NOTES

THE EXPRESSION OF k-STATISTIC k_{11} IN TERMS OF POWER SUMS AND SAMPLE MOMENTS

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The values of k_9 and k_{10} in terms of products of power sums s_r's ($s_r = \sum a^r$) have been published by the author ([7]). In this paper k_{11} is expressed in terms of s_r-products and sample moments, and is computed with the help of the tables of generalized k-statistics constructed by Abdel-Aty ([1]). From these tables

$$k_{11} = \frac{3628800}{n^{(11)}}[1^{11}] - \frac{19958400n}{n^{(10)}}[21^9] + \cdots \frac{[11]}{n},$$

where $n^{(11)} = n(n-1)(n-2)\cdots(n-10)$ and $[1^{11}]$, $[21^9]$ etc., are the augmented symmetric functions, which can be expressed in terms of s_r-products from the tables of symmetric functions given by David and Kendall [2]. Collecting all terms, k_{11} is expressed in terms of s_r-products.

As a check, the sum of the coefficients of all s_r's is $1/n$.

k_{11} is obtained in terms of sample moments m_r' by putting $s_1 = 0$ and $s_r = nm_r$ $(r > 1)$. Thus $s_{11} = nm_{11}$, $s_3s_2^4 = nm_3(nm_2)^4 = n^5m_2^4m_3$, etc.

The k-statistics have recently been applied in various fields by several writers such as Tukey [6], Hooke [4], Robson [5]. They are of interest to workers in the theory of sampling distributions and moment statistics. They are related also to certain aspects of the theory of numbers and combinatory analysis, as indicated by Dressel [3].

$$\begin{aligned}
k_{11} = \frac{1}{n^{(11)}}&[3628800s_1^{11} - 19958400ns_2s_1^9\\
&+ 39916800(n^2 - n)s_2^2s_1^7 - 34927200(n^3 - 3n^2 + 2n)s_2^3s_1^5\\
&+ 12474000(n^4 - 6n^3 + 11n^2 - 6n)s_2^4s_1^3\\
&- 1247400(n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)s_2^5s_1 + 6652800(n^2 + 8n)s_3s_1^8\\
&- 23284800(n^3 + 5n^2 - 6n)s_3s_2s_1^6\\
&\qquad\qquad + 24948000(n^4 + n^3 - 10n^2 + 8n)s_3s_2^2s_1^4\\
&- 8316000(n^5 - 4n^4 - n^3 + 16n^2 - 12n)s_3s_2^3s_1^2\\
&+ 415800(n^6 - 10n^5 + 35n^4 - 50n^3 + 24n^2)s_3s_2^4\\
&+ 3326400(n^4 + 8n^3 + 25n^2 - 34n)s_3^2s_1^5
\end{aligned}$$

