# A GENERALISATION OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS[1]

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**1. Introduction.** The partially balanced incomplete block (PBIB) designs were first defined by Bose and Nair [2] in 1939. Later on, in 1942, Nair and Rao [7] generalised the original definition to include some confounded factorial designs as well as many others in the class of PBIB designs. The class of PBIB designs was found to include most of the designs used in practice. In 1946, Harshberger [4] presented triple rectangular lattices, and Nair [6] proved that these designs were not in general PBIB designs, but that the duals of these designs were PBIB designs. So it was found that, except for the intra-inter-group balanced designs given by Nair and Rao [8], almost all the designs so far proposed, with limited number of distinct variances for elementary treatment comparisons, were either PBIB designs or duals of PBIB designs. Yet a need was felt to find a more general class of designs. In an attempt to find out why the PBIB designs with $m$ associate classes have $m$ distinct types of treatment comparisons, I came across a more general class of designs, which is given in this paper. The arguments which led to this generalisation are also put forward.

**2. Notation.** Let there be $v$ treatments, each replicated $r$ times in $b$ blocks of $k$ plots each. Let $\mathbf{N} = [n_{ij}]$ $(i = 1, 2, \cdots, v; j = 1, 2, \cdots, b)$ be the incidence matrix of the design, where $n_{ij}$ is equal to the number of times the $i$th treatment occurs in the $j$th block. It is assumed that $n_{ij}$ is 0 or 1. The assumed model is

$$y_{ij} = \mu + \beta_j + t_i + \epsilon_{ij}, \tag{2.1}$$

where $y_{ij}$ is the yield of the plot in the $j$th block to which the $i$th treatment is applied, $\mu$ is the general effect, $\beta_j$ is the effect of the $j$th block, $t_i$ is the effect of the $i$th treatment and $\epsilon_{ij}$'s are independent normal variates with mean 0 and variance $\sigma^2$. Let $T_i$ be the total yield of all the plots having the $i$th treatment, $B_j$ be the total yield of all the plots of the $j$th block and $t_i$ be a solution for $\hat{t}_i$ in the normal equations. Further denote the column vectors $\{T_1, T_2, \cdots, T_v\}$, $\{B_1, B_2, \cdots, B_b\}$, $\{t_1, t_2, \cdots, t_v\}$ and $\{\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v\}$ by $\mathbf{T}$, $\mathbf{B}$, $\mathbf{t}$ and $\hat{\mathbf{t}}$ respectively. It is well known that the reduced normal equations for the intra-block estimates of the treatment contrasts are

$$\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}, \tag{2.2}$$

where

$$\mathbf{Q} = \mathbf{T} - \frac{1}{k}\mathbf{NB} \tag{2.3}$$

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and

$$\mathbf{C} = r\mathbf{I}(v) - \frac{1}{k}\mathbf{NN}', \tag{2.4}$$

where $\mathbf{I}(v)$ is the $v \times v$ Identity matrix. The matric $\mathbf{C}$ defined in (2.4) will be called the $\mathbf{C}$-matrix of the design. Denote by $\mathbf{E}(m, n)$ the $m \times n$ matrix with all its elements equal to 1.

Lemma 2.1: *If the design is connected, the matrix* $\mathbf{C} + a\mathbf{E}(v, v)$ *is non-singular, where $a$ is any non-zero real number and* $\hat{\mathbf{t}} = [\mathbf{C} + a\mathbf{E}(v, v)]^{-1}\mathbf{Q}$ *is a solution of the equation* $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$.

Proof: Let $\theta_1, \theta_2, \cdots, \theta_v$ be the canonical roots of the $\mathbf{C}$-matrix and let $\mathbf{l}_1, \mathbf{l}_2, \cdots, \mathbf{l}_v$ be the corresponding canonical vectors. It is well known that the $\mathbf{C}$-matrix has one root 0 and that the corresponding canonical vector is $(v)^{-\frac{1}{2}}\mathbf{E}(v, 1)$; denote these by $\theta_1$ and $\mathbf{l}_1$ respectively. Then

$$\mathbf{C} = \sum_{i=2}^{v} \theta_i \mathbf{l}_i \mathbf{l}_i' \tag{2.7}$$

and

$$\mathbf{C} + a\mathbf{E}(v, v) = \sum_{i=2}^{v} \theta_i \mathbf{l}_i \mathbf{l}_i' + av\, \mathbf{l}_1 \mathbf{l}_1'. \tag{2.8}$$

