.e., } F = G,$$
$$A: \delta \neq 1, \quad \text{i.e., } F \neq G.$$

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Several nonparametric tests have been suggested for testing the hypothesis H', particularly by Mood [3]. The author [2] has considered some of these tests and discussed their asymptotic properties from the point of view of power considerations. These tests are based on what are known as generalised U-statistics and are reasonably efficient. But our main interest lies not in testing the hypothesis H' but H. However, once we have a class $\{W_N\}$ of tests for testing the hypothesis H', a class of tests $\{\hat{W}_N\}$ for testing the hypothesis H suggests itself. This class of tests may be obtained as follows. We obtain suitable estimates of the parameters ξ and η and then apply any of the tests of the class W_N to the deviations of the X's and the Y's from the respective estimates.

If the X's and the Y's come from normal populations, the usual test of significance for testing the hypothesis H is the variance ratio test based on the statistic

$$F = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}{\sum_{i=1}^{m} (X_i - \bar{X})^2} \cdot \frac{m-1}{n-1}, \tag{1.2}$$

which is also the most commonly used statistical test for comparing sample variances. Usually, however, since little is known about the populations from which the samples are drawn, this test is used as if the assumption of normality could be ignored. It appears, however, that this is not the case.

The sensitivity to non-normality of the F-test was first pointed out by E. S. Pearson [4] whose findings were later confirmed by Geary [5] and Gayen [6]. They showed that the F-test is particularly sensitive to changes in Kurtosis from the normal theory value of zero. It is easy to see that the F-statistic when suitably normalised is asymptotically distribution free. More recently, Box and Andersen ([7] and [8]) have studied this problem in great detail and have shown on the basis of extensive sampling experiments that the F-statistic so normalized is insensitive to departures from normality.

Since the tests considered in [2] are nonparametric and reasonable for normal alternatives, it appears that they might be more efficient for non-normal alternatives and also more stable for small samples. We propose, therefore, to investigate whether such tests, after modification by the introduction of estimates of parameters are asymptotically distribution free.

This is achieved by considering the asymptotic theory of generalised U-statistics modified by the introduction of estimates of parameters, which is given in Secs. 3 and 4. In Sec. 5, it is shown that the nonparametric test proposed in [2], after modification, is asymptotically distribution free for populations with bounded and symmetric probability densities. It turns out however that even under such restrictive conditions, the nonparametric test proposed by Mood, after modification is not asymptotically distribution free. Finally, the last section considers the small sample behavior of the proposed test for some particular alternatives.

2. Some definitions and known results.

DEFINITION 2.1. Let X_{ij}, $j = 1, 2, \cdots, n_i$ for a fixed i be independent random variables identically distributed with c.d.f. $F_i(x)$ and density function $f_i(x)$. Let i run from 1 to k and $s_1 \leqq n_1$, $s_2 \leqq n_2$, $\cdots$, $s_k \leqq n_k$. Further, let

$$\varphi(u_1, \cdots, u_{s_1}; v_1, \cdots, v_{s_2}; \cdots; w_1, w_2, \cdots, w_{s_k})$$

be a function symmetric in each set of its arguments. Then the statistic

$$U_N = \binom{n_1}{s_1}^{-1} \binom{n_2}{s_2}^{-1} \cdots \binom{n_k}{s_k}^{-1}$$
$$\cdot \sum \varphi(X_{1,\alpha_1}, \cdots, X_{1,\alpha_{s_1}}; X_{2,\beta_1}, \cdots, X_{2,\beta_{s_2}}; \cdots; X_{k,\delta_1}, \cdots X_{k,\delta_{s_k}}),$$

where the summation runs over all subscripts α, β, δ such that

$$1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_{s_1} \leqq n_1$$
$$1 \leqq \beta_1 < \beta_2 < \cdots < \beta_{s_2} \leqq n_2$$
$$\cdots$$
$$1 \leqq \delta_1 < \delta_2 < \cdots < \delta_{s_k} \leqq n_k$$

is called a generaliesd U-statistic.

Let $\rho_1, \rho_2, \cdots, \rho_k$ be k fixed numbers such that $n_i = N\rho_i$ and $\sum_{i=1}^k \rho_i = 1$. Then Lehmann [9] has shown that $\sqrt{N}[U_N - EU_N]$ is asymptotically normally distributed with mean zero and asymptotic variance σ^2 given by

$$\sigma^2 = \frac{s_1^2}{\rho_1} \zeta_{100\cdots0} + \frac{s_2^2}{\rho_2} \zeta_{010\cdots0} + \cdots + \frac{s_k^2}{\rho_k} \zeta_{00\cdots01},$$

where

$$\zeta_{00\cdots1\cdots0} = E\varphi_1\varphi_2 - [E\varphi_1]^2,$$

1 occurs at the ith place in $\zeta_{00\cdots1\cdots0}$,

$$\varphi_1 = \varphi(X_{11}, \cdots, X_{1s_1}; \cdots; X_{i1}, X_{i2}, \cdots, X_{is_i}; \cdots)$$

and φ_2 is obtained from φ_1 by replacing all the X_{jk} by X'_{jk} excepting X_{i1}, the primes denoting a new set of independent random variables. This result is a generalisation of the U-statistics considered by Hoeffding [10].

For the sake of simplicity, we shall restrict ourselves to the two sample problem only. The extension to k samples is straight forward.

