MAXIMAL INDEPENDENT STOCHASTIC PROCESSES[1]

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Introduction and summary. This paper concerns the following problem posed by R. Pyke (1958). What is the cardinality, M_k, of the maximal family of stochastically independent random variables defined on a given space Ω, of cardinality $\bar{\bar{\Omega}} = k$?

Since maximality is sought, the investigation is limited to two-valued, non-trivial (tvnt) random variables; and the σ-algebra of measurable subsets of Ω is taken to be that generated by the family of random variables. With these restrictions the problem is essentially one of cardinality.

The results are summarized in the table below.

Theorem Number	1	2	3	4
Cardinality, k, of space	$k < \aleph_0$	$k = \mathfrak{C}$	$k = k^{\aleph_0} > \aleph_0$	$k = \aleph_0$
Cardinality, M_k, of a maximal tvnt family	$[\log_2 k]$	$2^{\mathfrak{C}}$	2^k	$\aleph_0$

Theorem 1 follows from the fact that stochastic independence entails the non-vanishing of certain finite intersections of elementary sets. Theorem 2 is a result of Kakutani, Kodaira and Oxtoby [4, 5, 6]. Theorem 3 is a consequence of a set theoretic result of Tarski [11], and a theorem of Banach [1, 2, 10, 11], which results were used in proofs of Theorem 2. Theorem 4 follows from a construction and a lemma of Marczewski [9].

The paper is divided into five sections. Section 1 introduces the notation and terminology. Section 2 discusses two types of independence. Sections 3, 4, and 5 treat, respectively, the finite, non-countable and countable cases.

1. Terminology and notation. Let Ω be an arbitrary fixed abstract space. "$\bar{\bar{\Omega}}$" will denote the cardinality of Ω; and "$\emptyset$" will denote the empty subset of Ω.

If A is an arbitrary subset of Ω, "A^0" will denote the complement, $\Omega - A$, of A; and "A^1" will sometimes be used synonymously with A for notational convenience.

A "σ-algebra," $\mathcal{S}$, of subsets of Ω is a class containing Ω and closed under countable unions and complementation. A "probability measure" P on $\mathcal{S}$ is a countably

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additive, non-negative, real-valued set function such that $P(\Omega) = 1$; and the triplet $(\Omega, \mathcal{S}, P)$ is called a "probability measure space."

Since maximality is of prime interest here, it is desirable to restrict considerations to that of two-valued, non-trivial (tvnt) random variables $\{X_t, t \varepsilon T\}$, for which the following definitions and restrictions are natural. (a) "A_t" $= \{X_t = 1\}$; "A_t^0" $= \{X_t = 0\}$; "p_t" $= P(A_t) =$ "p_t^1" $=$ "$1 - p_t^{(0)}$"; $0 < p_t < 1$ for all $t \varepsilon T$, and (b) $\mathcal{S} = \mathcal{S}(\{A_t, t \varepsilon T\})$, i.e., the least σ-algebra with respect to which all the X_t are measurable.

A "tvnt random variable," is then a point function on Ω satisfying (a) and (b). In the sequel only tvnt random variables will be considered.

2. Two types of independence. A family $\{X_t, t \varepsilon T\}$ of tvnt random variables on Ω will be said to be "stochastically independent w.r.t. (with respect to) a collection $\{p_t, t \varepsilon T\}$" of probabilities if there exists on $\mathcal{S} = \mathcal{S}(\{A_t, t \varepsilon T\})$ a probability measure P such that $P(\bigcap_{n=1}^m A_{t_n}^{i_n}) = \Pi_{n=1}^m p_{t_n}^{i_n}$ for each finite subclass $\{A_{t_1}, \cdots, A_{t_m}\}$ of $\mathcal{S}$ and each sequence $\{i_n\}$ of 0's and 1's, P is called the "stochastic extension" of the $\{p_t\}$.

A closely related type of independence which will be useful is that of set independence.

The $\{X_t\}$ are said to be "σ-independent" if for each at-most-countable subclass $N = \{t_1, t_2, \cdots\}$ of T with $t_i \neq t_j$ for $i \neq j$ and each sequence $\{i_n\}$ of 0's and 1's, $\bigcap_{n=1}^\infty A_{t_n}^{i_n} \neq \emptyset$.

