SECOND ORDER ROTATABLE DESIGNS IN FOUR OR MORE DIMENSIONS[1]

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0. Introduction. The technique of fitting a response surface is one widely used (especially in the chemical industry) to aid in the statistical analysis of experimental work in which the "yield" of a product depends, in some unknown fashion, on one or more controllable variables. Before the details of such an analysis can be carried out, experiments must be performed at predetermined levels of the controllable factors, i.e., an experimental design must be selected prior to experimentation. Box and Hunter [2] suggested designs of a certain type, which they called rotatable, as being suitable for such experimentation. Such designs permit a response surface to be fitted easily and provide spherical information contours. A second order rotatable design aids the fitting of a second order (i.e., a quadratic) surface.

Let us assume that the measurements of the factors have been coded, permitting the use of cartesian axes in k-dimensional space to describe an experimental design for k factors. Suppose, in an experimental investigation with k factors, N (not necessarily distinct) combinations of level are employed. Thus the group of N experiments which arises can be described by the N points in k dimensions $(x_{1u}, x_{2u}, \cdots, x_{ku})$, $u = 1, 2, \cdots, N$, where, in the uth experiment, factor t is at level x_{tu}. This set of points is said to form a *rotatable arrangement* of the second order in k factors if

$$\sum_u x_{1u}^2 = \sum_u x_{2u}^2 = \cdots = \sum_u x_{ku}^4 = \lambda_2 N,$$

$$\sum_u x_{1u}^4 = \sum_u x_{2u}^4 = \cdots = \sum_u x_{ku}^4 = 3\sum_u x_{iu}^2 x_{ju}^2 = 3\lambda_4 N, \quad (i \neq j),$$

and all other sums of powers and products up to and including order four are zero, where all summations are over $u = 1$ to $u = N$. The point set is said to form a *rotatable design* of second order if the conditions above are satisfied *and* a certain matrix used in a consequent least squares estimation is non-singular. Box and Hunter [2] show that the necessary and sufficient condition for this to be so is $\lambda_4/\lambda_2^2 > k/(k + 2)$, a condition which may always be satisfied merely

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by the addition of points at the center (0, 0, 0) of the design. The inequality becomes an equality only when all the design points lie on a k-dimensional sphere.

When presenting a rotatable design, it is customary to "scale" it. By this it is meant that the scale of the coded controllable variables is chosen in such a way that $\lambda_2 = 1$. The reason for this is as follows. Given a second order design with a *specified* value of λ_4/λ_2^2, there are an infinite number of values possible for $\lambda_2 > 0$. Since these designs can be derived one from another merely by change of scale, we do not regard them as different. Thus the scaling condition $\lambda_2 = 1$ fixes a *particular* design and enables better comparison between two designs with different λ_4/λ_2^2's.

A previous paper by Bose and Draper [1] presented a new method for obtaining infinite classes of second order rotatable designs in three dimensions. In the present paper it is shown how the method previously employed may be used to obtain infinite classes of second order rotatable designs in dimensions higher than three by a suitable generation and combination of basic point sets. Also presented here is a method for adding to a second order rotatable design in $(k - 1)$ dimensions in order to convert it to a second order design in k dimensions—a method useful in situations where it is desired to add an extra variable while making use of data already obtained.

1. The generation of point sets in four or more dimensions. Let $(x_1, x_2, \cdots, x_k)$ be a point in k dimensions and let P_k be the symmetric group of order k, that is, the group of all permutations of k elements. Thus we obtain $k!$ points by operating upon $(x_1, x_2, \cdots, x_k)$ with the elements of P_k. Let R_{ik} be the transformation on k-space which takes the point $(x_1, x_2, \cdots, x_i, \cdots, x_k)$ into the point $(x_1, x_2, \cdots, -x_i, \cdots, x_k)$. From a single point $(x_1, x_2, \cdots, x_k)$, by an application of the $k!$ elements of P_k and/or the k transformations R_{ik}, $(i = 1, 2, \cdots, k)$, we can obtain a set of $2^k k!$ points all of which are distinct, provided that $x_1, x_2, \cdots, x_k$ are all non-zero and distinct. The set, which we shall call $H(x_1, x_2, \cdots, x_k)$ and which consists of the points

$$(\pm x_{i_1}, \pm x_{i_2}, \cdots, \pm x_{i_k})$$

where $i_1, i_2, \cdots, i_k$ run through every possible permutation of $1, 2, \cdots, k$, satisfies the following conditions:

$$\sum_u x_{iu}^2 = (k-1)!\, 2^k(x_1^2 + x_2^2 + \cdots + x_k^2),$$

$$\sum_u x_{iu}^4 = (k-1)!\, 2^k(x_1^4 + x_2^4 + \cdots + x_k^4),$$

$$\sum_u x_{iu}^2 x_{ju}^2 = (k-2)!\, 2^k \sum_{i,j=1}^{k} x_i^2 x_j^2, \qquad (i \neq j),$$

and all odd sums of squares and products up to and including order four are zero, where $i, j = 1, 2, \cdots, k$; and u is summed from 1 through N, the total number of points. Hence

$$\mathrm{Ex}[H(x_1, x_2, \cdots, x_k)] = (k-2)!\,2^k[(k-1)\sum_{i=1}^{k} x_i^4 - 3\sum_{i,j=1}^{k} x_i^2 x_j^2],$$

where $\mathrm{Ex}[H]$ is the *excess* of the point set H and was defined in [1]. For H as defined above, it is the amount by which the pure fourth moment $\sum_u x_{iu}^4$ exceeds three times the mixed fourth moment $\sum_u x_{iu}^2 x_{ju}^2$. The number of points in this set is too large for use in a design and it will be necessary to reduce the size of the set by making several of the x_i equal to one another and/or putting some of the x_i equal to zero.

We note that we could have begun this discussion by considering a set of only $2^{k-1}k!$ points. The group of all permutations has, as a subgroup, the group of all the *even* permutations. The set obtained from a point $(x_1, x_2, \cdots, x_k)$, when $x_1, x_2, \cdots, x_k$ are all distinct, by application of the even permutations only is such that its moments are symmetrical in the way we desire. However, nothing will eventually be gained by this procedure, for, once we make two of the x_i's equal in the more general set, we shall obtain double the set we would have obtained from the set generated by use of even permutations alone. Thus, except in the most general case when $x_1, x_2, \cdots, x_k$ are all distinct, no additional reduction will be achieved by our commencing with a half set. Note that when $k > 3$, a cyclic permutation of coordinates does not achieve symmetry.

When there are k factors, the number of constants to be estimated for a second order model is $1 + k + k + k(k-1)/2$ or $(k^2 + 3k + 2)/2$. For $4 \leqq k \leqq 7$, we have the following table:

k	4	5	6	7
$k^2 + 3k + 2$	15	21	28	36

To obtain a design consisting of a number of points equal to twice the number of constants to be estimated will be regarded here as a very desirable achievement. Unfortunately, because of the large number of moments to be balanced when selecting design points, such an achievement is rarely possible with the method of this paper. Thus some of the designs to be presented are useful only when a fairly large number of design points is allowable. In order to restrict the number of points in a generated set, we shall consider only cases where no more than three of $x_1, x_2, \cdots, x_k$ are distinct.

Consider the fraction of $H(p, \cdots, p; q, \cdots, q; r, \cdots, r)$ which contains all possible points once and once only. Let p occur x times, q occur y times, and r occur z times, so that $x + y + z = k$. Let ν be the number of zeros if any of p, q, and r are zero. For example, if $p \neq 0$, $q \neq 0$, $r = 0$, then $\nu = z$. Hence the desired fraction of the whole set, which may be denoted by $H(p^x, q^y, r^z)$, contains

$$\frac{k!}{x!\,y!\,z!}\,2^{k-\nu}$$

points. Therefore, the set may be written as $[x!\,y!\,z!\,2^{\nu}]^{-1} H(p^x, q^y, r^z)$, in notation consistent with earlier usage (see [1]).

This set has sums of powers and products as follows:

$$\sum_u x_{iu}^2 = \frac{(k-1)!}{x!\,y!\,z!} 2^{k-\nu} [xp^2 + yq^2 + zr^2],$$

$$\sum_u x_{iu}^4 = \frac{(k-1)!}{x!\,y!\,z!} 2^{k-\nu} [xp^4 + yq^4 + zr^4],$$

$$\sum_u x_{iu}^2 x_{ju}^2 = \frac{(k-2)!}{x!\,y!\,z!} 2^{k-\nu} [x(x-1)p^4 + y(y-1)q^4 + z(z-1)r^4 + 2xyp^2q^2 + 2yzq^2r^2 + 2zxr^2p^2],$$

and all other sums of powers and products up to and including order four are zero. Hence

$$\text{Ex}\,\{[x!\,y!\,z!\,2^{\nu}]^{-1} H(p^x, q^y, r^z)\} = \frac{(k-2)!}{x!\,y!\,z!} 2^{k-\nu} [x(k-3x+2)p^4 + y(k-3y+2)q^4 + z(k-3z+2)r^4 - 6xyp^2q^2 - 6yzq^2r^2 - 6zxr^2p^2]$$

is the excess of this generated set of $(k!/x!\,y!\,z!)2^{k-\nu}$ points.