Since the design is connected, the rank of the $\mathbf{C}$-matrix is $v - 1$, and therefore none of the $\theta_i$'s except $\theta_1$ is 0. Hence from (2.8) it follows that the matrix $\mathbf{C} + a\mathbf{E}(v, v)$ is non-singular and

$$[\mathbf{C} + a\mathbf{E}(v, v)]^{-1} = \sum_{i=2}^{v} \frac{1}{\theta_i} \mathbf{l}_i \mathbf{l}_i' + \frac{1}{av} \mathbf{l}_1 \mathbf{l}_1' \tag{2.9}$$

Also

$$\mathbf{C}[\mathbf{C} + a\mathbf{E}(v, v)]^{-1} = \sum_{i=2}^{v} \mathbf{l}_i \mathbf{l}_i' = \mathbf{I}(v) - \frac{1}{v} \mathbf{E}(v, v). \tag{2.10}$$

Hence, since $\sum Q_i = 0$, $[\mathbf{C} + a\mathbf{E}(v, v)]^{-1}\mathbf{Q}$ is a solution for $\hat{\mathbf{t}}$ in the equation $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$.

Lemma 2.2: *If* $\hat{\mathbf{t}} = \mathbf{AQ}$ *is a solution of* $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$, *then* $[\mathbf{A} + a\mathbf{E}(v, v)]\mathbf{Q}$ *is also a solution of* $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$.

**3. PBIB designs.** An incomplete block design is said to be partially balanced (PBIB) if it satisfies the following conditions (Bose and Shimamoto [3]):

(i) The experimental material is divided into $b$ blocks of $k$ plots each, different treatments being applied to the plots in the same block.

(ii) There are $v$ treatments each of which occurs in $r$ blocks.

(iii) There can be established relations of association between any two treatments satisfying the following requirements:

(a) Two treatments are either 1st, 2nd, $\cdots$, or $m$th associates.

(b) Each treatment has exactly $n_i$ $i$th associates ($i = 1, 2, \cdots, m$).

(c) Given any two treatments which are $i$th associates, the number of treatments common to the $j$th associates of the first and the $k$th associates of the second is $p^i_{jk}$ and is independent of the pair of treatments with which we start. Also $p^i_{jk} = p^i_{kj}$.

(iv) Two treatments which are the $i$th associates occur together in exactly $\lambda_i$ blocks.

Now further define each treatment to be its own 0th associate and the 0th associate of no other treatment. We may thus consistently write

$$\lambda_0 = r, \qquad n_0 = 1, \qquad p^0_{st} = \delta_{st}n_s, \qquad p^t_{0s} = p^t_{s0} = \delta_{st}, \tag{3.1}$$

where $\delta_{ij}$ is the Kronecker delta which is defined for all pairs of natural numbers $i, j$, as $\delta_{ij} = 1$, if $i = j$; and $\delta_{ij} = 0$, if $i \neq j$. Then the relations between the parameters are

$$\begin{gathered} bk = vr, \qquad \sum_{i=0}^{m} n_i = v, \\ \sum_{i=0}^{m} n_i\lambda_i = rk, \qquad \sum_{k=0}^{m} p^j_{ik} = n_i, \\ n_ip^i_{jk} = n_jp^j_{ik} = n_kp^k_{ij}, \qquad i, j, k = 0, 1, \cdots, m. \end{gathered} \tag{3.2}$$

