EXPECTATIONS OF FUNCTIONALS ON A STOCHASTIC PROCESS

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1. Introduction. Let $\{x(t), 0 \leqq t < \infty\}$ be a separable stochastic process with stationary, independent increments, for which $x(0) = 0$ and whose characteristic function is

$$E\{e^{i\xi x(t)}\} = e^{at(\cos\xi - 1)}, \qquad a > 0.$$

One may verify that, if $0 \leqq t_1 < t_2 < \cdots < t_k < \infty$ and m_j is an integer,

$$P\{x(t_k) = m_k, x(t_{k-1}) = m_{k-1}, \cdots, x(t_1) = m_1\}$$
$$= e^{-a(t_k - t_{k-1}) - \cdots - a(t_2 - t_1) - at_1} I_{m_k - m_{k-1}}[a(t_k - t_{k-1}] \cdots I_{m_2 - m_1}[a(t_2 - t_1]I_{m_1}[(at_1)],$$

where $I_n(x) = i^{-n} \cdot J_n(ix)$, $J_n(x)$ being the Bessel function of the first kind. By separability the sample functions, $x(t)$, of this process are simple functions which assume integral values on intervals. They may be interpreted as the monetary gain in coin tossing at random times. To be more precise, $x(t)$ is the sum of a random number, $N(t)$, of independent, identically distributed Bernoulli variables with distribution $P\{x = -1\} = P\{x = 1\} = \frac{1}{2}$, where $N(t)$ is the sample function of a Poisson process ([1], page 398). This process is important in the theory of collective risk and has been studied by Täcklind [2]. Certain similarities between it and the Wiener process led us to attempt to find the expected value of some functionals on this process using a method developed by Kac ([3], Section 3). The principal result of this paper is the following theorem.

Theorem. *Let*

$$\Psi_n = \int_0^\infty e^{-st} E\left\{\exp\left[-u \int_0^t V(x(\tau))\, d\tau\right], x(t) = n\right\} dt \tag{1.1}$$

where V is non negative. Then Ψ_n satisfies the difference system

$$\begin{gathered} \Psi_{n+1} - (2/a)(s + a + uV_n)\Psi_n + \Psi_{n-1} = -(2/a)\delta_{n,0}, \\ \Psi_n \to 0 \quad \text{as} \quad n \to \pm\infty, \end{gathered} \tag{1.2}$$

where V_n is the value of the function V when $x = n$. (Note: For any function K, $E\{K(x), x(t) = n\}$ means $E\{K(x)\chi(x)\}$ where $\chi(x) = 1$ if $x(t) = n$ and $\chi(x) = 0$ otherwise.)

In Section 2 we outline the proof of the theorem and in Section 3 we give some illustrative examples.

2. Proof of Theorem. In order that we may easily interchange the order of certain limits, we assume first that V is bounded. This restriction will be removed later in the proof. Following the method and notation of Kac we define inductively

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$$(2.1) \qquad Q_k(n, t) = \int_0^t \sum_{m=-\infty}^{\infty} V(m) e^{-a(t-\tau)} I_{n-m}[a(t-\tau)] Q_{k-1}(m, \tau)\, d\tau,$$

where $Q_0(n, t) = e^{-at} I_n(at)$. This gives

$$Q_k(n, t) = \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_k=-\infty}^{\infty} V(m_1) \cdots V(m_k)$$
$$\cdot \exp[-a(t-\tau_k) - a(\tau_k - \tau_{k-1}) - \cdots - a(\tau_2 - \tau_1) - a\tau_1]$$
$$\cdot I_{n-m_k}[a(t-\tau_k)] \cdot I_{m_k - m_{k-1}}[a(\tau_k - \tau_{k-1})] \cdots I_{m_2-m_1}[a(\tau_2 - \tau_1)] I_{m_1}(a\tau_1)\, d\tau_1 \cdots d\tau_k$$
$$= \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} E\{V[x(\tau_1)]V[x(t_2)] \cdots V[x(\tau_k)], x(t) = n\}\, d\tau_1 \cdots d\tau_k$$
$$= E\left\{\int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} V[x(\tau_1)]V[x(\tau_2)] \cdots V[x(\tau_k)]\, d\tau_1 \cdots d\tau_k, x(t) = n\right\}.$$