By giving specific values to p, q, r, x, y and z, we shall obtain the more useful sets of this type. In particular, we shall reject any set that contains more than 48 points in four dimensions. If p, q and r are distinct and all are non-zero, there are $4!\,2^4/2 = 192$ points in four dimensions. If $p \neq 0$, $q \neq 0$, $r = 0$, and $z = 1 = \nu$, there are $4!\,2^2/2 = 96$ points in four dimensions. Thus if there are three distinct values for p, q and r, we must put $r = 0$ and allow p and q to occur once only in order to maintain a reasonable number of points. This leads us to consider the generated set

$$S(p, q, 0^{k-2}) = [4(k-2)!]^{-1} H(p, q, 0, \cdots, 0)$$

obtained by setting $r = 0$, $x = y = 1$. The set has $4k(k-1)$ points and its excess is $4(k-1)(p^4 + q^4) - 24p^2q^2$. A short table of the number of points in this set follows:

k	4	5	6	7
$4k(k-1)$	48	80	120	168

$S(p, q, 0^{k-2})$ by itself forms a rotatable arrangement if

$$4(k-1)(p^4 + q^4) - 24p^2q^2 = 0$$

or

$$p^2/q^2 = [3 \pm \sqrt{9 - (k-1)^2}]/(k-1).$$

Since $k \geqq 4$, this is possible only when $k = 4$ and $p^2/q^2 = 1$. But if $p^2 = q^2$, the set can be reduced by half so that

$$(S(p, q, 0^{k-2}) = [8(k - 2)]^{-1}H(p, p, 0, \cdots, 0),$$

consisting of $2k(k - 1)$ points, forms a rotatable arrangement. The single design which arises from this calculation is already known and is called the extension of (25) by Gardiner, Grandage and Hader [3]. If we consider only $S(p, p, 0^{k-2})$ to begin with, this result is trivial, since the excess of $S(p, p, 0^{k-2})$ is $4(k - 4)p^4$ which can be zero *only* if $k = 4$, since $p \neq 0$.

Although it would be possible to use the $4k(k - 1)$ points of $S(p, q, 0^{k-2})$ in combination with other sets to form a rotatable design, we shall not do this because of the large number of points which would be involved. This leads us to mention one other point set that will not be used, given by $z = 0$, $x = 1$, $y = k - 1$, namely, $[(k - 1)!]^{-1}H(p, q^{k-1})$. This contains $k2^k$ points, too many for our purposes as the short table which follows shows.

k	4	5	6	7
$k2^k$	64	160	384	896

By the usual methods, it may be shown that when

$$p^2 = (3 + \sqrt{2k + 4})q^2$$

and

$$q^2 = N/2^k(2 + k + \sqrt{2k + 4}),$$

a rotatable design is obtained, a design already quoted by Gardiner, Grandage and Hader [3] as an extension of their design (23).

Thus it becomes clear that the only point sets which are a fraction of $H(p^x, q^y, r^z)$ and which obey all the required moment conditions except that

TABLE I

Selected point sets

Set	Points of Set	No. of points $N(k)$	Value of $N(k)$				Excess
			$k = 4$	$k = 5$	$k = 6$	$k = 7$	
S_1	$(\pm a, \pm a, \cdots, \pm a)$	2^k	16	32	64	128	$-2^{k+1}a^4$
$\frac{1}{2}S_1$ ($k \geqq$ 5 only)	one half replicate of S_1	2^{k-1}		16	32	64	$-2^k a^4$
S_2	$(\pm c, 0, \cdots, 0)$ and permutations	$2k$	8	10	12	14	$2c^4$
S_3	$(0, \pm f, \cdots, \pm f)$ and permutations	$k2^{k-1}$	32	80	192	448	$-(2k - 5)2^{k-1}f^4$
S_4	$(\pm p, \pm p, 0, \cdots, 0)$ and permutations	$2k(k - 1)$	24	40	60	84	$4(k - 4)p^4$
S_5	$(\pm p, \pm p, \pm p, 0, \cdots, 0)$	$4k(k - 1)(k - 2)/3$	32	80	160	280	$4(k - 2)(k - 7)p^4$