Finally, the $\{X_t\}$ are said to be "finitely independent" if every finite intersection $\bigcap_{n=1}^m A_{t_n}^{i_n} \neq \emptyset$.

Relations between the types of independence form the basis for the proofs of Theorems 1, 2, and 3. The fundamental result in this direction [1, 2, 8, 10, 11] is

LEMMA 1. *If $\{X_t\}$ is a σ-independent family, then for arbitrary given probabilities $\{p_t\}$, $\{X_t\}$ is stochastically independent* w.r.t. *the $\{p_t\}$.*

Now, since any set of positive measure is non-empty; and since, when $0 < p_t < 1$ for all $t \varepsilon T$, each $\prod_{n=1}^m p_{t_n}^{i_n} \neq 0$, the following partial converse of Lemma 1 is valid.

LEMMA 2. *If a family $\{X_t\}$ is stochastically independent* w.r.t. *some given $\{p_t\}$ $(0 < p_t < 1)$, then the $\{X_t\}$ are finitely independent.*

Since for a finite family $\{X_t\}$ σ-independence and finite independence are equivalent, one can use Lemmas 1 and 2 and an elementary algebraic formula to find the maximum cardinality M_k for finite k.

3. Finite spaces. If Ω has finite cardinality, then any family $\{X_t\}$ of independent random variables on Ω is necessarily finite. But for any finite class $\{A_1, A_2, \cdots, A_s\}$ there are exactly 2^s sets of the form $\bigcap_{n=1}^s A_n^{i_n}$, which sets are mutually disjoint and non-empty whenever the $\{X_t\}$ are finitely independent.

However, as previously mentioned, for the finite family $\{X_t\}$ finite independence is equivalent to σ-independence. Hence Lemmas 1 and 2 and an elementary construction yield,

THEOREM 1. *If $\bar{\bar{\Omega}} = k < \aleph_0$, then*

(α) *for each* $m \leqq \log_2 k$, *there exists a family of* m tvnt *stochastically independent random variables on* Ω;

(β) *if* $m > \log_2 k$, *there exists no such family*; *and consequently*

(γ) $M_k = [\log_2 k]$ *for* $k < \aleph_0$.

Since the proofs for non-countable spaces also make use of the relationships between the types of independence, it is feasible to treat them next.

4. Non-countable spaces. An indirect result of the Kakutani-Kodaira-Oxtoby non-separable extension of Lebesgue measure [4, 5, 6] is the following.

THEOREM 2. *If* $\bar{\bar{\Omega}} = \mathfrak{c}$, *the power of the continuum, then there exists a family of* $2^{\mathfrak{c}}$ tvnt *stochastically independent random variables on* Ω; *and, therefore,* $M_{\mathfrak{c}} = 2^{\mathfrak{c}}$.

The proof of Theorem 2 is based on a lemma of Tarski ([12], Hilfsatz 3.16, p. 61), which lemma can be used to obtain the solution for a special class of cardinal numbers.

TARSKI'S LEMMA. *If* $\bar{\bar{\Omega}} = k^{\underline{m}} = k \geqq \aleph_0$ (*where* $k^{\underline{m}} = \sum_{r<m} k^r$), *then there exists a class,* $\mathcal{U}$ *of subsets of* Ω *such that*

(1) $\bar{\bar{\mathcal{U}}} = 2^k$; *and*

(2) *for each pair* $\mathcal{L}$ *and* $\mathcal{M}$ *of disjoint subclasses* (*of* $\mathcal{U}$) *with cardinalities less than* m,

$$\left[\bigcap_{A \in \mathcal{L}} A\right] \not\subset \left[\bigcup_{B \in \mathcal{M}} B\right].$$

In view of Lemma 1 the interest here lies in a formulation of Tarski's Lemma in terms of σ-independence. It is readily established that

LEMMA 3. *If* $\bar{\bar{\Omega}} = k^{\underline{m}} = k \geqq \aleph_0$ *for some* $m > \aleph_0$, *then there exists a family* $\{X_t, t \in T\}$ *of two-valued random variables on* Ω *such that*

(1) $\bar{\bar{T}} = 2^k$, i.e., *the family is of maximal cardinality*; *and*

(2) *the family* $\{X_t\}$ *is* σ-*independent.*

The maximal independence theorem now follows from Lemmas 1 and 3, and the fact that $k^{\underline{\aleph_1}} = k^{\aleph_0}$ for $k \geqq 2$.