Now consider $v(v + 1)/2$ treatment pairs $(i, j)$ $(i, j = 1, 2, \cdots, v)$, assuming that $(i, j)$ is identical with $(j, i)$. Partition them into $(m + 1)$ disjoint classes and corresponding to the $t$th class $(t = 0, 1, \cdots, m)$, define the $v \times v$ matrix $\mathbf{B}_t = [B^t_{ij}]$, where $B^t_{ij} = 1$, if the pair $(i, j)$ belongs to the $t$th class and $B^t_{ij} = 0$ otherwise. The classes can be called the association classes and the corresponding matrices, the association matrices. As there is one to one correspondence between the association classes and matrices defined above, either of them will uniquely determine the other. It can be seen that each $\mathbf{B}_t$ is symmetric. Since every pair must belong to one of the association classes, it is obvious that

$$\sum_{i=0}^{m} \mathbf{B}_i = \mathbf{E}(v, v). \tag{3.3}$$

THEOREM 3.1: *The necessary and sufficient conditions, that* $(m + 1)$ *association matrices* $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$ *determine an association scheme for an m associate class* PBIB *design, are that*

$$\mathbf{B}_0 = \mathbf{I}(v), \tag{3.4}$$

*and*

$$\mathbf{B}_t\mathbf{B}_x = \sum_{i=0}^{m} p^i_{tx}\mathbf{B}_i, \qquad t, x = 0, 1, \cdots, m. \tag{3.5}$$

The proof of the above theorem follows immediately from the definition of a PBIB design given by Bose and Shimamoto [3].

The idea of association matrices was also developed by Bose and Mesner [1] independently and became available after submission of the manuscript of this paper. The reader may note that the concept introduced by Bose and Mesner is slightly different from one given here. The idea of association classes and matrices as introduced by the former is confined only to PBIB designs, whereas, my interest being the generalisation of PBIB designs, the association classes and matrices are defined in terms of partitioning of $v(v+1)/2$ combinations of $v$ objects (an object may occur more than once) taken two at a time, into $(m+1)$ mutually exclusive and exhaustive classes. Theorem 3.1 gives a set of necessary and sufficient conditions for such a scheme of partitioning to be an association scheme of a PBIB. Lemma 3.1 of Bose and Mesner [1] proves the necessary part of the condition; the sufficiency is proved by Lemma 5.1 of [1].

Before deriving further results, it is necessary to prove the following matrix theorem.

THEOREM 3.2: *If* $\mathbf{A}$ *is a* $v \times v$ *positive definite matrix, such that all the non-negative integral powers of* $\mathbf{A}$ *are of the form*

$$\mathbf{A}^N = \sum_{i=0}^{m} u_{Ni}\mathbf{B}_i, \qquad N = 0, 1, 2, \cdots, \tag{3.6}$$

*where* $u_{Ni}$ *are scalar constants and* $\mathbf{B}_i$ *are fixed* $v \times v$ *matrices and* $\mathbf{A}^0$ *means* $\mathbf{I}(v)$, *then the matrix* $\mathbf{A}^{-1}$ *must also be of the form* $\sum d_i\mathbf{B}_i$, *where* $d_i$ *are scalar constants.*

PROOF: Let $\theta$ be the maximum of the canonical roots of the matrix $\mathbf{A}$. Then the canonical roots of the matrix $\mathbf{B} = \mathbf{I}(v) - \{1/(\theta+1)\}\mathbf{A}$ lie within the range 0 and 1. Now consider the series

$$\mathbf{D} = \sum_{N=0}^{\infty} \mathbf{B}^N. \tag{3.7}$$

The above series converges because the series $\sum x^N$ converges for $-1 < x < 1$ and the canonical roots of $\mathbf{B}$ lie within the range (Macdufee [5]). Also it can be shown that

$$\mathbf{AD} = (\theta + 1)\mathbf{I}(v) = \mathbf{DA}, \tag{3.8}$$

hence

$$\mathbf{A}^{-1} = \frac{1}{\theta+1}\mathbf{D}. \tag{3.9}$$

Now since every power of $\mathbf{A}$ is a linear combination of matrices $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$, the same is true for every power of $\mathbf{B}$ and hence $\mathbf{D}$ is also a linear combination of the matrices $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$.