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$$- 5544000(n^5 + 2n^4 + 5n^3 - 44n^2 + 36n)s_3^2s_2s_1^3$$
$$+ 1663200(n^6 - 5n^5 + 15n^4 - 55n^3 + 104n^2 - 60n)s_3^2s_2^2s_1$$
$$+ 369600(n^6 + 25n^4 - 30n^3 - 116n^2 + 120n)s_3^3s_1^2$$
$$- 92400(n^7 - 9n^6 + 55n^5 - 195n^4 + 304n^3 - 156n^2)s_3^3s_2$$
$$- 1663200(n^3 + 29n^2 + 42n)s_4s_1^7$$
$$+ 4989600(n^4 + 22n^3 - 17n^2 - 6n)s_4s_2s_1^5$$
$$- 4158000(n^5 + 14n^4 - 67n^3 + 88n^2 - 36n)s_4s_2^2s_1^3$$
$$+ 831600(n^6 + 5n^5 - 85n^4 + 295n^3 - 396n^2 + 180n)s_4s_2^3s_1$$
$$- 1386000(n^5 + 20n^4 + 65n^3 - 14n^2 - 72n)s_4s_3s_1^4$$
$$+ 1663200(n^6 + 10n^5 - 15n^4 - 40n^3 + 44n^2)s_4s_3s_2s_1^2$$
$$- 207900(n^7 - n^6 - 25n^5 + 85n^4 - 96n^3 + 36n^2)s_4s_3s_2^2$$
$$- 138600(n^7 + 3n^6 + 15n^5 + 25n^4 - 256n^3 + 212n^2)s_4s_3^2s_1$$
$$+ 138600(n^6 + 25n^5 + 225n^4 - 385n^3 + 854n^2 - 720n)s_4^2s_1^3$$
$$- 103950(n^7 + 11n^6 + 55n^5 - 655n^4$$
$$+ 2104n^3 - 2956n^2 + 1440n)s_4^2s_2s_1$$
$$+ 11550(n^8 - n^7 + 67n^6 - 355n^5 + 1084n^4 - 1804n^3 + 1008n^2)s_4^2s_3$$
$$+ 332640(n^4 + 71n^3 + 396n^2 + 36n)s_5s_1^6$$
$$- 831600(n^5 + 56n^4 + 101n^3 - 374n^2 + 216n)s_5s_2s_1^4$$
$$+ 498960(n^6 + 40n^5 - 135n^4 - 70n^3 + 524n^2 - 360n)s_5s_2^2s_1^2$$
$$- 41580(n^7 + 23n^6 - 265n^5 + 925n^4 - 1296n^3 + 612n^2)s_5s_2^3$$
$$+ 221760(n^6 + 45n^5 + 145n^4 + 435n^3 - 1346n^2 + 720n)s_5s_3s_1^3$$
$$- 166320(n^7 + 27n^6 - 105n^5 + 385n^4$$
$$- 1576n^3 + 2708n^2 - 1440n)s_5s_3s_2s_1$$
$$+9240(n^8 + 11n^7 - 89n^6 + 785n^5 - 2936n^4 + 4244n^3 - 2016n^2)s_5s_3^2$$
$$- 41580(n^7 + 39n^6 + 395n^5 - 115n^4 - 396n^3 + 1516n^2 - 1440n)s_5s_4s_1^2$$
$$+ 13860(n^8 + 17n^7 + 49n^6 - 805n^5 + 2614n^4 - 3532n^3 + 1656n^2)s_5s_4s_2$$
$$+ 2772(n^8 + 38n^7 + 652n^6 - 1510n^5 + 9199n^4$$
$$- 33088n^3 + 54948n^2 - 30240n)s_5^2s_1$$
$$- 55440(n^5 + 146n^4 + 1871n^3 + 2086n^2 - 1080n)s_6s_1^5$$
$$+ 110880(n^6 + 115n^5 + 765n^4 - 2095n^3 + 854n^2 + 360n)s_6s_2s_1^3$$

$$
\begin{aligned}
&- 41580(n^7 + 83n^6 - 65n^5 - 1975n^4 + 6544n^3 - 7468n^2 + 2880n)s_6s_2^2s_1 \\
&- 27720(n^7 + 87n^6 + 395n^5 + 1085n^4 - 1836n^3 - 4052n^2 + 4320n)s_6s_3s_1^2 \\
&+ 9240(n^8 + 53n^7 - 275n^6 + 1175n^5 - 4406n^4 + 7412n^3 - 3960n^2)s_6s_3s_2 \\
&+ 4620(n^8 + 62n^7 + 484n^6 + 1970n^5 - 13001n^4 \\
&\qquad + 41168n^3 - 60924n^2 + 30240n)s_6s_4s_1 \\
&- 462(n^9 + 38n^8 + 652n^7 - 1510n^6 + 9199n^5 \\
&\qquad - 33088n^4 + 54948n^3 - 30240n^2)s_6s_5 \\
&+ 7920(n^6 + 270n^5 + 6295n^4 + 18810n^3 - 8816n^2 - 1440n)s_7s_1^4 \\
&- 11880(n^7 + 207n^6 + 2795n^5 - 5515n^4 \\
&\qquad - 7836n^3 - 20428n^2 - 10080n)s_7s_2s_1^2 \\
&+ 1980(n^8 + 143n^7 + 355n^6 - 8275n^5 \\
&\qquad + 28444n^4 - 37228n^3 + 16560n^2)s_7s_2^2 \\
&+ 2640(n^8 + 146n^7 + 976n^6 + 110n^5 + 11899n^4 \\
&\qquad - 60736n^3 + 77844n^2 - 30240n)s_7s_3s_1 \\
&- 330(n^9 + 86n^8 + 316n^7 + 5450n^6 - 35201n^5 \\
&\qquad + 115424n^4 - 176796n^3 + 90720n^2)s_7s_4 \\
&- 990(n^7 + 459n^6 + 16795n^5 + 91785n^4 \\
&\qquad - 11756n^3 - 67044n^2 + 30240n)s_8s_1^3 \\
&+ 990(n^8 + 332n^7 + 7054n^6 - 8380n^5 - 63851n^4 \\
&\qquad + 159248n^3 - 124644n^2 + 30240n)s_8s_2s_1 \\
&- 165(n^9 + 206n^8 + 1636n^7 - 5230n^6 + 58999n^5 \\
&\qquad - 236896n^4 + 332484n^3 - 151200n^2)s_8s_3 \\
&+ 110(n^8 + 713n^7 + 36277n^6 + 292115n^5 + 92434n^4 \\
&\qquad - 519628n^3 + 340008n^2 - 60480n)s_9s_1^2 \\
&- 55(n^9 + 458n^8 + 12472n^7 - 11530n^6 - 186701n^5 \\
&\qquad + 555392n^4 - 581772n^3 + 211680n^2)s_9s_2 \\
&- 11(n^9 + 968n^8 + 60082n^7 + 595760n^6 + 371569n^5 \\
&\qquad - 1594648n^4 + 1261788n^3 - 332640n^2)s_{10}s_1 \\
&+ (n^{10} + 968n^9 + 60082n^8 + 595760n^7 + 371569n^6 \\
&\qquad -1594648n^5 + 1261788n^4 - 332640n^3)s_{11}].
\end{aligned}
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REFERENCES

