A PROPERTY OF ADDITIVELY CLOSED FAMILIES OF DISTRIBUTIONS

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1. Introduction. The property that a linear combination of independent χ^2 variables with coefficients other than unity (or zero) is not distributed as χ^2 has for long been tacitly understood or explicitly stated in studies of the distribution of quadratic forms, the Behrens-Fisher problem, and the precision of estimates of variance components, and in the derivation of tests for the analysis of variance of unbalanced designs. The earliest explicit statement known to the author occurs as a special case of a corollary given without proof by Cochran [2]. A proof depending on the form of the moment-generating function of χ^2 was given by James [6]. The purpose of this note is to state and prove the analogous property for a general class of closed families of distributions, on the basis of work by Teicher [8].

Definition. A one-parameter family of univariate cumulative distribution functions $F(x; \lambda)$ is *additively closed*, if, for any two members $F(x; \lambda_1)$ and $F(x; \lambda_2)$,

$F(x; \lambda_1) * F(x; \lambda_2) \overset{x}{\equiv} F(x; \lambda_1 + \lambda_2)$.

2. Principal Result.

Theorem. *Consider a one-parameter additively closed family of univariate cumulative distribution functions $F(x; \lambda)$, where λ is* (i) *any positive integer,* (ii) *any positive rational, or* (iii) *any positive real number* (*except that in case* (iii) *it is required that $\phi(t; \lambda)$, the characteristic function of $F(x; \lambda)$, be either continuous in λ or real-valued for real t*). *Let three cumulants with orders j, $j + h$, $j + 2h$ (j, h* positive integers) *exist and be non-zero. If j is odd or both j and h are even, also let $F(x; \lambda) = 0$ for $x < 0$ and $F(x; \lambda) > 0$ for $x > 0$. Then the only linear combinations of a finite number of independent variables with distributions in the family, $\sum_{r=1}^{k} c_r X_r$, ($c_r \neq 0$, real), whose distributions are also in the family are those with all $c_r = 1$.*

Proof. According to Theorems 1 and 2 of [8] the characteristic function of a member of the family is of the form $[f(t)]^{\lambda}$, where $f(t)$ is a characteristic function not depending on λ. Let λ_r be the value of the parameter of the distribution of X_r. If $\sum_{r=1}^{k} c_r X_r$ is to have its distribution in the family for some λ, then

$$\prod_{r=1}^{k} [f(c_r t)]^{\lambda_r} = [f(t)]^{\lambda}. \tag{1}$$

Since the cumulants of order through $j + 2h \equiv m$ exist,

$$\log f(t) = \sum_{\nu=1}^{m} \frac{\kappa_\nu}{\nu!} (it)^\nu + o(t^m) \tag{2}$$

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in some neighborhood of $t = 0$, where the κ_ν are the cumulants of the distribution corresponding to $f(t)$. Hence

$$\sum_{r=1}^{k} \lambda_r \sum_{\nu=1}^{m} \frac{\kappa_\nu}{\nu!} (ic_r t)^\nu + o(t^m) = \lambda \sum_{\nu=1}^{m} \frac{\kappa_\nu}{\nu!} (it)^\nu + o(t^m),$$

or

$$\sum_{\nu=1}^{m} \frac{\kappa_\nu}{\nu!} (it)^\nu \left(\sum_{r=1}^{k} \lambda_r c_\nu^r - \lambda \right) + o(t^m) = 0. \tag{3}$$

For this to be true as t approaches zero the coefficients of $t, t^2, \cdots, t^m$ must be zero. Since κ_j, κ_{j+h}, and κ_{j+2h} are not zero,

$$\sum_{r=1}^{k} \lambda_r c_r^j = \lambda, \qquad \sum_{r=1}^{k} \lambda_r c_r^{j+h} = \lambda, \qquad \sum_{r=1}^{k} \lambda_r c_r^{j+2h} = \lambda. \tag{4}$$

Multiplying both sides of the second equation by 2 and subtracting them respectively from the sums of the corresponding sides of the first and third equations[1] give

$$\sum_{r=1}^{k} \lambda_r c_r^j (1 - c_r^h)^2 = 0. \tag{5}$$

Since $\lambda_r > 0$ and the c_r are real and not zero, this equation and an even value of j imply that $c_r^h = 1$. If in addition h is odd, then $c_r = 1$. If both j and h are even or if j is odd, then the conditions $F(x; \lambda) = 0$ for $x < 0$ and $F(x; \lambda) > 0$ for $x > 0$ imply that $c_r > 0$ as shown below. Hence in these cases also it follows that all c_r are unity.