COROLLARY 3.2.1: *If there exist matrices* $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$, *such that* $\mathbf{I}(v)$, $\mathbf{E}(v, v)$, *and all the positive integral powers of the* $\mathbf{C}$*-matrix of a connected design are linear combinations of the matrices* $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$, *then there exists a solution* $\hat{\mathbf{t}} = \mathbf{AQ}$ *of the equation* $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$, *such that the matrix* $\mathbf{A}$ *is a linear combination of the matrices* $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$, *and also*

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I}(v) - \frac{1}{v}\,\mathbf{E}(v, v). \tag{3.10}$$

The proof of Corollary 3.2.1 follows immediately from Theorem 3.2 and Lemma 2.1.

The $\mathbf{C}$-matrix of a PBIB design can be written in the form

$$\mathbf{C} = \frac{r(k-1)}{k}\,\mathbf{B}_0 - \sum_{i=1}^{m} \frac{\lambda_i}{k}\,\mathbf{B}_i, \tag{3.11}$$

where $\mathbf{B}_t$ is the association matrix corresponding to the $t$th associate class ($t = 0, 1, \cdots, m$). Using relation (3.5) and mathematical induction, it can be proved that all the powers of $\mathbf{C}$ are linear combinations of $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$, also $\mathbf{I}(v) = \mathbf{B}_0$ and $\mathbf{E}(v, v) = \sum_0^m \mathbf{B}_i$. Hence, by Corollary 3.2.1, it follows that a solution $\hat{\mathbf{t}} = \mathbf{AQ}$ exists, such that

$$\mathbf{A} = \sum_{i=0}^{m} d_i \mathbf{B}_i. \tag{3.12}$$

With a little algebra, it can be shown that the $d_i$'s are the solutions of the equations

$$\begin{aligned} \sum_{i=0}^{m}\sum_{j=0}^{m} p_{ij}^{l} c_i\, d_j &= 1 - \frac{1}{v}, && \text{if } l = 0; \\ &= -\frac{1}{v}, && \text{if } l = 1, 2, \cdots, m, \end{aligned} \tag{3.13}$$

where

$$c_0 = \frac{r(k-1)}{k}, \qquad c_i = -\frac{\lambda_i}{k}, \qquad i = 1, 2, \cdots, m. \tag{3.14}$$

Since the $m + 1$ equations in (3.13) are not independent, any $m$ of them can be taken and solved with an additional convenient restriction like $\sum d_i = 0$, or, for some $j$, $d_j = 0$. It can be verified that the solutions obtained by taking $d_j = 0$ will be identical with those obtained by Bose and Nair [2].

## 4. Restrictions on association matrices.

LEMMA 4.1: *If* $\mathbf{C} = \sum_{i=0}^{m} c_i \mathbf{B}_i$ *and if*

$$\mathbf{B}_t\mathbf{B}_x + \mathbf{B}_x\mathbf{B}_t = 2\sum_{i=0}^{m} q_{tx}^{i}\mathbf{B}_i, \tag{4.1}$$

*for all* $x, t = 0, 1, \cdots, m$, *then*

$$\mathbf{C}^N = \sum_{i=0}^{m} u_{Ni}\mathbf{B}_i, \tag{4.2}$$

*for all positive integral values of* $N$.

PROOF: The theorem is true for $N = 1$. Assuming the result to be true for $N$, it can be proved for $N + 1$ as follows:

Since a matrix commutes with its powers,

$$\mathbf{C}^{N+1} = \mathbf{C}^N\mathbf{C} = \mathbf{C}\mathbf{C}^N. \tag{4.3}$$

Therefore

$$\mathbf{C}^{N+1} = \tfrac{1}{2}(\mathbf{C}^N\mathbf{C} + \mathbf{C}\mathbf{C}^N). \tag{4.4}$$

On applying (4.2) for $N$, (4.4) becomes

$$\mathbf{C}^{N+1} = \tfrac{1}{2}\sum_{j=0}^{m}\sum_{i=0}^{m} u_{Ni}c_j(\mathbf{B}_i\mathbf{B}_j + \mathbf{B}_j\mathbf{B}_i). \tag{4.5}$$

Hence substituting for $\mathbf{B}_i\mathbf{B}_j + \mathbf{B}_j\mathbf{B}_i$ from (4.1),

$$\mathbf{C}^{N+1} = \sum_{t=0}^{m}\left\{\sum_{i=0}^{m}\sum_{j=0}^{m} u_{Ni}c_jq_{ij}^{t}\right\}\mathbf{B}_t\,. \tag{4.6}$$

Hence by mathematical induction Lemma 4.1 is proved.