DEFINITION 2.2. As before, let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples drawn from populations with c.d.f.'s $F(x - \xi)$ and $G(x - \eta)$ respectively. Further, let

$\hat{\xi}(X_1, \cdots, X_m)$ and $\hat{\eta}(Y_1, \cdots, Y_n)$ be estimates of ξ

and η, the two location parameters. Then the generalised U-statistic with the observations centered at the respective location parameters and the modified generalised U-statistic for the two sample problem are respectively,

$$U_N = \binom{m}{s_1}^{-1} \binom{n}{s_2}^{-1} \sum_{\alpha,\beta} \varphi(X_{\alpha_1} - \xi, \cdots, X_{\alpha_{s_1}} - \xi; Y_{\beta_1} - \eta, \cdots, Y_{\beta_{s_2}} - \eta),$$

$$\hat{U}_N = \binom{m}{s_1}^{-1} \binom{n}{s_2}^{-1} \sum_{\alpha,\beta} \varphi(X_{\alpha_1} - \hat{\xi}, \cdots, X_{\alpha_{s_1}} - \hat{\xi}; Y_{\beta_1} - \hat{\eta}, \cdots, Y_{\beta_{s_2}} - \hat{\eta}),$$

where $\varphi(u_1, \cdots, u_{s_1}; v_1, \cdots, v_{s_2})$ is a function symmetric in u and in v and the summation runs over all subscripts α, β such that

$$1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_{s_1} \leqq m,$$

$$1 \leqq \beta_1 < \beta_2 < \cdots < \beta_{s_2} \leqq n.$$

DEFINITION 2.3. Let $\hat{W}_N$ be a test based on the statistic $\hat{U}_N$. If the asymptotic distribution of $\hat{U}_N$ is independent of the original populations from which the X's and the Y's are drawn under the null hypothesis, the test $\hat{W}_N$ will be said to be asymptotically distribution free.

Finally we define a quantity L_N required in the study of the asymptotic behavior of modified generalised U statistics.

DEFINITION 2.4.

$$L_N = \binom{m}{s_1}^{-1} \binom{n}{s_2}^{-1}$$

$$\sum\nolimits_{\alpha,\beta} [\varphi(X_{\alpha_1} - \hat{\xi}, \cdots, X_{\alpha_{s_1}} - \hat{\xi}; Y_{\beta_1} - \hat{\eta}, \cdots, Y_{\beta_{s_2}} - \hat{\eta}) - A(\hat{\xi} - \xi, \hat{\eta} - \eta)],$$

where

$$A(t_1 - \xi, t_2 - \eta) = E\varphi(X_1 - t_1, \cdots, X_{s_1} - t_1; Y_1 - t_2, \cdots, Y_{s_2} - t_2),$$

expectation being taken with respect to all the X's and the Y's.

3. The limiting distribution of L_N. In this section, we will prove theorems, giving the conditions under which L_N and U_N have the same asymptotic normal distribution. We will start with one sample problem and then extend the result to two samples. In what follows, we write $\mathcal{L}(X_n) \rightarrow \mathcal{L}(X)$ (read: the distribution law of X_n converges to the distribution law of X), or $\lim_{n\to\infty} \mathcal{L}(X_n) = \mathcal{L}(X)$ if $F_n(a) \rightarrow F(a)$ at every point a of continuity of F where F_n and F are the c.d.f.'s of X_n and X, respectively.

THEOREM 3.1. *Let $X_1, X_2, \cdots, X_n$ be n independent identically distributed* *random variables with c.d.f.* $F(x - \xi)$. *Let* $\varphi(u_1, u_2, \cdots, u_s)$ *with* $s \leqq n$ *be a real valued symmetric function of its arguments such that if*

$$W(x_1, x_2, \cdots, x_s, t) = \varphi(x_1 - t, \cdots, x_s - t) - A(t - \xi), \tag{3.0}$$

where $A(t - \xi) = E\varphi(X_1 - t, \cdots, X_s - t)$, *the following conditions are satisfied.*

$$(B_1) \quad \begin{aligned} &|W(x_1, x_2, \cdots, x_s, t)| \leqq M_1, \text{ and } E|W(X_1, \cdots, X_s; t + h) \\ &\quad - W(X_1, \cdots, X_s; t)| \leqq M_2 h, \; M_1 \text{ and } M_2 \text{ being fixed constants} \end{aligned}$$

There exists a sequence $\{t_j\}$ *such that for each set of x's*

$$(B_2) \quad \begin{aligned} &\sup_{0 \leqq t_j \leqq k} | W(x_1, \cdots, x_s, t_j) - W(x_1, \cdots, x_s, 0) | \\ &\qquad = \sup_{0 \leqq t \leqq k} | W(x_1, \cdots, x_s, t) - W(x_1, \cdots, x_s, 0) |. \end{aligned}$$

Further, let $\hat{\xi}(X_1, X_2, \cdots, X_n)$ *be an estimate of* ξ *such that given* $\Sigma_1 > 0$, *there exists a number b such that for n sufficiently large*

$$P\left\{| \hat{\xi} - \xi | \geqq \frac{b}{\sqrt{n}}\right\} \leqq \Sigma_1. \tag{3.1}$$

Define

$$U_n = \binom{n}{s}^{-1} \sum \varphi(X_{\alpha_1} - \xi, \cdots, X_{\alpha_s} - \xi), \tag{3.2}$$

the summation being taken over all subscripts α *such that*

$$1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leqq n$$

and

$$L_n = \binom{n}{s}^{-1} \sum [\varphi(X_{\alpha_1} - \hat{\xi}, \cdots, X_{\alpha_s} - \hat{\xi}) - A(\hat{\xi} - \xi)].$$

Then

$$(3.3) \quad \begin{aligned} \lim_{n \to \infty} \mathcal{L}(\sqrt{n} L_n) &= \lim_{n \to \infty} \mathcal{L}(\sqrt{n} [U_n - EU_n]) \\ &= N(0, s^2 \zeta_1), \end{aligned}$$

where

$$(3.4) \quad \begin{aligned} \zeta_1 &= E\varphi_1^2(X_1 - \xi) - E^2\varphi(X_1 - \xi, \cdots, X_s - \xi), \\ \varphi_1(x_1 - \xi) &= E\varphi(x_1 - \xi, X_2 - \xi, \cdots, X_s - \xi). \end{aligned}$$

PROOF. For the sake of simplicity we may, without loss of generality, assume $\xi = 0$. However, before we proceed further, we shall first prove the following lemma, which we shall use in the proof of the theorem.