their excess is not zero and which, in addition, contain what we shall consider a reasonable number of points are obtained by setting $z = 0$, $q = 0$ (i.e., letting the coordinates take two distinct values, one of which is zero) or setting $z = y = 0$ (i.e., allowing only one possible value for the coordinate). Proceeding in this way, we consider the five sets listed in Table I as suitable for combination with one another for the formation of rotatable designs. We shall not use the fractional set notation here because of its unwieldiness. Other possible sets are neglected on the grounds that they contain too many points for our purposes. Several features of the sets above are immediately noticeable:

S_1 and S_3 have negative excess.

S_5 has negative excess if $k < 7$, positive excess if $k > 7$ and zero excess if $k = 7$. (Thus if $k = 7$, the points of S_5 form a rotatable arrangement; the design thus formable will be derived later).

S_2 has positive excess.

S_4 has positive excess if $k > 4$.

These facts determine the combinations of sets we shall choose to form several infinite classes of rotatable designs analogous to those formed in three dimensions.

2. Infinite classes of second order designs in four or more dimensions. The generated sets may now be combined in the same way as was done previously in the three dimensional case [1]. Six of the more useful combinations are presented in Table II. All of the previously known designs (apart from the two mentioned separately in Section 1) occur as special cases of the classes in the table.

3. A method of constructing a second order design in k dimensions using a second order design in $(k - 1)$ dimensions. Select a second order rotatable arrangement of points in $(k - 1)$ dimensions to which the scaling condition $\lambda_2 = 1$ has not yet been applied. Then we shall have (say) N' points

$$(x_{1u}, x_{2u}, \cdots, x_{k-1,u}) \qquad 1 \leqq u \leqq N',$$

for which

$$\sum_u x_{iu}^2 = A \neq N',$$

$$\sum_u x_{iu}^4 = 3\sum_u x_{iu}^2 x_{ju}^2 = 3C, \qquad \text{say, } (i \neq j), \quad i, j = 1, 2, \cdots, (k-1),$$

and all odd sums of powers and products up to and including order four are zero. Consider all the points obtained by adding a further coordinate $x_{ku} = \pm b$ to the coordinates of the $(k - 1)$ dimensional points. Thus we obtain a point set in k dimensions

$$(x_{1u}, x_{2u}, \cdots, x_{k-1,u}, \pm b), \qquad u = 1, 2, \cdots, N' \tag{3.1}$$

consisting of $2N'$ points.

TABLE II

Classes of rotatable designs

Set	αS_1	S_2	S_3	S_4	S_5	$N-n_o$ Number of points in each class for $4 \leq k \leq 7$ (assuming full replicate of S_1, $\alpha = 1$)				Range of first parameter ratio on which class depends	Second parameter ratio in terms of first	Design point coordinate values in terms of N and the parameter ratios	Value of λ_4/λ_2^2
Parameter	a	c	f	p	p								
No. of sets used for class						k=4	k=5	k=6	k=7				
	1	2				32	52	88	156	$0 \leq x \leq \sqrt{2^{k-1}\alpha}$	$y = \sqrt{2^k\alpha - x^2}$	$a = [N/2(x+y+\alpha 2^{k-1})]^{1/2}$ $c_1 = x^{1/2}a, \quad c_2 = y^{1/2}a$	$\alpha 2^k a^4/N$
	2	1				40	74	140	270	$0 \leq x \leq \sqrt{2^{-k}/\alpha_1}$	$y = \sqrt{(2^{-k} - \alpha_1 x^2)/\alpha_2}$	$c = [N/2(1+2^{k-1}(\alpha_1 x + \alpha_2 y))]^{1/2}$ $a_1 = x^{1/2}c, \quad a_2 = y^{1/2}c$	c^4/N
		2	1			48	100	216	476	$0 \leq x \leq \sqrt{(2k-5)2^{k-3}}$	$y = \sqrt{(2k-5)2^{k-2} - x^2}$	$f = [N/2(x+y+(k-1)2^{k-2}]^{1/2}$ $c_1 = x^{1/2}f, \quad c_2 = y^{1/2}f$	$(k-2)2^{k-1}f^4/N$
	2			1		-	104	188	340	$0 \leq x \leq \sqrt{2^{1-k}(k-1)/\alpha_1}$	$y = [2^{1-k}(k-4) - \alpha_1 x^2]^{1/2}/\alpha_2$	$p = [N/(4(k-1)+2^k(\alpha_1 x + \alpha_2 y))]^{1/2}$ $a_1 = x^{1/2}p, \quad a_2 = y^{1/2}p$	$2(k-2)p^4/N$
	1	1		1		-	82	136	226	$0 \leq x \leq \sqrt{\alpha 2^k}$	$y = [\alpha 2^k - x^2]^{1/2}/2(k-4)$	$a = [N/(2x+4(k-1)y+\alpha 2^k)]^{1/2}$ $c = x^{1/2}a, \quad p = y^{1/2}a$	$(4y+\alpha 2^k)a^4/N$
		2			1	48	100	184	308	$0 \leq x \leq \sqrt{(k-2)(7-k)}$	$y = \sqrt{2(k-2)(7-k) - x^2}$	$p = [N/2(x+y+(k-1)(k-2))]^{1/2}$ $c_1 = x^{1/2}p, \quad c_2 = y^{1/2}p$	$8(k-2)p^4/N$

