TABLES FOR UNBIASED TESTS ON THE VARIANCE OF A NORMAL POPULATION

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1. Summary. Tables of critical values defining an unbiased test are given for testing the null hypothesis $\sigma^2 = \sigma_0^2$ against the two-sided alternative hypothesis $\sigma^2 \neq \sigma_0^2$ where σ^2 is the variance of a normal population. Use of the tabulated values leads to the logarithmically shortest confidence limits for σ^k, $k > 0$. The critical values have been found to five significant figures for $\alpha = .01, .05, .10$ where α is the size of the critical region and for $\nu = 1(1)20, 24, 30, 40, 60, 120$ where ν equals the degrees of freedom of the chi-square distribution. A least squares equation is given which may be used to find the critical values when $\nu \geqq 10$ for $\alpha = .01, .05, .10$.

Since submitting a revision of the present paper for publication, the article by Tate and Klett [6] appeared necessitating a second revision. An explanation of the overlap of the present paper with [6] is included in Section 5. In addition, a brief discussion of [2], which was called to the writer's attention by the editor, has been added in Section 5.

2. The Problem. Suppose that a random sample $x_1, x_2, \cdots, x_n$ is taken from a normal population having mean μ and variance σ^2 with the thought of either testing the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the two-sided alternative hypothesis $H_1 : \sigma^2 \neq \sigma_0^2$, or constructing two-sided confidence intervals for σ^2. The usual (equal-tail) procedure is to reject H_0 at significance level α if $(n-1)s^2/\sigma_0^2 \leqq \chi^2_{n-1,1-\alpha/2}$, or $(n-1)s^2/\sigma_0^2 \geqq \chi^2_{n-1,\alpha/2}$ where

$$(n-1)s^2 = \textstyle\sum_{i=1}^n (x_i - \bar{x})^2, \; n\bar{x} = \sum_{i=1}^n x_i,$$

and where $\chi^2_{\nu,\beta}$ is the upper β quantile of the chi-square distribution with ν degrees of freedom. A set of confidence intervals for σ^2 with confidence coefficient $1 - \alpha$ is then $(n-1)s^2/\chi^2_{n-1,\alpha/2} \leqq \sigma^2 \leqq (n-1)s^2/\chi^2_{n-1,1-\alpha/2}$. It is well-know that such a procedure leads to a biased test. For a discussion of the choice of a critical region and unbiased tests, see [3] and [5].

3. The Solution. Let $f_\nu(t)$ denote the p.d.f. of χ^2_ν, let $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = \int_0^A f_\nu(t)\,dt$, $\alpha_2 = \int_B^\infty f_\nu(t)\,dt$, and let $P(\lambda)$ be the power of the test based on the critical values A and B when $\lambda = \sigma^2/\sigma_0^2$, then

$$P(\lambda) = 1 - \int_{A/\lambda}^{B/\lambda} f_\nu(t)\,dt.$$

It has been shown, ([3], [5]), that if we choose A and B so that $P(\lambda)$ is a minimum at $\lambda = 1$, subject to

$$\int_A^B f_\nu(t)\,dt = 1 - \alpha, \tag{1}$$

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we are led to

$$Af_\nu(A) = Bf_\nu(B). \tag{2}$$

In the last paragraph of Section 2 on page 8 of [3] Neyman and Pearson give their personal reasons for recommending an unbiased test; they also give in their Table I on page 19 the values of A and B and the corresponding values of α_1 and α_2 for the following five cases: $\alpha = .10$, $\nu = 2, 9$; $\alpha = .02$, $\nu = 2, 3, 9$. Scheffé [5] has shown that choosing A and B in this manner makes the ratio B/A a minimum which leads to the logarithmically shortest confidence intervals for σ^k, $k > 0$. In addition to being unbiased, K. V. Ramachandran [4] has shown that the power of such a test procedure has the monotonicity property.