LEMMA 3.2. *Let*

$$H_{r,n}(x_1, x_2, \cdots, x_s, t) = \sup_{\frac{r\delta}{\sqrt{n}} \leqq z \leqq t} \left| W(x_1, \cdots, x_s, z) - W\left(x_1, \cdots, x_s, \frac{r\delta}{\sqrt{n}}\right) \right| \tag{3.5}$$

and

$$S_{r,n}(t) = \sqrt{n} \binom{n}{s}^{-1} \sum_{\alpha} \left[W\left(x_{\alpha_1}, \cdots, x_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(x_{\alpha_1}, \cdots, x_{\alpha_s}, \frac{\delta r}{\sqrt{n}}\right)\right]. \tag{3.6}$$

Then, if $r\delta \leqq t \leqq (r+1)\delta$ *and* n *is sufficiently large,*

$$\text{(i)}\quad EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \leqq \frac{M_2 \delta}{\sqrt{n}}; \tag{3.7}$$

$$\begin{aligned} \text{(ii)}\quad E\Bigg\{\Bigg[& H_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\ & - EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\Bigg] \\ & \cdot \Bigg[H_{r,n}\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\ & - EH_{r,n}\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\Bigg]\Bigg\} \to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.8}$$

where

$$1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leqq n,$$

$$1 \leqq \beta_1 < \beta_2 < \cdots < \beta_s \leqq n;$$

$$\text{(iii)}\quad E \mid S_{r,n}(t) \mid^2 \leqq \frac{d(t - r\delta)}{\sqrt{n}}, \tag{3.9}$$

where d *is a fixed constant and higher powers of* $1/\sqrt{n}$ *are neglected.*

PROOF. (i) and (ii) are easily obtained as consequences of conditions (B_1)

and (B_2) of Theorem 3.1. To prove (iii) we have

$$E \mid S_{r,n}(t) \mid^2$$

$$= n \binom{n}{s}^{-2} \sum_{\alpha,\beta} E\left\{\left[W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right.$$

$$\left. \cdot \left[W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right\}.$$

Consider a typical term; with c integers common to the two terms. We then have

$$E\left\{\left[W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right.$$

$$\left. \cdot \left[W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right\}$$

$$\leqq E\left|\left[W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right.$$

$$\left. \cdot \left[W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{r\delta}{\sqrt{n}}\right)\right]\right|$$

$$\leqq 2M_1 E\left| W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{r\delta}{\sqrt{n}}\right)\right|$$

$$\leqq 2M_1 M_2 \frac{(t - r\delta)}{\sqrt{n}}.$$

The total contribution of such terms to

$$E|S_{r,n}(t)|^2 \leqq n \binom{n}{s}^{-2} \binom{n}{2s - c} \cdot A(t - r\delta) / \sqrt{n},$$

A being some fixed constant. It follows that

$$E \mid S_{r,n}(t) \mid^2 \sim \frac{1}{n^{c-\frac{1}{2}}} (t - r\delta).$$

When $c = 0$, the expectation of the product is zero. Retaining only powers of $1/\sqrt{n}$, the result now follows. Q.E.D.

PROOF OF THEOREM 3.1. Let

$$S_n^{(t)} = \sqrt{n} \binom{n}{s}^{-1} \sum_{\alpha} \left[W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) - W\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, 0\right)\right].$$

Then it is easily seen that

$$S_n(t) = S_{r,n}(t) + S_{0,n}(r\delta).$$

Let $\epsilon > 0$, $\delta = \epsilon/2M_2$, and t be such that $r\delta \leqq t \leqq (r+1)\delta$. Then it is seen that

$$
\begin{aligned}
|S_{r,n}(t)| &\leqq \sqrt{n}\binom{n}{s}^{-1} \sum_{\alpha} H_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{t}{\sqrt{n}}\right) \\
&\leqq \sqrt{n}\binom{n}{s}^{-1} \sum_{\alpha} H_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\
&\leqq \sqrt{n}\binom{n}{s}^{-1} \sum_{\alpha} \left[H_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\right. \\
&\quad \left. - EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\right] \\
&\quad + \sqrt{n}EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\
&\leqq \sqrt{n}\binom{n}{s}^{-1} \sum_{\alpha} \left| H_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\right. \\
&\quad \left. - EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\right| + M_2\delta \\
&= D_1 + M_2\delta,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 = \sqrt{n}\binom{n}{s}^{-1} \sum_{\alpha} \Bigg[H_{r,n}&\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\
&- EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\Bigg].
\end{aligned}
$$

Now

$$
\begin{aligned}
ED_1^2 = n\binom{n}{s}^{-2} \sum_{\alpha,\beta} E\Bigg\{\Bigg[H_{r,n}&\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) \\
&- EH_{r,n}\left(X_{\alpha_1}, \cdots, X_{\alpha_s}, \frac{(r+1)\delta}{\sqrt{n}}\right)\Bigg] \\
\cdot\left[H_{r,n}\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{(r+1)\delta}{\sqrt{n}}\right) - EH_{r,n}\left(X_{\beta_1}, \cdots, X_{\beta_s}, \frac{\delta(r+1)}{\sqrt{n}}\right)\right]&\Bigg\},
\end{aligned}
$$

the summation having the same meaning as before. Considering again a typical term with c integers common to the two terms, we find that the total contribution of such terms to ED_1^2 is