Note: Set $\alpha = \frac{1}{2}$ for a half replicate of S_1 ($k \geqq 5$ only).

Consider the point sets

$$(0, 0, \cdots, 0, \pm p) \tag{3.2}$$

$$(0, 0, \cdots, 0, \pm q). \tag{3.3}$$

Then the values of p and q may be so adjusted that the three sets (3.1), (3.2) and (3.3), together with any center points which may be added, form a second order rotatable design in k dimensions. The number of points in the derived design is $N = 2N' + 4 + n_0$.

This may be shown as follows. The addition of the extra coordinate x_k to the $(k - 1)$ dimensional point set contributes only to $\sum_u x_{ku}^2$, $\sum_u x_{ku}^4$ and $\sum_u x_{iu}^2 x_{ku}^2$, $i = 1, 2, \cdots, (k - 1)$. It is clear that moments which were previously zero remain zero and that odd sums of powers and products involving x_k are zero because x_k is constant $(\pm b)$. Thus these sums of powers and products will be zero either for each set of N' points separately or else for the two sets combined. Thus for all of the $N = 2N' + 4 + n_0$ points,

$$\begin{aligned}
&\sum_u x_{iu}^2 = 2A, && 1 \leqq i \leqq k - 1,\\
&\sum_u x_{iu}^4 = 6C, && 1 \leqq i \leqq k - 1,\\
&\sum_u x_{iu}^2 x_{ju}^2 = 2C, && 1 \leqq i \neq j \leqq k - 1,\\
&\sum_u x_{ku}^2 = 2N'b^2 + 2(p^2 + q^2), &&\\
&\sum_u x_{ku}^4 = 2N'b^4 + 2(p^4 + q^4), &&\\
&\sum_u x_{iu}^2 x_{ku}^2 = 2b^2 A, && 1 \leqq i \leqq k - 1.
\end{aligned} \tag{3.4}$$

where u is summed from 1 through N, and all other sums of powers and products up to and including order four are zero.

There will be symmetry in the moments up to fourth order provided that

$$\begin{aligned}
p^2 + q^2 + N'b^2 &= A,\\
p^4 + q^4 + N'b^4 &= 3C,\\
Ab^2 &= C.
\end{aligned} \tag{3.5}$$

Thus, if these conditions can be satisfied by choice of p, q and b, the N points will automatically form a second order rotatable arrangement, since the equations above imply that $\sum_{u=1}^N x_{iu}^4 = 3\sum_{u=1}^N x_{iu}^2 x_{ju}^2 = 6C$ for $i \neq j$ and $i, j =$

$1, 2, \cdots, k$. From (3.5)

$$
\begin{aligned}
b^2 &= C/A, \\
p^2 + q^2 &= (A^2 - N'C)/A, \qquad (3.6)\\
p^4 + q^4 &= C(3A^2 - N'C)/A^2
\end{aligned}
$$

Solving the simultaneous equations of (3.6) we obtain

$$p^2, q^2 = [(A^2 - N'C) \pm \sqrt{2C(3A^2 - N'C) - (A^2 - N'C)^2}]/2A. \tag{3.7}$$