THEOREM 4.1: *If the* $\mathbf{C}$*-matrix of a connected design is* $\mathbf{C} = \sum_0^m c_i\mathbf{B}_i$, *and the matrices* $\mathbf{B}_0, \mathbf{B}_1, \cdots, \mathbf{B}_m$ *are the association matrices of the design satisfying conditions* $\mathbf{B}_0 = \mathbf{I}(v)$ *and* $\mathbf{B}_t\mathbf{B}_x + \mathbf{B}_x\mathbf{B}_t = 2\sum_0^m q_{tx}^{i}\mathbf{B}_i$, *then the analysis of the design will be identical with that of a* PBIB *design.*

PROOF: From Corollary 3.2.1, and Lemma 4.1, it follows that a solution $\hat{\mathbf{t}} = \mathbf{AQ}$ of $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$ exists such that

$$\mathbf{A} = \sum_{i=0}^{m} e_i\mathbf{B}_i \tag{4.7}$$

and

$$\begin{aligned}\mathbf{I}(v) - \frac{1}{v}\,\mathbf{E}(v, v) &= \mathbf{AC} = \mathbf{CA},\\ &\tfrac{1}{2} = (\mathbf{AC} + \mathbf{CA}).\end{aligned} \tag{4.8}$$

Simplifying both the sides in terms of $\mathbf{B}_t$'s, we get

$$\mathbf{B}_0 - \frac{1}{v}\sum_{t=0}^{m}\mathbf{B}_t = \sum_{t=0}^{m}\left\{\sum_{j=0}^{m}\sum_{i=0}^{m} c_ie_jq_{ij}^{t}\right\}\mathbf{B}_t\,. \tag{4.9}$$

Hence, on equating the coefficients of the matrices $\mathbf{B}_t$ on both sides of the equation, the $e_i$'s are given by a solution of the equations

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{m} q_{ij}^{t}c_ie_j &= 1 - \frac{1}{v}, &\quad \text{if } t = 0;\\ &= -\frac{1}{v}, &\quad \text{if } t = 1, 2, \cdots, m\end{aligned} \tag{4.10}$$

On comparing equations (4.10) and (3.13), they are seen to be identical except for a change of notation. This implies that one can obtain exactly the same analysis as that of a PBIB design (Bose and Nair [2]), even if the condition (3.5) is replaced by the less stringent condition (4.1).

The combinatorial implication of the condition (4.1) is the following: If two treatments are $i$th associates, then the number of treatments common between the $j$th associates of the first and the $k$th associates of the second, plus the number of treatments common between the $k$th associates of the first and the $j$th associates of the second, is equal to $2q_{jk}^{i}$, and is the same for all the pairs of treatments which are $i$th associates.

Hence the above condition can replace the condition (iiic) of the definition of a PBIB design given by Bose and Shimamoto [3], and the analysis of the design will remain the same. In the case of two associate classes the two conditions are equivalent, but in general they are not.

*Example* 4.1: Consider the following design with parameters: $v = 6$, $b = 9$, $r = 3$, $k = 2$, $m = 4$, $n_1 = n_2 = n_3 = 1$, $n_4 = 2$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$. The plan of the design is given in the Table 4.1 and the association scheme in Table 4.2.

Now consider the treatments 1 and 3. The number of treatments common between the 1st associates of 1 and the 2nd associates of 3 is one, whereas there is no treatment common between the 1st associates of 3 and the 2nd associates of 1. Hence it is clear that this design is not a PBIB as defined by [3], but it can be verified that the design satisfies the condition given in (4.1) and that some of the $q_{jk}$ are

$$q_{12}^{4} = \tfrac{1}{2} = q_{23}^{4} = q_{31}^{4}\,. \tag{4.11}$$

One observes that the above example is obtained by taking two $X$-replications and one $Y$-replication of a $3 \times 2$ simple rectangular lattice design (Harshberger [5]). A similar result will be obtained for any design formed by taking $r_1$ $X$-replications and $r_2$ $Y$-replications ($r_1 \neq r_2$) of a $p(p-1)$ simple rectangular lattice design; but, in general, when $p > 3$, there will be five associate classes.