$$\leqq n\binom{n}{s}^{-2}\binom{n}{2s-c}E\left|\left[H_{r,n}\left(X_{\alpha_1},\cdots,X_{\alpha_s},\frac{(r+1)\delta}{\sqrt{n}}\right) - EH_{r,n}\left(X_{\alpha_1},\cdots,X_{\alpha_s},\frac{(r+1)\delta}{\sqrt{n}}\right)\right]\right.$$
$$\left.\cdot\left[H_{r,n}\left(X_{\beta_1},\cdots,X_{\beta_s},\frac{(r+1)\delta}{\sqrt{n}}\right) - EH_{r,n}\left(X_{\beta_1},\cdots,X_{\beta_s},\frac{(r+1)\delta}{\sqrt{n}}\right)\right]\right|$$
$$\sim\frac{A}{n^{c-1}}E\left|\left[H_{r,n}\left(X_{\alpha_1},\cdots,X_{\alpha_s},\frac{(r+1)\delta}{\sqrt{n}}\right) - EH_{r,n}\left(X_{\alpha_1},\cdots,X_{\alpha_s},\frac{(r+1)\delta}{\sqrt{n}}\right)\right]\right.$$
$$\left.\cdot\left[H_{r,n}\left(X_{\beta_1},\cdots,X_{\beta_s},\frac{(r+1)\delta}{\sqrt{n}}\right) - EH_{r,n}\left(X_{\beta_1},\cdots,X_{\beta_s},\frac{(r+1)\delta}{\sqrt{n}}\right)\right]\right|,$$

which tends to zero by Lemma 3.2 for $c \geqq 1$. When $c = 0$, the expectation of the product is identically equal to zero.

Hence $ED_1^2 \to 0$ as $n \to \infty$. It follows that

$$P\{\mid \sup_{r\delta\leqq t\leqq(r+1)\delta} S_{r,n}(t)\mid > \epsilon\} \to 0.$$

for every r

Also $E|S_{0,n}(r\delta)|^2 \leqq 2M_1M_2r\delta\,/\,\sqrt{n} \to 0$ as $n \to \infty$, therefore,

$$S_{0,n}(r\delta) \xrightarrow{P} 0.$$

It follows that $\sup_{t\epsilon C}|S_n(t)| \to 0$, C being some bounded set. Hence,

$$S_n(\sqrt{n}\,\xi) \xrightarrow{P} 0,$$

that is,

$$\sqrt{n}\,L_n - \sqrt{n}\,[U_n - EU_n] \xrightarrow{P} 0,$$

therefore,

$$\lim_{n\to\infty}\mathcal{L}(\sqrt{n}\,L_n) = \lim_{n\to\infty}\mathcal{L}(\sqrt{n}\,[U_n - EU_n]).$$

But by Hoeffding's Theorem 7.1, page 305 of [10], U_n is asymptotically normally distributed, whence the required result follows. Q.E.D.

We complete this section by stating without proof the generalization of the above result to the two sample problem. The proof goes more or less along the same lines as that of Theorem 3.1.

THEOREM 3.3. *Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples drawn from populations with c.d.f.'s $F(x - \xi)$ and $G(x - \eta)$ respectively. Further, let $\varphi(u_1, \cdots, u_{s_1}; v_1, \cdots, v_{s_2})$ with $s_1 \leqq m$ and $s_2 \leqq n$ be a real-valued* *function symmetric in u and in v separately such that if*

$$
\begin{aligned}
&W(x_1, x_2, \cdots, x_{s_1}, y_1, \cdots, y_{s_2}, t_1, t_2) \\
&= \varphi(x_1 - t_1, \cdots, x_{s_1} - t_1; y_1 - t_2, \cdots, y_{s_2} - t_2) - A(t_1 - \xi, t_2 - \eta),
\end{aligned} \tag{3.10}
$$

the following conditions are satisfied:

$$
\begin{aligned}
&|W(x_1, x_2, \cdots, x_{s_1}, y_1, y_2, \cdots, y_{s_2}, t_1, t_2| \leqq M_{11} \\
&E|W(X_1, \cdots, X_{s_1}, Y_1, \cdots, Y_{s_2}, t_1 + h, t_2) \\
&\qquad - W(X_1, \cdots, X_{s_1}, Y_1, \cdots Y_{s_2}, t_1, t_2)| \leqq M_{21}h \\
&E|W(X_1, \cdots X_{s_1}, Y_1, \cdots, Y_{s_2}, t_1, t_2 + k) \\
&\qquad - W(X_1, \cdots, X_{s_1}, Y_1, \cdots Y_{s_2}, t_1, t_2)| \leqq M_{22}k,
\end{aligned} \tag{B$_3$}
$$

where M_{11}, M_{21} and M_{22} are certain fixed constants.

There exist sequences $\{t_j\}$ and $\{l_j\}$ such that for every set of x's *and y*'s,

$$
\begin{aligned}
&\sup_{\substack{0 \leqq t_j \leqq k_1 \\ 0 \leqq l_j \leqq k_2}} |W(x_1, \cdots, x_{s_1}; y_1, \cdots, y_{s_2}, t_j, l_j) \\
&\qquad - W(x_1, x_{s_1}; y_1, \cdots, y_{s_2}, 0, 0)| \\
&= \sup_{\substack{0 \leqq t \leqq k_1 \\ 0 \leqq l \leqq k_2}} |W(x_1, \cdots, x_{s_1}; y_1, \cdots, y_{s_2}, t, l) \\
&\qquad - W(x_1, \cdots, x_{s_1}; y_1, \cdots, y_{s_2}, 0, 0)|.
\end{aligned} \tag{B$_4$}
$$