We now apply the scaling condition $\lambda_2 = 1$, which gives $2A = N$ or $A = N/2$, and this must be substituted into the expressions for p^2 and q^2 above. Hence,

$$p^2, q^2 = [(N^2 - 4N'C) \pm \sqrt{8C(3N^2 - 4N'C) - (N^2 - 4N'C)^2}]/8A. \tag{3.8}$$

In order that both p^2 and q^2 should be real and non-negative, i.e., in order that a new design should be obtainable, the original design must satisfy the condition

$$2 \geqq \phi \geqq 1, \tag{3.9}$$

where $\phi = (A^2 - N'C)^2/C(3A^2 - N'C)$. It is necessary to determine in the usual way for individual cases whether or not the addition of center points is required.

As an illustration of the method we now derive a second order design in four dimensions from a second order design in three dimensions. Consider the well known cube plus octahedron arrangement in three dimensions with no center points, given by

$$
\begin{array}{ccc}
(\pm a, & \pm a, & \pm a), \\
(\pm c, & 0, & 0), \\
(\ 0, & \pm c, & 0), \\
(\ 0, & 0, & \pm c),
\end{array} \tag{3.10}
$$

In the notation of this section,

$$3C = 8a^4 + 2c^4 = 24a^4 = 3(C).$$

Hence

$$c^4 = 8a^4 = (2^{3/4}a)^4 = (1.682a)^4,$$

so that

$$
\begin{aligned}
C &= 8a^4, \\
A &= 8a^2 + 2c^2 = 4(2 + \sqrt{2})a^2, \quad \text{and} \qquad (3.11)\\
N' &= 14.
\end{aligned}
$$

Thus $\phi = 1.55$, and use of the method is possible.

Consider the point set in four dimensions given by

$$\begin{array}{rrrr}
(\pm a, & \pm a, & \pm a, & \pm b), \\
(\pm c, & 0, & 0, & \pm b), \\
(0, & \pm c, & 0, & \pm b), \\
(0, & 0, & \pm c, & \pm b), \\
(0, & 0, & 0, & \pm p), \\
(0, & 0, & 0, & \pm q).
\end{array}$$

These points form a second order rotatable arrangement if the solutions for p^2 and q^2 which result from substitution of (3.11) into (3.7) are real and non-negative, which they are, since $2 \geqq \phi \geqq 1$. Performing the calculation, we find that $p^2 = 4.196400\ a^2$, $q^2 = 1.259446\ a^2$, so that $p = 2.049\ a$, $q = 1.122\ a$. We recall that $c = 1.682\ a$, while $b = \sqrt{C/A} = 0.765\ a$. Thus we have a second order rotatable arrangement in four dimensions with 32 points given by

$$\begin{array}{rrrr}
(\pm a, & \pm a, & \pm a, & \pm 0.765\ a), \\
(\pm 1.682\ a, & 0, & 0, & \pm 0.765\ a), \\
(0, & \pm 1.682\ a, & 0, & \pm 0.765\ a), \\
(0, & 0, & \pm 1.682\ a, & \pm 0.765\ a), \\
(0, & 0, & 0, & \pm 2.049\ a), \\
(0, & 0, & 0, & \pm 1.122\ a),
\end{array}$$

where a is to be determined by application of the scaling condition $\lambda_2 = 1$. The separate sets which comprise the arrangement have radii $\sqrt{3a^2 + b^2}$, $\sqrt{c^2 + b^2}$ p and q, that is $a\sqrt{3.585}$, $a\sqrt{3.414}$, p and q, or $1.189\ a$, $1.848\ a$, $2.049\ a$ and $1.122\ a$. Thus the arrangement is not spherical, and it can be used as a design without addition of center points. However,

$$\lambda_4/\lambda_2^2 = 16a^4/N = .02144N,$$

where $N = 32 + n_0$. Hence $\lambda_4/\lambda_2^2 = .686$, when $n_0 = 0$. This is greater than the singular value of .667 (for $k = 4$), but not very much so; it would therefore be preferable to use a few center points with this design. When, for example, $n_0 = 4$, $\lambda_4/\lambda_2^2 = .772$. After deciding on the number of center points to be used, we can determine the value of a which specifies the design points from the scaling condition. This gives

$$a^2 = (2 - \sqrt{2})N/16 = .03661\ N.$$

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