Thus

$$(2.2) \quad Q_k(n, t) = E\left\{\frac{1}{k!}\left(\int_0^t V[x(\tau)]\, d\tau\right)^k, x(t) = n\right\} \leqq \frac{1}{k!} t^k M^k P\{x(t) = n\},$$

where M is an upper bound for V. We define

$$(2.3) \qquad Q(n, t, u) = \sum_{k=0}^{\infty} (-1)^k u^k Q_k(n, t).$$

Using (2.2), we obtain

$$(2.4) \qquad Q(n, t, u) = E\left\{\exp\left[-u \int_0^t V(x(\tau))\, d\tau\right], x(t) = n\right\}.$$

We see immediately that

$$(2.5) \qquad Q(n, t, u) \leqq P\{x(t) = n\} = e^{-at} I_n(at).$$

From (2.1) and (2.3), we find

$$(2.6) \qquad \begin{aligned} &Q(n, t, u) - Q_0(n, t) \\ &\qquad = -u \sum_{m=-\infty}^{\infty} \int_0^t V(m) e^{-a(t-\tau)} I_{n-m}[a(t-\tau)] Q(m, \tau, u)\, d\tau. \end{aligned}$$

Now let, [see (1.1)], $\Psi_n = \int_0^\infty e^{-st} Q(n, t, u)\, dt$, and take the Laplace Transform of both sides in (2.6). This gives (see [4] page 131, Formula 6)

$$(2.7) \qquad \psi_n = \frac{A^{|n|}}{c} - \frac{u}{c} \sum_{m=-\infty}^{\infty} A^{|n-m|} \psi_m V_m,$$

where $c = (s^2 + 2as)^{\frac{1}{2}}$ and $A = a/(s + a + c)$. From (2.7) it can be shown that, for $n \neq 0$,

$$\Psi_{n+1} + \Psi_{n-1} = [(A + A^{-1}) + (u/c) V_n (A^{-1} - A)]\Psi_n,$$

with a similar formula for $n = 0$. The difference system (1.2) now follows easily, the boundary conditions coming from the estimate in (2.5).

Now suppose that $V(x)$ is unbounded and define $V_M(x)$ as $V(x)$ if $V(x) \leqq M$ and 0 otherwise. We have then from (1.2) the difference equation

$$\Psi_{M,n+1} - \frac{2}{a}(s + a + uV_{M,n})\Psi_{M,n} + \Psi_{M,n-1} = -\frac{2}{a}\delta_{n,0}, \tag{2.8}$$

where

$$\Psi_{M,n} = \int_0^\infty e^{-st}E\left\{\exp\left[-u\int_0^t V_M(x(\tau))\,d\tau\right], x(t) = n\right\} dt.$$

By bounded convergence $\lim_{M\to\infty} \Psi_{M,n} = \Psi_n$. Thus, taking limits on both sides of (2.8), we obtain the desired result.

3. Examples.

(a) Let $V(x) = 0$ if $-p < x < q$ and 1 otherwise where p and q are positive integers. We define $\Psi_n^* = \lim_{u\to\infty} \Psi_n$ and note that

$$\Psi_n^* = \int_0^\infty e^{-st}P\{-p < x(\tau) < q \text{ for } 0 \leqq \tau \leqq t, x(t) = n\}\, dt.$$

We observe that, for $-p < n < q$, Ψ_n^* satisfies the difference equation in (1.2) corresponding to this V; hence,

$$\Psi_n^* = \begin{cases} 0 & n \geqq q \\ D_1A^n + D_2A^{-n} & 0 \leqq n \leqq q \\ E_1A^n + E_2A^{-n} & -p \leqq n \leqq 0 \\ 0 & n \leqq -p \end{cases} \tag{3.1}$$

where D_1, D_2, E_1, and E_2 are suitable constants, and

$$A = a/[s + a + (s^2 + 2as)^{\frac{1}{2}}].$$

Let

$$\Psi = \sum_{n=-\infty}^{+\infty} \Psi_n^* = \int_0^\infty e^{-st}P\{-p < x(\tau) < q \text{ for } 0 \leqq \tau \leqq t\}\, dt.$$

Using (3.1), we obtain

$$\Psi = 1/s\cdot(1 - A^p)(1 - A^q)/(1 - A^{p+q}). \tag{3.2}$$

In the special case where $p = \infty$, (3.2) is easily inverted giving

$$P\{\sup_{0\leqq\tau\leqq t} x(\tau) < q\} = 1 - q\int_0^t e^{-a\tau}(I_q(a\tau)/\tau)\, d\tau,$$

a result obtained by Baxter and Donsker in ([5], Section 4).