Further, let $\hat{\xi}(X_1, \cdots X_m)$ and $\hat{\eta}(Y_1, \cdots, Y_n)$ be estimates of ξ and η respectively such that given $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exist numbers b_1 and b_2 such that for m and n sufficiently large

$$P\left\{ |\hat{\xi} - \xi| \geqq \frac{b_1}{\sqrt{n}} \right\} \leqq \epsilon_1, \tag{3.11}$$

$$P\left\{ |\hat{\eta} - \eta| \geqq \frac{b_2}{\sqrt{n}} \right\} \leqq \epsilon_2. \tag{3.12}$$

Define

$$U_N = \binom{m}{s_1}^{-1} \binom{n}{s_2}^{-1} \sum_{\alpha, \beta} \varphi(X_{\alpha_1} - \xi, \cdots, X_{\alpha_{s_1}} - \xi; Y_{\beta_1} - \eta, \cdots, Y_{\beta_{s_2}} - \eta), \tag{3.13}$$

the summation being taken over all subscripts α, β such that

$$
\begin{aligned}
&1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_{s_1} \leqq m, \\
&1 \leqq \beta_1 < \beta_2 < \cdots < \beta_{s_2} \leqq n.
\end{aligned} \tag{3.14}
$$

Then if $m = N\rho$ *and* $n = N(1 - \rho)$,

$$\lim_{N\to\infty} \mathfrak{L}(\sqrt{N}\, L_N) = \lim_{N\to\infty} \mathfrak{L}(\sqrt{N}\,(U_N - EU_N)) = N(0, \sigma^2), \tag{3.15}$$

where σ^2 *is the asymptotic variance of* U_N *and is given by*

$$\sigma^2 = \frac{s_1^2}{\rho}\zeta_{10} + \frac{s_2^2}{1-\rho}\zeta_{01}, \tag{3.16}$$

where ζ_{10} *and* ζ_{01} *have the same meaning as in* (2.1).

4. The asymptotic distribution of modified generalised U-statistics. We are now in a position to consider the statistic $\hat{U}_N$ and obtain conditions under which it has the same asymptotic normal distribution as the statistic U_N. This result is contained in Theorem 4.1.

THEOREM 4.1. *If in addition to the conditions of Theorem* 3.1,

(i) $\sqrt{n}(\hat{\xi} - \xi)$ *has a limiting distribution*

and

(ii) $A(t) = E[\varphi(X_1 - t, \cdots, X_s - t) \mid \xi = 0]$ *has a derivative continuous in the neighbourhood of the origin, then*

(a) *If* $A'(0) = 0$, *where* $A'(t) = \frac{d}{dt}A(t)$,

$$\lim_{n\to\infty} \mathfrak{L}(\sqrt{n}\,[\hat{U}_n - EU_n]) = \lim_{n\to\infty} \mathfrak{L}(\sqrt{n}\,[U_n - EU_n]) = N(0, s^2\zeta_1).$$

(b) *If* $A'(0) \neq 0$, $\hat{\xi}$ *is asymptotically normally distributed and the joint distribution of* $\hat{\xi}$ *and* U_n *is asymptotically normal, then* $\sqrt{n}(\hat{U}_n - EU_n)$ *is asymptotically normally distributed.*

PROOF. We have

$$\sqrt{n}[\hat{U}_n - EU_n] = \sqrt{n}[\hat{U}_n - A(\hat{\xi} - \xi)] + \sqrt{n}[A(\hat{\xi} - \xi) - EU_n].$$

But $A(\hat{\xi} - \xi) = A(0) + (\hat{\xi} - \xi)A'(h)$ where $h = \Delta(\hat{\xi} - \xi)$, $|\Delta| < 1$. Therefore

$$\sqrt{n}[\hat{U}_n - EU_n] = \sqrt{n}[\hat{U}_n - A(\hat{\xi} - \xi)] + \sqrt{n}(\hat{\xi} - \xi) \cdot A'(h).$$

Since $\sqrt{n}(\hat{\xi} - \xi)$ has a limiting distribution and $A'(0) = 0$, it follows from the continuity considerations and Slutsky's theorem that $\sqrt{n}[\hat{U}_n - EU_n]$ and $\sqrt{n}[\hat{U}_n - A(\hat{\xi} - \xi)]$ have the same asymptotic distribution. But by Theorem 3.1, $\sqrt{n}[\hat{U}_n - A(\hat{\xi} - \xi)]$ and $\sqrt{n}[U_n - EU_n]$ have the same asymptotic normal distribution. It follows that $\sqrt{n}[\hat{U}_n - EU_n]$ and $\sqrt{n}[U_n - EU_n]$ have the same asymptotic normal distribution. This proves (a).

To prove (b), it is sufficient to remark that because of Theorem 3.1 and Slutsky's Theorem, the joint distribution of $\sqrt{n}(\hat{\xi} - \xi)$ and $\sqrt{n}[\hat{U}_n - A(\hat{\xi} - \xi)]$ is asymptotically normal. Q.E.D.

In the preceding theorem we make the following observations.
(1) If $A'(0) = 0$, then $\sigma^2(\hat{U}_n) = \sigma^2(U_n)$.
(2) If $A'(0) \neq 0$, then $\sigma^2(\hat{U}_n) = \sigma^2(U_n)$, if and only if

$$A'(0) = \frac{-2\sigma(U_n, \hat{\xi})}{\sigma^2(\hat{\xi})},$$

where $\sigma(U_n, \hat{\xi})$ is the asymptotic covariance between U_n and $\hat{\xi}$ and $\sigma^2(\hat{\xi})$ is the asymptotic variance of $\hat{\xi}$.

For the sake of simplicity we will now consider the special case when $s = 1$, $\hat{\xi}$ is the sample median and $f(x)$ is symmetric about the median which may be taken to be the origin.