To show that the conditions $F(x; \lambda) = 0$ for $x < 0$ and $F(x; \lambda) > 0$ for $x > 0$ imply that no c_r can be negative, we first note that if all c_r were negative then $\Sigma c_r X_r$ would be negative with probability one and hence could not have its distribution in the family. We therefore suppose that there are exactly p negative values of c_r with $0 < p < k$, say $c_1, c_2, \cdots, c_p$. Let

$$X = -\sum_{r=1}^{p} c_r X_r, \qquad Y = \sum_{r=p+1}^{k} c_r X_r. \tag{6}$$

The cumulative distribution functions of X and Y are, say,

$$\begin{aligned} G(x) &= F\left(-\frac{x}{c_1}; \lambda_1\right) * \cdots * F\left(-\frac{x}{c_p}; \lambda_p\right), \\ H(y) &= F\left(\frac{y}{c_{p+1}}; \lambda_{p+1}\right) * \cdots * F\left(\frac{y}{c_k}; \lambda_k\right), \end{aligned} \tag{7}$$

and thus possess the properties of $F(x; \lambda)$:

$$\begin{aligned} G(x) &= 0 \text{ for } x < 0, \qquad G(x) > 0 \text{ for } x > 0; \\ H(y) &= 0 \text{ for } y < 0, \qquad H(y) > 0 \text{ for } y > 0. \end{aligned} \tag{8}$$

[1] This method of combination, simpler than that used initially, was pointed out by Professor Arne Magnus.

Then

$$\text{(9)} \qquad \Pr\left(\sum_{r=1}^{k} c_r X_r < 0\right) = \Pr(Y < X) = \int_0^\infty H(x)\, dG(x).$$

$G(x)$ is not a degenerate distribution since $c_r \neq 0$ and its second cumulant, for example, is not zero. Hence $G(x)$ has a positive increase over some interval in which $H(x) > 0$. Hence there is a positive probability that $\Sigma c_r X_r < 0$. But this is impossible for any member of the family. Hence no c_r can be negative.

3. Discussion of theorem. The requirement that the initial point of increase of the distributions be zero can be dropped by restricting consideration to positive c_r.

The theorem is satisfied with a minimum number of cumulants required if the first three—the mean, the variance, and the "skewness" measure κ_3—are not zero, provided that $F(x, \lambda) = 0$ for $x < 0$ and $F(x, \lambda) > 0$ for $x > 0$. Beyond this proviso, only the requirement $\kappa_3 \neq 0$ need be stated explicitly since this implies $\kappa_2 \neq 0$, and since a non-negative, non-degenerate random variable must have $\kappa_1 \neq 0$. However, the condition $\kappa_3 \neq 0$ is not necessary for the conclusion of the theorem, as shown by the example of the additively closed family of binomial distributions with $p = \frac{1}{2}$ and parameter the sample size. Although $\kappa_3 = 0$ in this case, the theorem applies with κ_2, κ_4, and κ_6 all non-zero.

If the three non-zero cumulants used in the theorem include κ_2, it need not be explicitly stated that $\kappa_2 \neq 0$ since $\kappa_{j+2h} \neq 0$ implies $\kappa_2 \neq 0$.

A requirement that $\kappa_1 \neq 0$ would by itself exclude two cases for which the conclusion of the theorem is false: normal with mean zero and variance λ; Cauchy with median zero and semi-interquartile range λ. However, even $\kappa_1 \neq 0$ and the further conditions $\kappa_2 \neq 0$ and $c_r > 0$ are not sufficient for the conclusion of the theorem. Consider the one-parameter family of normal distributions with variance λ and mean $\gamma\lambda$, where $\lambda > 0$ and $\gamma \neq 0$. The distribution of $c_1X_1 + c_2X_2$ is normal with mean γ times the variance if

$$c_1 = \tfrac{1}{2}[1 + (1 + 4a/\lambda_1)^{\frac{1}{2}}], \qquad c_2 = \tfrac{1}{2}[1 \pm (1 - 4a/\lambda_2)^{\frac{1}{2}}],$$

where $0 < a \leqq \lambda_2/4$. Thus this family does not satisfy the conclusion of the theorem although $\kappa_1 \neq 0$, $\kappa_2 \neq 0$, $c_1 > 0$, $c_2 > 0$.

This example leads to the conclusion that, among distributions with moments of all orders, the condition that some three cumulants of the form κ_j, κ_{j+h}, and κ_{j+2h} not be zero, although it has not been shown necessary for the conclusion of the theorem, is little stronger than necessary. Specifically, we can prove that any additively closed family of non-degenerate distributions with all moments existing (and, in case (iii), characteristic function continuous in λ) which satisfies the conclusion of the theorem must have at least one $\kappa_j \neq 0$ with $j > 2$. For suppose this were not true. Since all moments exist, so do all cumulants. All cumulants beyond κ_2 would be zero. Consequently the distributions would be normal. By Teicher's Theorem 1 [8] a one-parameter additively closed family of non-degenerate normal distributions with, in case (iii), characteristic functions continuous in $\lambda(\lambda > 0)$ must have those characteristic functions of the form

(10) $$[f(t)]^{\lambda} = e^{\mu it - \sigma^2 t^2/2},$$

where $\sigma > 0$ and $f(t)$ is a characteristic function not depending on λ. Hence

(11) $$\sigma^2 = \alpha\lambda, \qquad \mu = \gamma\lambda,$$

where $\alpha > 0$ and γ is real. (In case (iii) we may let $\alpha = 1$ without loss of generality.) Equations (10) and (11) would thus be true if the statement in question were not true. But, as shown in the preceding paragraph (where we may take the variance as $\alpha\lambda > 0$ also), this family does not satisfy the conclusion of the theorem, contrary to the hypothesis. Hence the statement in question must be true.