[1] ADBEL-ATY, "Tables of generalised k-Statistics," *Biometrika*, Vol. 41 (1954), pp. 253–260.

[2] F. N. DAVID, AND M. G. KENDALL, "Tables of symmetric functions," *Biometrika*, Vol. 36 (1949), pp. 431–441.

[3] P. L. DRESSEL, "Statistical semi-invariants and their estimates with particular emphasis on their relation to algebraic invariants," *Ann. Math. Stat.*, Vol. 11 (1940), pp. 33–57.
[4] R. HOOKE, "Some applications of bipolykays to the estimation of variance components and their moments," *Ann. Math. Stat.*, 27 (1956), pp. 80–98.
[5] D. S. ROBSON, "Application of multivariate polykays to the theory of unbiased ratio-type estimation," *J. Amer. Stat. Assn.*, Vol. 52 (1957), pp. 511–522.
[6] J. W. TUKEY, "Variances of variance components, I. balanced designs," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 722–736.
[7] M. ZIAUD-DIN, "Expression of the k-statistics k_9 and k_{10} in terms of power sums and sample moments," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 800–803.

A GENERALIZATION OF THE GLIVENKO-CANTELLI THEOREM

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A theorem referred to as the Glivenko theorem or the Glivenko-Cantelli theorem states that if $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent, identically distributed random variables with any common distribution function $F(x)$, then the sequence $\{F_n(x)\}$ of empirical distribution functions converges uniformly to $F(x)$ with probability one. (See Loève [3] and Gnedenko [2].) The assumption of independence is not necessary for this theorem, and it is readily observed that the same conclusion holds if the sequence of random variables is a strictly stationary, ergodic (or metrically transitive) sequence. The purpose of this note is to prove a generalization of this theorem in the case where the sequence of random variables is strictly stationary, not necessarily ergodic, and with the same assumption that the common distribution function is arbitrary.

It is assumed that the reader is familiar with strictly stationary stochastic processes (with discrete time) and is acquainted with the notion of measure-preserving set transformation determined by the process and the notion of random variable transformation determined by this set transformation. Information on these concepts is available in Doob [1] and Loève [3]. The principal result to be used in the proof of the theorem is the ergodic theorem for random variables (see Loève [3], p. 434), which can be stated as follows:

Let S be a measure-preserving set transformation over the probability space $(\Omega, \mathcal{A}, P)$, let T be the random variable transformation determined by S, and let $\mathfrak{I}$ be the invariant sub-sigma-field of $\mathcal{A}$ determined by S. If X is any random variable for which $E \mid X \mid < \infty$, then

$$P\{n^{-1}(X + TX + \cdots + T^{n-1}X) \rightarrow E(X \mid \mathfrak{I})\} = 1.$$

By means of the ergodic theorem in this form the following theorem is obtained.

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