Now

$$\begin{aligned} A(t) &= E\varphi(X - t) \\ &= \int \varphi(x - t) f(x)\, dx \\ &= \int \varphi(y) f(y + t)\, dy. \end{aligned}$$

If there exists an integrable function $g(y)$ such that

$$\left| \frac{f(y + t) - f(y + t_0)}{t - t_0} \right| \leqq g(y) \tag{4.1}$$

and the derivative of f exists almost everywhere except for a set of measure zero, then

$$A'(0) = \int \varphi(y) f'(y)\, dy. \tag{4.2}$$

Also it has been shown in [11] that the joint distribution of U_n and ξ is asymptotically normal and that

$$\sigma^2(\xi) = \frac{1}{4nf^2(0)} \tag{4.3}$$

and

$$\sigma(U_n, \xi) = \frac{1}{2nf(0)} \int_0^\infty [\varphi(x) - \varphi(-x)] f(x)\, dx. \tag{4.4}$$

Hence $\sigma^2(\hat{U}_n) = \sigma^2(U_n)$ if and only if

$$\int_0^\infty \left[4f(0) + \frac{f'(x)}{f(x)} \right] [\varphi(x) - \varphi(-x)] f(x)\, dx \underset{f}{\equiv} 0. \tag{4.5}$$

We will now show that the condition (4.5) implies that $\varphi(x) - \varphi(-x) = 0$ almost everywhere. To show this, it is enough to consider the subfamily of probability densities given by

$$f(x, \theta) = \frac{1}{2\theta} e^{-|x|/\theta}. \tag{4.6}$$

We observe that the derivative of f exists everywhere except at the origin. Also, we have

$$\left| \frac{e^{-|x+h|/\theta} - e^{-|x|/\theta}}{h/\theta} \right| \leqq c e^{-|x|/\theta}, \tag{4.7}$$

for h sufficiently small, c being a fixed constant. Condition (4.1) is thus satisfied for the family of distributions (4.6). On substitution, condition (4.5) becomes

$$\int_0^\infty e^{-x/\theta}[\varphi(x) - \varphi(-x)]\, dx \underset{\theta}{\equiv} 0, \tag{4.8}$$

whence it follows from the unicity of the unilateral Laplace transform that $\varphi(x) - \varphi(-x) = 0$ almost everywhere, in which case $A'(0) = 0$ and condition (2) reduces to condition (1).

It is now clear that $A'(0) = 0$ is a necessary and sufficient condition that $\hat{U}_n$ and U_n have the same asymptotic normal distribution.

We will now extend the results of Theorem 4.1 to the two-sample problem

THEOREM 4.2. *If in addition to the conditions of Theorem* 3.3,

(i) $\sqrt{N}(\hat{\xi} - \xi)$ *and* $\sqrt{N}(\hat{\eta} - \eta)$ *have limiting distributions*

and

(ii) $A(t_1, t_2) = E[\varphi(X_1 - t_1, \cdots, X_{s_1} - t_1, Y_1 - t_2, \cdots, Y_{s_2} - t_2) \mid \xi = \eta = 0]$

possesses first order partial derivatives continuous in the neighborhood of the origin, then

(a) *If*

$$\left.\frac{\partial A(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} = \left.\frac{\partial A(t_1, t_2)}{\partial t_2}\right|_{t_1 = t_2 = 0} = 0,$$

$$\begin{aligned}\lim_{N\to\infty} \mathcal{L}(\sqrt{N}\,(\hat{U}_N - EU_N) &= \lim_{N\to\infty} \mathcal{L}(\sqrt{N}\,[U_N - EU_N]) \\ &= N(0, \sigma^2),\end{aligned}$$

where σ^2 *is the asymptotic variance of* U_N.

(b) If the above condition is not satisfied, $\hat{\xi}$ and $\hat{\eta}$ are asymptotically normally distributed and the joint distribution of $\hat{\xi}$, $\hat{\eta}$ and the U statistic is asymptotically normal, then $\sqrt{N}[\hat{U}_N - EU_N]$ is asymptotically normally distributed.

PROOF. The proof of this theorem goes in exactly the same lines as that of Theorem 4.1 and is fairly obvious. Q.E.D.

It may be remarked here that the results of Secs. 3 and 4 can be extended to random vectors as also to functions of several U-statistics. The proof follows in exactly the same way as the theorem on the asymptotic distribution of a function of moments follows from the fact of their asymptotic normality [12]. We shall content ourselves by stating an analogue of Theorem 4.2 as applied to several U-statistics.

THEOREM 4.3. *With reference to the two sample problem, let*

$$\varphi(u_1, u_2, \cdots, u_{s_1(\gamma)}; \text{for}, v_1, v_2, \cdots, v_{s_2(\gamma)}), \qquad \gamma = 1, \cdots, g,$$

with $s_1(\gamma) \leqq m$ *and* $s_2(\gamma) \leqq n$ *be* g *real valued functions symmetric in* u *and in* v. *Further, let*

$$W^{(\gamma)}(x_{\alpha_1}, \cdots, x_{\alpha_{s_1(\gamma)}}; y_{\beta_1}, \cdots, y_{\beta_{s_2(\gamma)}}, t_1, t_2) = \varphi^{(\gamma)}(x_{\alpha_1}, \cdots, x_{\alpha_{s_1(\gamma)}}; y_{\beta_1}, \cdots, y_{\beta_{s_2(\gamma)}}) - A^{(\gamma)}(t_1, t_2),$$

where

$$A^{(\gamma)}(t_1, t_2) = E[\varphi^{(\gamma)}(X_{\alpha_1} - t_1, \cdots, X_{\alpha_{s_1(\gamma)}} - t_1; Y_{\beta_1} - t_2, \cdots, Y_{\beta_{s_2(\gamma)}} - t_2) \mid \xi = \eta = 0]$$