4. Method of Solution. Equations (1) and (2) were solved simultaneously for A and B to five significant figures for $\nu = 1(1)20(2)120$ using Newton's iteration method. Table I contains the values of A and B for $\alpha = .01, .05, .10$ and $\nu = 1(1)20, 24, 30, 40, 60, 120$. For convenience, the corresponding values of α_1 and α_2, which are of interest in themselves, are given in Table II.

The results for $\nu = 12(2)120$ were standardized and used to fit a least squares equation of the form:

$$Y = a_0 + a_1 m^{-\frac{1}{2}} + a_2 m^{-1} + a_3 m^{-\frac{3}{2}} \tag{3}$$

where $m = \nu/2$, $Y = (t - m)m^{-\frac{1}{2}}$, where $t = A/2$ or $B/2$ depending on whether we want the lower or upper critical values, respectively. The form (3) was chosen since there is a well-known asymptotic expansion for the percentage points of the chi-square distribution in powers of $\nu^{-\frac{1}{2}}$ derived by Campbell, [1]. Equation (3) was tested for $\nu = 10(2)120$, $\alpha = .01, .05, .10$ and found to give results which are accurate to at least four significant figures. The least squares coefficients are given in the following table.

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
	Lower	Upper	Lower	Upper	Lower	Upper
a_0	−2.5753	2.5760	−1.9598	1.9600	−1.6448	1.6449
a_1	2.2023	2.2098	1.2772	1.2805	.90037	.90181
a_2	−.62981	.69167	−.35077	.36871	−.24991	.25783
a_3	.051724	.23691	.10022	.18198	.092008	.13944

5. Related Tables. Ramachandran [4] gives to two decimals the values of A and B for $\alpha = .05$ and $\nu = 2(1)8(2)24, 30, 40, 60$ (Table 744).

Tate and Klett [6] give to four decimals the values of A and B for $\alpha = .10, .05, .01, .005, .001$ and $\nu = 2(1)29$ (Table 680). Values in Table I which also appear in [6] are those for $\alpha = .10, .05, .01$ and $\nu = 2(1)20, 24$. Values in Table I but not appearing in [6] are those for $\alpha = .10, .05, .01$ and $\nu = 1, 30, 40, 60, 120$. Rather than delete the points which overlap, it was thought better to leave Table I as originally computed for completeness and for comparison with values

TABLE I

Values for unbiased tests on the variance of a normal population

ν	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
	A	*B*	*A*	*B*	*A*	*B*
1	.00013422	11.345	.0031593	7.8168	.012116	6.2595
2	.017469	13.285	.084727	9.5303	.16763	7.8643
3	.10105	15.127	.29624	11.191	.47639	9.4338
4	.26396	16.901	.60700	12.802	.88265	10.958
5	.49623	18.621	.98923	14.369	1.3547	12.442
6	.78565	20.296	1.4250	15.897	1.8746	13.892
7	1.1221	21.931	1.9026	17.392	2.4313	15.314
8	1.4978	23.533	2.4139	18.860	3.0173	16.711
9	1.9068	25.106	2.9532	20.305	3.6276	18.087
10	2.3444	26.653	3.5162	21.729	4.2582	19.446
11	2.8069	28.178	4.0994	23.135	4.9063	20.789
12	3.2912	29.683	4.7005	24.525	5.5696	22.119
13	3.7949	31.170	5.3171	25.900	6.2462	23.436
14	4.3161	32.641	5.9477	27.263	6.9348	24.742
15	4.8530	34.097	6.5908	28.614	7.6339	26.039
16	5.4041	35.540	7.2453	29.955	8.3427	27.326
17	5.9683	36.971	7.9100	31.285	9.0603	28.605
18	6.5444	38.390	8.5842	32.607	9.7859	29.876
19	7.1316	39.798	9.2670	33.921	10.519	31.140
20	7.7289	41.197	9.9579	35.227	11.259	32.398
24	10.207	46.706	12.791	40.383	14.276	37.372
30	14.138	54.762	17.206	47.958	18.943	44.697
40	21.094	67.793	24.879	60.275	26.987	56.645
60	35.967	92.907	40.965	84.178	43.698	79.926
120	84.347	164.51	92.106	153.03	96.258	147.36