THEOREM 3. *If* $\Omega = k = k^{\aleph_0} > \aleph_0$, *then there exists a family of* 2^k tvnt *stochastically independent random variables on* Ω. *Consequently,* $M_k = 2^k$.

At this point one notices that although the proofs have some aspects in common the results for $k < \aleph_0$ and $k > \aleph_0$ are quite different in nature. For the case $k = \aleph_0$, not only does the result differ from the two preceding results, but also the nature of the proof is different. In this last case one employs the results of Marczewski [9] concerning purely atomic measures.

5. Countable spaces. If $(\Omega, \mathcal{S}, P)$ is a probability measure space, then $B \in \mathcal{S}$ is said to be an *atom* of P if (α) $P(B) > 0$ and, (β) whenever $B \supset E \in \mathcal{S}$, $P(E) = 0$ or $P(E) = P(B)$.

Further, if Ω is the union of atoms, P is called *purely atomic.*

Marczewski [9] has essentially proved that

LEMMA 4. *If* $\{X_n\}$ *is countable family of* tvnt *random variables stochastically independent* w.r.t. $\{p_n\}$, *then the stochastic extension* P *is purely atomic if and only if* $\sum_{n=1}^{\infty} \min(p_n, 1 - p_n) < \infty$.

Now, if $\{X_t, t \varepsilon T\}$ is a family of stochastically independent random variables on a *countable* space Ω, then clearly

(i) each probability measure P on $\mathcal{S}$ is purely atomic;

(ii) for each countable $N \subset T$, $\sum_{t \varepsilon N} \min (p_t, 1 - p_t) < \infty$; and, therefore,

(iii) at most countably many of the probabilities $\{\min (p_t, 1 - p_t), t \varepsilon T\}$ are non-zero.

Consequently, one concludes

LEMMA 5. *Any family of* tvnt *stochastically independent random variables on a countable space* Ω *is at most countable.*

In order to complete the solution it is sufficient to construct a countable tvnt family on an arbitrary countable space Ω.

In his constructive proof of the necessity of Lemma 4, Marczewski [9] demonstrates essentially that

LEMMA 6. *There exists a countable set* $\{p_n\}$ *of probabilities; a space* Y; *and a countable family* $\{Z_n\}$ *of* tvnt *random variables on* Y (*with* $B_n = \{Z_n = 1\}$ *and* $p_n = P(B_n)$) *such that*

(α) *the* $\{Z_n\}$ *are stochastically independent* w.r.t. $\{p_n\}$;

(β) $0 < p_n \leqq \frac{1}{2}$ *for all* n; *and* $\sum_{n=1}^{\infty} p_n < \infty$;

(γ) *the stochastic extension,* P, *is purely atomic; and*

(δ) *the atoms of* P *are exactly those sets of the form* $\bigcap_{n=1}^{\infty} B_n^{i_n}$ *where* $\sum_{n=1}^{\infty} i_n < \infty$.

The space Y above is not necessarily countable. However, it is purely atomic and has countably many atoms. Hence, there exists a $1 - 1$ mapping, φ, of the atoms of P onto the single points of any given countable space Ω. The naturally induced measure; σ-algebra; and random variables on Ω constitute the desired construction. (A construction with a Markov process is given by Blackwell [13].)

One, therefore, concludes

THEOREM 4. *If* Ω *is an arbitrary countable space,*

(a) *there exists a countable family of stochastically independent* tvnt *random variables on* Ω;

(b) *there exists no non-countable family of stochastically independent* tvnt *random variables on* Ω; *and, hence,*

(c) $M_{\aleph_0} = \aleph_0$.

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