**5. Further generalisation.** From the foregoing arguments, we can see that an analysis almost similar to that of a PBIB can be derived from only the assumptions that association matrices satisfy the condition (4.1) and that the **C**-matrix and $\mathbf{I}(v)$ are linear combinations of the association matrices. Hence, instead of taking $\mathbf{B}_0 = \mathbf{I}(v)$, we can think of some association matrices yielding $\mathbf{I}(v)$ as their linear combination. This will lead to partitioning treatments into several groups and finally, to the following definition:

*Definition* 5.1: In an incomplete block design, partial balance over intra- and inter-group treatment comparisons will be achieved, if the following conditions are satisfied:

(i) The experimental material is divided into $b$ blocks of $k$ plots each, different treatments being applied to the units in the same block.

TABLE 4.1.
*Plan of the design*

| Replication | 1 | | | 2 | | | 3 | | |
|---|---|---|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Treatments | 1<br>2 | 3<br>4 | 5<br>6 | 1<br>2 | 3<br>4 | 5<br>6 | 1<br>6 | 2<br>3 | 4<br>5 |

TABLE 4.2.
*Association scheme*

| Treatment | Associates | | | |
|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th |
| 1 | 2 | 6 | 4 | 3, 5 |
| 2 | 1 | 3 | 5 | 4, 6 |
| 3 | 4 | 2 | 6 | 1, 5 |
| 4 | 3 | 5 | 1 | 2, 6 |
| 5 | 6 | 4 | 2 | 1, 3 |
| 6 | 5 | 1 | 3 | 2, 4 |

(ii) There are $v$ treatments divided into $h$ groups of $n_1, n_2, \cdots, n_h$ treatments respectively; the treatments of the $i$th group occur in exactly $r_i$ blocks.

(iii) There can be established relations of association between any two treatments satisfying the following requirements:

(a) A treatment of the $i$th group and a treatment of the $j$th group are either $ij$:1th, $ij$:2th, $\cdots$, or $ij$:$m_{ij}$th associates $(i, j = 1, 2, \cdots, h)$; $ij$:$t$th associates are the same as $ji$:$t$th associates.

(b) Each treatment of the $i$th group has exactly $n_{ij:t}ij$:$t$th associates $(j = 1, 2, \cdots, h, t = 1, 2, \cdots, m_{ij})$ and has zero $1k$:$t$th associates $(l \neq i, k \neq i)$.

(c) Given any two treatments which are the $ij$:$t$th associates, the number of treatments common to the $i_1j_1$:$t_1$th associates of the first and $i_2j_2$:$t_2$th associates of the second plus the number of treatments common to the $i_2j_2$:$t_2$th associates of the first and $i_1j_1$:$t_1$th associates of the second is $2\, q_{ij:t}(i_1j_1:t_1, i_2j_2:t_2)$ and is independent of the pair of the treatments with which we start.

(iv) Two treatments which are $ij$:$t$th associates occur together in exactly $\lambda_{ij:t}$ blocks.

Because of the treatment groupings the condition (iiic) of Definition 5.1 can be expressed as follows:

(d) Given any two treatments which are the $ij$:$t$the associates $(i \neq j)$, the first belonging to the $i$th group and the second belonging to the $j$th group, the number of treatments common to $ik{:}t_1$th associates of the first and $jk{:}t_2$th associates of the second is equal to 2 $q_{ij:t}(ik{:}t_1, jk{:}t_2)$ and is independent of the pair of treatments with which we start. Also given any two treatments which are the $ii{:}t$th associates, the number of treatments common to the $ik{:}t_1$ associates of the first and $ik{:}t_2$th associates of the second plus the number of treatments common to the $ik{:}t_2$th associates of the first and $ik{:}t_1$th associates of the second is equal to 2 $q_{ii:t}(ik{:}t_1, ik{:}t_2)$ and is independent of the pair of treatments with which we start.