possess partial derivatives continuous in the neighborhood of the origin and $W^{(\gamma)}$ *satisfy the conditions* (B$_3$) *and* (B$_4$) *of Theorem* 3.3 *for* $\gamma = 1, \cdots, g$. *Also let* $\sqrt{N}(\hat{\xi} - \xi)$ *and* $\sqrt{N}(\hat{\eta} - \eta)$ *have limiting distributions where the estimates* $\hat{\xi}$ *and* $\hat{\eta}$ *satisfy the conditions* (3.11) *and* (3.12) *of Theorem* 3.3. *Define*

$$U_N^{(\gamma)} = \binom{m}{s_1(\gamma)}^{-1} \binom{n}{s_2(\gamma)}^{-1} \cdot \sum_{\alpha,\beta} \varphi^{(\gamma)}(X_{\alpha_1} - \xi, \cdots, X_{\alpha_{s_1(\gamma)}} - \xi; Y_{\beta_1} - \eta, \cdots, Y_{\beta_{s_2(\gamma)}} - \eta),$$

the summation having the same meaning as before. Then

(i) *a necessary and sufficient condition that the joint asymptotic distribution of*

$$\sqrt{N}(\hat{U}_N^{(1)} - EU_N^{(1)}), \qquad \cdots, \qquad \sqrt{N}(\hat{U}_N^{(g)} - EU_N^{(g)})$$

be the same as the joint asymptotic distribution of

$$\sqrt{N}(U_N^{(1)} - EU_N^{(1)}), \qquad \cdots, \qquad \sqrt{N}(U_N^{(g)} - EU_N^{(g)})$$

is that

$$\left.\frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=0} = \left.\frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_2}\right|_{t_1=t_2=0} = 0$$

for $\gamma = 1, 2, \cdots, g$.

(ii) *A necessary and sufficient condition that the asymptotic distribution of* $\sqrt{N}\sum_{\gamma=1}^{g} C_\gamma[\hat{U}_N^{(\gamma)} - EU_N^{(\gamma)}]$ *be the same as the asymptotic distribution of*

$$\sqrt{N}\sum_{\gamma=1}^{g} C_\gamma(U_N^{(\gamma)} - EU_N^{(\gamma)})$$

is that

$$\sum_{\gamma=1}^{g} C_\gamma \frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_1}\bigg|_{t_1=t_2=0} = 0$$

and

$$\sum_{\gamma=1}^{g} C_\gamma \frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_2}\bigg|_{t_1=t_2=0} = 0.$$

5. Consequences of Theorem 4.2. In this section, we will consider some of the tests of a class $\{\hat{W}_N\}$ based on a class of statistics $\{\hat{U}_N\}$ for testing the hypothesis that two populations differ only in location and investigate whether they are asymptotically distribution free.

Consider first the test statistic T proposed in [2] based on a sample of m X's and n Y's. The test statistics may be defined as

$$T = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} K(x_i, Y_j), \tag{5.1}$$

where

$$K(X, Y) = 1 \quad \text{if} \begin{cases} \text{either } 0 < X < Y, \\ \text{or} \quad Y < X < 0, \end{cases} \tag{5.2}$$
$$= 0 \quad \text{otherwise.}$$

A corresponding modified test is then based on the statistic

$$\hat{T} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} K(X_i - \tilde{X}, Y_j - \tilde{Y}), \tag{5.3}$$

$\tilde{X}$ and $\tilde{Y}$ being the sample medians. Let $\xi = \eta = 0$. We then have

$$\begin{aligned} A(t_1, t_2) &= EK(X - t_1, Y - t_2) \\ &= \int_0^\infty [1 - G(x + t_2)]\, dF(x + t_1) + \int_{-\infty}^0 G(x + t_2)\, dF(x + t_1). \end{aligned} \tag{5.4}$$

Also $W(x, y, t_1, t_2) = K(x - t_1, y - t_2) - A(t_1, t_2)$. It can then be shown that $E \mid W(X, Y, t_1, 0) - W(X, Y, 0, 0)|$

$$\leqq 3 \int_{-\infty}^{0} | F(x + t_1) - F(x) |\, dG(x) + 2 | F(t_1) - F(0) |$$
$$\leqq 5at_1$$

if the distribution function F has a derivative F' bounded in absolute value by a. Similarly, it can be shown that

$$E|W(X, Y, 0, t_2) - W(X, Y, 0, 0)| \leqq 5bt_2,$$

provided the distribution function G has a derivative G' bounded in absolute value by b.

Hence the condition (B_3) of Theorem 3.3 is satisfied. Observing that K can be expressed as a difference of two monotone functions, it is easy to see that condition (B_4) is also satisfied. Again, we have

$$\frac{\partial A(t_1, t_2)}{\partial t_1} = -f(t_1)[2G(t_2) - 1] + \int_0^\infty f(x + t_1)\, dG(x + t_2) - \int_{-\infty}^0 f(x + t_1)\, dG(x + t_2),$$

$$\frac{\partial A(t_1, t_2)}{\partial t_2} = -\int_0^\infty g(x + t_2)\, dF(x + t_1) + \int_{-\infty}^0 g(x + t_2)\, dF(x + t_1).$$

Clearly,

$$\left.\frac{\partial A(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} = \left.\frac{\partial A(t_1, t_2)}{\partial t_2}\right|_{t_1 = t_2 = 0} = 0$$

if $f(x)$ and $g(x)$ are symmetric about the origin. Conditions (a) of Theorem 4.2 are satisfied. Hence $\hat{T}$ has the same asymptotic normal distribution as the statistic T. This consequence is stated in Theorem 5.1.