(b) Let $V(x) = x^2$. The difference equation in (1.2) then becomes

$$\Psi_{n+1} - \frac{2}{a}(s + a + un^2)\Psi_n + \Psi_{n-1} = -\frac{2}{a} \cdot \delta_{n,0}.$$

We define $\Psi(\xi) = \sum_{n=-\infty}^{\infty} e^{2in\xi}\Psi_n = \sum_{n=-\infty}^{\infty} \Psi_n \cos 2n\xi$. Then $\Psi(\xi)$ satisfies the differential system

$$\Psi''(\xi) - [(4/u)(s + a) - (4a/u)\cos 2\xi]\Psi(\xi) = -4/u, \tag{3.3}$$
$$\Psi'(0) = \Psi'(\pi/2) = 0.$$

To solve (3.3) we consider the differential equation

$$\Psi''(\xi) + [\mu - (4/u)(s + a) + (4a/u)\cos 2\xi]\Psi(\xi) = 0 \tag{3.4}$$

with the same boundary conditions as in (3.3). The Green's function $G(\xi, \eta)$ for (3.4) is given by

$$G(\xi, \eta) = \sum_{k=0}^{\infty} \phi_k(\xi)\phi_k(\eta)/\mu_k \tag{3.5}$$

where μ_k and $\phi_k(\xi)$ are the eigenvalues and normalized eigenfunctions of (3.4) respectively. By Mercer's Theorem ([6], p. 138) the convergence is uniform in ξ and η, the μ_k's all being positive (at least for large s). The solution for $\Psi(\xi)$ in (3.3) is thus given by

$$\Psi(\xi) = (4/u)\int_0^{\pi/2} G(\xi, \eta)\, d\eta = (4/u)\sum_{k=0}^{\infty} \frac{\phi_k(\xi)}{\mu_k}\int_0^{\pi/2} \phi_k(\eta)\, d\eta. \tag{3.6}$$

On the other hand, if we let $\lambda = \mu - (4/u)(s + a)$, (3.4) is seen to be Mathieu's equation. Using the notation of ([4], p. 46), we find that

$$\phi_k(\xi) = b_k ce_{2k}(\xi) = b_k \sum_{n=0}^{\infty} A_{2k,2n} \cos 2n\xi$$

where $b_k = (2/\pi)^{\frac{1}{2}}$ if $k = 0$ and $b_k = 2/(\pi)^{\frac{1}{2}}$ if $k \neq 0$.
Upon substituting in (3.6), we obtain

$$\Psi(\xi) = 2\pi/u \sum_{k=0}^{\infty} b_k^2 A_{2k,0}/\mu_k \sum_{n=0}^{\infty} A_{2k,2n} \cos 2n\xi$$
$$= (4/u)\sum_{k=0}^{\infty} A_{2k,0}/\mu_k \sum_{n=-\infty}^{+\infty} A_{2k,2n} \cos 2n\xi$$

where $A_{2k,2n} = A_{2k,-2n}$ for $n < 0$. After interchanging the order of summation, we have

$$\Psi(\xi) = (4/u)\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \{A_{2k,0}A_{2k,2n}/[\lambda_k + (4/u)(s + a)]\} \cos 2n\xi.$$

By the uniqueness of the Fourier coefficients, it follows that

$$\Psi_n = (4/u) \sum_{k=0}^{\infty} A_{2k,0} A_{2k,2n} / [\lambda_k + (4/u)(s + a)].$$

Inverting with respect to s, we obtain

$$E\left\{\exp\left[-u \int_0^t [x(\tau)]^2 \, d\tau\right], x(t) = n\right\} = \sum_{k=0}^{\infty} A_{2k,0} A_{2k,2n} e^{-(a+u\lambda_k/4)t}.$$

REFERENCES

1. J. L. Doob, *Stochastic Processes*, John Wiley and Sons, New York, 1953.
2. Sven Täcklind, "Sur le risque de ruine dans des jeux inéquitables," *Skand. Aktuarietids.*, Vol. 25, (1942), pp. 1–42.
3. M. Kac, "On some connections between probability theory and differential and integral equations," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1951, pp. 189–215.
4. Wilhelm Magnus and Fritz Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Chelsea, New York, 1949.
5. Glen Baxter and M. D. Donsker, "On the distribution of the supremum functional for processes with stationary independent increments," *Trans. Amer. Math. Soc.*, Vol. 85, (1957), pp. 73–87.
6. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953.