In these designs the total number of associate classes '$m$' is given by

$$m = \sum_{i \geqq j=1}^{h} m_{ij} . \tag{5.5}$$

The relations between the parameters are

$$\begin{aligned}
&\sum_{i=1}^{h} n_i = v, && \\
n_j = &\sum_{t=0}^{m_{jj}} n_{jj:t} = \sum_{t=1}^{m_{ij}} n_{ij:t}, && i \neq j; \\
&\sum_{l=1}^{m_{ik}} q_{ii:t}(ik{:}t_1, ik{:}l) = n_{ik:t_1}, && \\
&2\sum_{l=1}^{m_{jk}} q_{ij:t}(ik{:}t_1, jk{:}l) = n_{ik:t_1}, && \text{if } i \neq j. \\
&n_{ij:t}q_{ij:t}(ik{:}l, jk{:}t_1) = n_{ik:l}q_{ik:l}(ij{:}t, jk{:}t_1), && \\
& && \text{if } i \neq j,\ i \neq k.
\end{aligned} \tag{5.6}$$

If $\mathbf{B}_{ij:t}$ denotes the association matrix corresponding to the $ij{:}t$th associate class, then

$$\mathbf{B}_{i_1j_1:t_1}\mathbf{B}_{i_2j_2:t_2} + \mathbf{B}_{i_2j_2:t_2}\mathbf{B}_{i_1j_1:t_1} = 2\sum\nolimits^* q_{ij:t}(i_1j_1{:}t_1, i_2j_2{:}t_2)\mathbf{B}_{ij:t}, \tag{5.7}$$

where $\sum^*$ denotes the summation over all the possible values of $ij{:}t$.

Also, the $\mathbf{C}$ matrix can be written in the form

$$\mathbf{C} = \sum\nolimits^* c_{ij:t}\mathbf{B}_{ij:t}, \tag{5.8}$$

where

$$\begin{aligned}
c_{ij:t} &= r_i(k-1)/k, && \text{if } i = j \text{ and } t = 0; \\
&= -\lambda_{ij:t}/k, && \text{otherwise.}
\end{aligned} \tag{5.9}$$

Hence, by Lemma 4.1 and Corollary 3.2.1, the solution of the normal equations is given by $\hat{\mathbf{t}} = \mathbf{AQ}$ where the matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \sum\nolimits^* d_{ij:t}\mathbf{B}_{ij:t}, \tag{5.10}$$

and the constants $d_{ij:t}$ are given by a solution of the equations

$$(5.11) \quad \sum' q_{ij:t}(i_1j_1:t_1, i_2j_2:t_2)c_{i_1j_1:t_1} d_{i_2j_2:t_2} = 1 - v^{-1}, \text{ if } i = j \text{ and } t = 0; \\ = -v^{-1} \text{ otherwise,}$$

where $\sum'$ represents the summation over all the values of $i_1j_1:t_1$ and $i_2j_2:t_2$ Now, from Lemmas 2.1 and 2.2, it can be assumed that a solution, such that **A** is orthogonal to the vector $\mathbf{E}(v, 1)$, exists, and then

$$(5.10) \qquad d_{ii:0} + \sum_{j=1}^{h} \sum_{t=1}^{m_{ij}} n_{ij:t}\, d_{ij:t} = 0.$$

Hence, using $h$ equations of (5.10), $(m + h)$ equations of (5.11) can be reduced to $m$ equations in $m$ unknowns. So it seems that the analysis of the designs given in Definition 5.1 is similar to that of a PBIB design with $m$ associate classes.

In general, these designs involve a large number of associate classes and consequently their analysis is complicated. The minimum number of classes $m$ is 3, when $h = 2$; the analysis for this design is given by Nair and Rao [9].

Another simple case is the one for which $m_{ij} = 1$ and $\lambda_{ij:1} = \lambda$ for all $i \neq j$. In this case the inverse of $\mathbf{C} + (\lambda/k)\mathbf{E}(v, v)$ can be obtained by working out the inverses of $h$ diagonal sub-matrices. Further, if $m_{ii} = 1$ or 2, the computational work will be reduced considerably.

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