Furthermore, any asymmetrical distribution with characteristic function expansible in a convergent Maclaurin series must have *some* non-zero κ_j for $j > 2$ (or non-zero central moment) of odd order; this follows from the formula for the cumulative distribution function in terms of the characteristic function.

The distributions of some additively closed families are members of the Pearson system. By use of Kendall's recurrence relation for the cumulants of Pearson curves [7] it can be shown that any Pearson-type distribution except the normal for which the recurrence relation is valid has at least three non-zero cumulants of the form κ_j, κ_{j+h}, and κ_{j+2h}. A family of Pearson Type III distributions with left-hand endpoint at zero and the non-additive parameter fixed (the family of all χ^2 distributions for example) is thus an additively closed family that satisfies the theorem.

An example showing that the conditions $F(x;\lambda) = 0$ for $x < 0$ and $F(x;\lambda) > 0$ for $x > 0$ are not implicit in the conclusion of the theorem is the family of Poisson distribution functions $F(x;\lambda, b, a)$ where $\lambda > 0$, $b \neq 0$, $a \neq 0$, and F is a step-function with a jump equal to $\lambda^{\nu}e^{-\lambda}/\nu!$ at $x = \lambda b + \nu a$, $\nu = 0, 1, 2, \cdots$ [3]. None of the cumulants is zero, so that the theorem can be applied without invoking the above conditions. Such translation can be applied more generally to additively closed families of distributions; the corresponding slight extension of the theorem is omitted.

It may be questioned whether the existence of any moments is necessary to assure the conclusion of the theorem. In cases (ii) and (iii) the general form of the characteristic function of an infinitely divisible distribution is available [8] and might be thought applicable. No appreciable results have been derived therefrom, however. The above example of Cauchy distributions shows that a restriction of some sort must be placed on an additively closed family whose moments do not exist in order to assure the conclusion.

4. Further examples. The generalized Poisson distributions associated with an arbitrary but fixed distribution [4] form a one-parameter additively closed family of distributions. The generalized Poisson distributions include the negative binomial distributions [1] and Neyman's contagious distributions [4].

The example in section 3 of the additively closed family of normal distributions not satisfying the conclusion of our theorem can be generalized to certain families of stable distributions. Suppose that we have an additively closed family of stable distributions with additive parameter λ of any of the three types in the theorem, $\phi(t; \lambda)$ being continuous in λ for each t if λ is of type (iii). It follows from the general form of the characteristic function of a stable distribution [5] and from Teicher's Theorem 1 that

$$\begin{aligned}\log(\phi(t; \lambda)) &= \beta(\lambda)it - \theta(\lambda)\,|t|^{\alpha(\lambda)}\left[1 + i\delta(\lambda)\frac{t}{|t|}\,\omega(t; \lambda)\right]\\ &= \lambda\beta(1)it - \lambda\theta(1)\,|t|^{\alpha(1)}\left[1 + i\delta(1)\frac{t}{|t|}\,\omega(t; 1)\right],\end{aligned}$$

where $\alpha(\lambda)$, $\beta(\lambda)$, $\delta(\lambda)$, $\theta(\lambda)$ are real functions of λ satisfying $0 < \alpha(\lambda) \leqq 2$, $|\delta(\lambda)| \leqq 1$, and $\theta(\lambda) \geqq 0$, and where $\omega(t; \lambda) = \tan[\pi\alpha(\lambda)/2]$ or $(2/\pi)\log|t|$ according as $\alpha(\lambda) \neq 1$ or $\alpha(\lambda) = 1$.

By equating real and imaginary parts and simple computations, it is readily established that

$$\begin{aligned}\theta(\lambda) &= \lambda\theta(1), &\quad \alpha(\lambda) &= \alpha(1),\\ \beta(\lambda) &= \lambda\beta(1), &\quad \delta(\lambda) &= \delta(1).\end{aligned}$$

With these conditions the corresponding family of stable distributions is indeed additively closed. Every stable distribution is in at least one of the infinitely many such families of stable distributions. If X_1 and X_2 are independent random variables whose distributions are in the above family, it can be shown that there exist constants c_1 and c_2 unequal to 0 or 1 such that $c_1X_1 + c_2X_2$ is also in the family. Thus one-parameter additively closed families of stable distributions, with $\phi(t; \lambda)$ continuous in λ in case (iii) of the theorem, cannot satisfy the conclusion of the theorem.

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