SUCCESSIVE RECURRENCE TIMES IN A STATIONARY PROCESS

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Let $X_0, X_1, X_2, \cdots$ be a stationary sequence of random variables. Let B be any linear Borel set for which $P(X_0 \varepsilon B) > 0$. We are concerned with the successive recurrence times $\nu_1, \nu_2, \cdots$ of B; their time averages and their expectations. Without loss of generality, we shall assume the basic probability space Ω to be the collection of all sequences $\omega = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ and X_n to be the coordinate variables, i.e., $X_n(\omega) = x_n$. Let T be the shift transformation. The nth coordinate of $T\omega$ is the $(n + 1)$th coordinate of ω. Then T is 1–1 and preserves the probability measure P. For any

$$\omega = \{\cdots x_{-1}, x_0, x_1, \cdots\},$$

if there are infinitely many positive integers n with $x_n \varepsilon B$, let

$$\nu_1, \nu_1 + \nu_2, \cdots, \nu_1 + \nu_2 + \cdots + \nu_k, \cdots$$

be the successive positive integers for which $x_{\nu_2+\cdots-\nu_k} \varepsilon B$. If there are finitely many, say K, positive integers n with $x_n \varepsilon B$, define $\nu_1, \cdots, \nu_K$ as before but define $\nu_{K+1} = \nu_{K+2} = \cdots = \infty$. In this paper, Theorem 1 is concerned with the time average of the successive recurrence times, the ν's. In Theorem 2 the successive recurrence times are proved to be stationary given $X_0 \varepsilon B$. Theorem 3 may be considered as a generalization of a theorem of M. Kac in which he proved the formula (7) for the first recurrence time ν_1 ([2], pp. 1006).

THEOREM 1: *For almost all* ω

$$\lim_{k\to\infty} \frac{\nu_1(\omega) + \cdots + \nu_k(\omega)}{k} \tag{1}$$

exists. The limit may be finite or infinite. It is finite for almost all $\omega \varepsilon E$ *where* $E = [X_0 \varepsilon B]$. *In particular, if* T *is ergodic, the limit is equal to* $1/P(E)$ *with probability one.*

PROOF: Let I_E be the indicator function of E, i.e., $I_E(\omega) = 1$ if $\omega \varepsilon E$ and $I_E(\omega) = 0$ if $\omega \not\varepsilon E$. By the ergodic theorem, for almost all ω

$$\lim_{k\to\infty} \frac{I_E(T\omega) + \cdots + I_E(T^n\omega)}{n} = f(\omega), \tag{2}$$

where $f(\omega) > 0$ for almost all $\omega \varepsilon E$. If T is ergodic $f(\omega) \equiv P(E)$.

In fact, $[I_E(T\omega) + \cdots + I_E(T^n\omega)]n^{-1}$ is the relative frequency of occurrence of B. If the limit of the relative frequency, as $n \to \infty$, is positive, B must occur infinitely often; therefore, $\nu_1(\omega), \nu_2(\omega), \cdots$ are all finite. Thus all successive recurrence times are finite for almost all $\omega \varepsilon E$. In particular, if T is ergodic, they are all finite for almost all $\omega \varepsilon \Omega$.

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Let Ω' be the collection of all ω for which $\nu_1(\omega), \nu_2(\omega), \cdots$ are all finite. Let $\omega \varepsilon \Omega'$. For any positive integer k, let $n_k = \nu_1(\omega) + \cdots + \nu_k(\omega)$. Then

$$\frac{\nu_1(\omega) + \cdots + \nu_k(\omega)}{k} = \frac{n_k}{I_E(T\omega) + \cdots + I_E(T^{n_k}\omega)}.$$

Therefore, for almost all $\omega \varepsilon \Omega'$, there exists the limit

$$\lim_{k\to\infty} \frac{\nu_1(\omega) + \cdots + \nu_k(\omega)}{k} = \frac{1}{f(\omega)}. \tag{3}$$

The limit is finite or infinite according as $f(\omega) > 0$ or $f(\omega) = 0$. If $\omega \not\varepsilon \Omega'$ then there is a positive integer K for which $\nu_{K+1}(\omega) = \nu_{K+2}(\omega) = \cdots = \infty$. Therefore

$$\lim_{k\to\infty} \frac{\nu_1(\omega) + \cdots + \nu_k(\omega)}{k} = \infty.$$

It is clear that $I_E(T\omega) + \cdots + I_E(T^n\omega) \leqq K$ for all n and that

$$\lim_{n\to\infty} \frac{I_E(T\omega) + \cdots + I_E(T^n\omega)}{n} = 0.$$

Therefore (3) again holds true. Hence (3) holds true with probability one. If T is ergodic, $1/f(\omega) \equiv 1/p(E)$.

Let P_E, $E = [X_0 \varepsilon B]$, be the conditional probability measure given $X_0 \varepsilon B$, i.e., for any measurable set F,

$$P_E(F) = P(E \cap F)/P(E). \tag{4}$$

Then $\nu_1, \nu_2, \cdots$ are finite valued with probability one under the probability measure P_E.

Theorem 2. *The random variables $\nu_1, \nu_2, \cdots$ are stationary under the conditional probability measure P_E, i.e.,*

$$P_E(\nu_1 = i_1, \cdots \nu_k = i_k) = P_E(\nu_{m+1} = i_1, \cdots \nu_{m+k} = i_k) \tag{5}$$

for any positive integers, m, k, and any k-tuple of positive integers, $(i_1, \cdots i_k)$.