TABLE II

Values of α_1 and α_2 associated with the unbiased tests

ν	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
	α_1	α_2	α_1	α_2	α_1	α_2
1	.009243	.000757	.044824	.005176	.087647	.012353
2	.008697	.001303	.041479	.008521	.080398	.019602
3	.008289	.001711	.039266	.010734	.075954	.024046
4	.007980	.002020	.037717	.012283	.072964	.027036
5	.007739	.002261	.036570	.013430	.070796	.029204
6	.007547	.002453	.035680	.014320	.069137	.030863
7	.007389	.002611	.034965	.015035	.067816	.032184
8	.007256	.002744	.034376	.015624	.066735	.033265
9	.007144	.002856	.033879	.016121	.065827	.034173
10	.007046	.002954	.033453	.016547	.065053	.034947
11	.006960	.003040	.033083	.016917	.064381	.035619
12	.006884	.003116	.032757	.017243	.063792	.036208
13	.006816	.003183	.032468	.017532	.063269	.036730
14	.006756	.003244	.032208	.017792	.062802	.037198
15	.006700	.003300	.031974	.018026	.062381	.037619
16	.006650	.003350	.031761	.018239	.061998	.038002
17	.006604	.003396	.031567	.018433	.061649	.038351
18	.006561	.003439	.031388	.018612	.061329	.038671
19	.006522	.003478	.031223	.018777	.061034	.038966
20	.006485	.003515	.031070	.018930	.060760	.039240
24	.006362	.003638	.030555	.019445	.059840	.040160
30	.006223	.003777	.029981	.020019	.058817	.041183
40	.006064	.003936	.029325	.020675	.057649	.042352
60	.005873	.004127	.028540	.021460	.056256	.043745
120	.005620	.004380	.027509	.022491	.054430	.045569

given in [2], [4], and [6]. Since five significant figures are given in Table I whereas four decimals are given in [6], the two tables supplement each other.

Fertig and Proehl [2] give to four decimals the values of P for $\nu = 1(1)50$ and $k = .435(.005).500(.010).700(.051).500(.3)3.0$ where P is the probability of a more extreme result than the one observed in the sample when H_0 is true, using an unbiased critical region. Specifically, P is the probability of a smaller r than the one observed, where $r = tf_\nu(t)$, $t = \nu x$, $x = s^2/\sigma_0^2$. The procedure in using the table in [2] is as follows: First compute $x = s^2/\sigma_0^2$ from the sample, then k which is defined by $k = (x - \log x)/\log 10$ is found from the graph on page 197 (Fig. 1) as a function of x, and finally P is found in Table 1 as a function of k and ν. If P is less than some preassigned α, we reject H_0 at the $100\alpha\%$ level. The table in [2] does not give the critical values corresponding to a specified α such as .05, etc., but gives the probability of a result more extreme than the one observed.

6. Conclusions. Using the new values given in Table I it turns out that for all cases computed $\alpha_1 > \alpha/2$. See Table II. Both, $P(\lambda)$, the power curve based on the newly computed critical values and, $P^*(\lambda)$, the power curve based on the equal tail areas were computed and compared. These results are not included since they would be too space consuming. However, on page 21 (Figure 4) of [3] there is a comparison of $P(\lambda)$ and $P^*(\lambda)$ for the two cases: $\alpha = .10$, $\nu = 9$ and $\alpha = .02$, $\nu = 2$. In all cases computed it turned out that $P(\lambda) > P^*(\lambda)$ when $\lambda < 1$ and $P(\lambda) \leqq P^*(\lambda)$ when $\lambda \geqq 1$.

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