THEOREM 5.1. *If the X's and the Y's are distributed symmetrically about the respective medians and have bounded density functions, the test of the hypothesis H based on the statistic $\hat{T}$ is asymptotically distribution free.*

Consider now the test statistic suggested by Mood [3]. The test statistic may be defined as

$$M = \sum_{i=1}^{n} \left(r_i - \frac{m + n + 1}{2}\right)^2, \tag{5.5}$$

where r_i is the rank of Y_i in the combined sample of $(m + n)$ observations. Noting that

$$r_i = 1 + \sum_{j=1}^{m} \varphi(X_j, Y_i) + \sum_{k=1}^{n} \varphi(Y_k, Y_i), \tag{5.6}$$

where

$$\begin{aligned} \varphi(u, v) &= 1 \quad \text{if} \quad u < v, \\ &= 0 \quad \text{otherwise}, \end{aligned}$$

it is easy to see that if $m + n = N$, and

$$\begin{aligned} \psi(u, v, w) &= 1 \quad \text{if} \quad u < w \quad \text{and} \quad v < w, \\ &= 0 \quad \text{otherwise}, \end{aligned}$$

$$\frac{M}{N^3} = C_1 U_N^{(1)} + C_2 U_N^{(2)} + C_3 U_N^{(3)} + P\left(\frac{1}{N}\right), \tag{5.7}$$

where, C_1, C_2, C_3 are certain known fixed constants, $P(1/N)$ is a third-degree polynomial in $1/N$ and

$$
\begin{aligned}
U_N^{(1)} &= \binom{m}{2}^{-1}\binom{n}{1}^{-1}\sum_{i}^{n}\sum_{j\neq k}^{m}\psi(X_j, X_k, Y_i),\\
(5.8)\qquad U_N^{(2)} &= \binom{m}{1}^{-1}\binom{n}{2}^{-1}\sum_{j}^{m}\sum_{k\neq i}^{n}\psi(X_j, Y_k, Y_i),\\
U_N^{(3)} &= \binom{m}{1}^{-1}\binom{n}{1}^{-1}\sum_{i}\sum_{j}\varphi(X_j, Y_i)
\end{aligned}
$$

are three generalised U-statistics so that

$$
(5.9)\qquad \frac{\hat{M}}{N^3} = C_1\hat{U}_N^{(1)} + C_2\hat{U}_N^{(2)} + C_3\hat{U}_N^{(3)} + P\left(\frac{1}{N}\right),
$$

where $\hat{U}_N^{(i)}$ is obtained from $U_N^{(i)}$ by centering the observations at the respective sample medians. Consider the statistic $\hat{U}_N^{(3)}$. We have,

$$
\begin{aligned}
A^{(3)}(t_1, t_2) &= E\varphi(X - t_1, Y - t_2)\\
&= \int F(x + t_1)\, dG(x + t_2),\\
W^{(3)}(x, y, t_1, t_2) &= \varphi(x - t_1, y - t_2) - A^{(3)}(t_1, t_2).
\end{aligned}
$$

It can then be shown that

$$
E|W^{(3)}(X, Y, t_1, 0) - W^{(3)}(X, Y, 0, 0)| \leqq 2at_1
$$

and

$$
E|W^{(3)}(X, Y, 0, t_2) - W^{(3)}(X, Y, 0, 0)| \leqq 2bt_2 .
$$

Condition (B_3) of Theorem 3.3 is thus satisfied. Exactly in the same manner, it can be shown that the condition (B_3) is also satisfied by the statistics $\hat{U}_N^{(1)}$ and $\hat{U}_N^{(2)}$. Condition (B_4) is also easily seen to be satisfied. Also we have

$$
\begin{aligned}
A^{(2)}(t_1, t_2) &= E\psi(X_i - t_1, Y_j - t_2, Y_k - t_2)\\
&= \int F(x + t_1)G(x + t_2)\, dG(x + t_2),\\
A^{(1)}(t_1, t_2) &= E\psi(X_i - t_1, X_j - t_1, Y_k - t_2)\\
&= \int F^2(x + t_1)\, dG(x + t_2),\\
\left.\frac{\partial A^{(1)}(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} &= 2\int F(x)f(x)g(x)\, dx,\\
\left.\frac{\partial A^{(2)}(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} &= \int G(x)f(x)g(x)\, dx,\\
\left.\frac{\partial A^{(3)}(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} &= \int f(x)g(x)\, dx.
\end{aligned}
$$

Clearly,

$$\sum_{\gamma=1}^{3} C_\gamma \frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_1}\bigg|_{t_1=t_2=0} \neq 0.$$

Similarly, it is easy to see that

$$\sum_{\gamma=1}^{3} C_\gamma \frac{\partial A^{(\gamma)}(t_1, t_2)}{\partial t_2}\bigg|_{t_1=t_2=0} \neq 0.$$

Hence, it follows as a consequence of Theorem 4.3 that the statistics $\hat{M}$ and M do not have the same asymptotic normal distribution. It follows that the test based on the statistic $\hat{M}$ is not asymptotically distribution free.

6. Small sample behavior of the proposed test. It was shown in the previous section that the test statistic $\hat{T}$ is asymptotically distribution free. We will now give some idea regarding the small sample behavior of this test by considering the simplest possible case, namely $m = n = 3$. The computations involved even in this relatively simple case are very extensive. We will consider the one sided test of the hypothesis

$$H:\delta = 1,$$

$$A:\delta > 1.$$

We will consider some special alternatives and obtain the size and the relative efficiency of the Test $\hat{T}$ with respect to the corresponding best test for each of these alternatives. These results are presented in Table 1.

TABLE 1

Population	Size of $\hat{T}$ test	Relative efficiency of $\hat{T}$ test w. r. t. the corresponding best test for—		
		$\delta = 2$	$\delta = 3$	$\delta = 4$
Normal	0.23	0.83	0.76	
Uniform	0.25	0.70	0.68	0.68
Double exponential	0.25	0.92	0.81	0.81

From the above results we see that the size of the test remains more or less constant. The test is highly efficient for exponential alternatives and moderately so for normal and uniform alternatives.

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