Proof: We shall proceed by induction on the integer m. Let $F_{i_1}, \cdots {i_k} = [\nu_1 = i_1, \cdots \nu_k = i_k]$, and let $E' = \Omega - E$. Then

$$\begin{aligned}
P_E[\nu_2 &= i_1, \cdots \nu_{k+1} = i_k] \\
&= \sum_{n=1}^{\infty} P_E[\nu_1 = n, \nu_2 = i_1, \cdots \nu_{k+1} = i_k] \\
&= \sum_{n=1}^{\infty} P_E[T^{-1}E' \cap \cdots \cap T^{-(n-1)}E' \cap T^{-n}E \cap T^{-n}F_{i_1,\cdots i_k}] \\
&= \frac{1}{P(E)} \sum_{n=1}^{\infty} P[E \cap T^{-1}E' \cap \cdots \cap T^{-(n-1)}E' \cap T^{-n}E \cap T^{-n}F_{i_1,\cdots i_k}] \\
&= \frac{1}{P(E)} \sum_{n=1}^{\infty} P[T^nE \cap T^{(n-1)}E' \cap \cdots \cap TE' \cap E \cap F_{i_1,\cdots i_k}] \\
&= \frac{1}{P(E)} P\left[\left(\bigcup_{n=1}^{\infty} T^nE\right) \cap E \cup F_{i_1,\cdots i_k}\right]
\end{aligned}$$

The Poincaré recurrence theorem ([1], pp. 10) asserts that

$$P\left[\left(\bigcup_{n=1}^{\infty} T^n E\right) \cap E\right] = P(E).$$

Hence

$$P_E[\nu_2 = i_1, \cdots, \nu_{k+1} = i_k] = \frac{1}{P(E)} P[E \cap F_{i_1, \cdots, i_k}] = P_E[F_{i_1}, \cdots i_k]$$
$$= P_E[\nu_1 = i_1, \cdots \nu_k = i_k].$$

Hence (5) is true for $m = 1$ and any k and any k-tuple $(i_1, \cdots i_k)$. Now assume that (5) holds true for all $m \leqq M$.

$$PE[\nu_{M+2} = i_1, \cdots \nu_{M+1+k} = i_k] = \sum_{n=1}^{\infty} P_E[\nu_{M+1} = n, \nu_{M+2} = i_1, \cdots, \nu_{M+1+k} = i_k]$$
$$= \sum_{n=1}^{\infty} P_E[\nu_1 = n, \nu_2 = i_1, \cdots, \nu_{1+k} = i_k] = P_E[\nu_2 = i_1, \cdots, \nu_{k+1} = i_k]$$
$$= P_E[\nu_1 = i_1, \cdots, \nu_k = i_k].$$

Hence (5) is true for all m.

THEOREM 3: *Let $f(\omega)$ be defined by (2), i.e., $f(\omega)$ is the limit, as $n \to \infty$, of the relative frequency of occurrence of B. Then, for any k,*

$$\int \nu_k(\omega) P_E(d\omega) = \int \frac{1}{f(\omega)} P_E(d\omega). \tag{6}$$

The conditional expectation of the kth recurrence time given $X_0 \varepsilon B$ is finite if and only if $1/f(\omega)$ is integrable with respect to P_E. In particular, if the shift transformation T is ergodic, then

$$\int \nu_k(\omega)\, P_E(d\omega) = \frac{1}{P(E)}. \tag{7}$$

PROOF. By Theorem 1, the set of all ω such that

$$\lim_{k\to\infty} \frac{\nu_1(\omega) + \cdots + \nu_k(\omega)}{k} = \frac{1}{f(\omega)}$$

has P_E measure 1. Since the process $\nu_1, \nu_2, \cdots$ is stationary under P_E by Theorem 2, the conditional expectations $\int \nu_k(\omega)P_E\,(d\omega)$ are the same for all k. If $\int \nu_k(\omega)P_E\,(d\omega) < \infty$, since $\{\nu_k\}$ is stationary, (6) follows easily from the ergodic theorem. If $\int \nu_k(\omega)P_E\,(d\omega) = \infty$, let

$$\nu_k^N(\omega) = \begin{cases} \nu_k(\omega), & \nu_k(\omega) \leqq N, \\ N, & \text{otherwise.} \end{cases}$$

Then the process $\nu_1^N, \nu_2^N, \cdots$ is again stationary under P_E, and therefore the set of ω for which $\lim_{K\to\infty} (\nu_1^N(\omega) + \cdots + \nu_k^N(\omega)/k$ exists has P_E measure 1. Let $g_N(\omega)$ be the limit. We have

$$\int \nu_k^N(\omega) P_E(d\omega) = \int g_N(\omega) P_E(d\omega).$$

But $g_N(\omega) \leqq 1/f(\omega)$, hence

$$\int \nu_k^N(\omega) P_E(d\omega) \leqq \int \frac{1}{f(\omega)} P_E(d\omega).$$

Since

$$\lim_{N\to\infty} \int \nu_k^N(\omega) P_E(d\omega) = \int \nu_k(\omega) P(d\omega) = \infty,$$

hence

$$\int \frac{1}{f(\omega)} P_E(d\omega) = \infty,$$

and again (7) is true.

REFERENCES

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(2) M. Kac, "On the Notion of Recurrence in Discrete Processes," *Bull. Amer. Math. Soc.*, Vol. 53 (1947), pp. 1002–1010.