A TRANSIENT DISCRETE TIME QUEUE WITH FINITE STORAGE

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Introduction. Suppose a queue at which i customers arrive during each unit time interval with probability $a(i)$, $a(0) > 0$, $a(n) > 0$, $a(s) = 0$ when $s > n$. Service occurs as follows: at the end of each unit interval k customers are served with probability $b(k)$ if more than k customers are present. If k customers are present, all of them are served with probability $\sum_{j \geqq k} b(j)$. At the end of service, if more than N customers remain, all but N are lost. It is assumed that $b(0) > 0$, $b(m) > 0$, $b(s) = 0$ for $s > m$. Let

$$A(z) = \sum_{i=0}^{n} a(i) z^i, \qquad B(z) = \sum_{i=0}^{m} b(i) z^{-i},$$

$$P(z) = A(z)B(z) = \sum_{i=-m}^{n} p(i) z^i.$$

Let $X(1), X(2), \cdots$ be independent random variables such that

$$\Pr\{X(i) = k\} = p(k), \qquad S(t) = S(0) + \sum_{i=1}^{t} X(i),$$

where $S(0)$ is a random variable, $0 \leqq S(0) \leqq N$, $E\{z^{S(0)}\} = K(z) = \sum k(i) z^i$.

In the first section of this paper [2] we find the generating functions of the first hit probabilities of $S(t)$ above N and below 0, formulae (1.1a) and (1.1b) by using results of a previous paper. In subsequent sections the methods used to examine recurrent events are used first to find the first hits of $S(t)$ outside the interval $[0, N]$ then to find the distribution of the queue during a busy period, finally to find the distribution of the transient queue. In the last section we derive the generating function of the steady state.

The problem of allocation of storage space in digital computers motivated consideration of finite storage in a discrete time queue.

At the time [2] was published the author did not recognize that Theorem 2 was a version of Wald's identity [3].

I. The absorbing barrier process. The following notation will be used. For $a > 0$, $k > 0$, $N \geqq i \geqq 0$, let

$$g(u, -a, t) = \Pr\{S(t) = -a, 0 \leqq \min_{0<j<t} S(j) \mid S(0) = u\},$$

$$h(u, N + k, t) = \Pr\{S(t) = N + k, \max_{0<j<t} S(j) \leqq N \mid S(0) = u\},$$

$$g(N, u, -a, t) = \Pr\{S(t) = -a, 0 < \min_{0<j<t} S(j),$$

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$$\max_{0<j<t} S(j) \leqq N \mid S(0) = u\},$$

$$h(N, u, N+k, t) = \Pr\{S(t) = N+k, 0 \leqq \min_{0<j<t} S(j),$$

$$\max_{0<j<t} S(j) \leqq N \mid S(0) = u\},$$

$$f(N, u, i, t) = \Pr\{S(t) = i, 0 \leqq \min_{0<j<t} S(j),$$

$$\max_{0<j<t} S(j) \leqq N \mid S(0) = u\},$$

$$G(u, -a, w) = \sum_{t>0} g(u, -a, t)w^t,$$

$$H(u, N+k, w) = \sum_{t>0} h(u, N+k, t)w^t,$$

$$F(N, u, z, t) = \sum_{i=0}^{N} f(N, u, i, t)z^i,$$

$$\mathfrak{F}(N, u, z, w) = \sum_{t>0} F(N, u, z, t)w^t,$$

$$G(N, u, -a, w) = \sum_{t>0} g(N, u, -a, t)w^t,$$

$$H(N, u, N+k, w) = \sum_{t>0} h(N, u, N+k, t)w^t,$$

$$\mathcal{G}(u, z, w) = \sum_{a>0} G(u, -a, w)z^{-a},$$

$$\mathcal{H}(u, z, w) = \sum_{k>0} H(u, N+k, w)z^{N+k},$$

$$\mathcal{G}(N, u, z, w) = \sum_{a>0} G(N, u, -a, w)z^{-a},$$

$$\mathcal{H}(N, u, z, w) = \sum_{k>0} H(N, u, N+k, w)z^{N+k},$$

$$\mathcal{R}(N, u, z, w) = z^n - \mathcal{G}(N, u, z, w) - \mathcal{H}(N, u, z, w).$$

It is shown in [2] that

$$\text{(1.1a)} \qquad G(u, -a, w) = \sum_{b=1}^{m} A(a, b)\lambda_b^u(w),$$

where $\lambda_1(w), \cdots, \lambda_m(w)$ are the solutions of $1 = wP(\lambda(w))$ within the unit circle for $|w| < 1$ and $A = \| A(a, b) \|_{1 \leqq a,b \leqq m}$ is the inverse of $L = \| L(a, b) \| = \| \lambda_a^{-b}(w) \|_{1 \leqq a,b \leqq m}$.

Let $\gamma_1(w), \cdots, \gamma_n(w)$ be the solutions of $1 = wP(1/\gamma(w))$ within the unit circle for $|w| < 1$ and let $\Gamma = \| \Gamma(a, b) \| = \| \gamma_a^{-b}(w) \|_{1 \leqq a,b \leqq n}$, $C = \| C(a, b) \| = \Gamma^{-1}$. The same argument that leads to (1.1a) yields

$$\text{(1.1b)} \qquad H(u, N+k, w) = \sum_{a=1}^{n} C(k, a)\gamma_a^{N-u}(w).$$

The $G(u, -a, w)$ and $H(u, N+k, w)$ will be considered known and the $G(N, u, -a, w)$ and $H(N, u, N+k, w)$ will be expressed in terms of them.

By a standard decomposition

$$
\begin{aligned}
&\{S(t) = -a, 0 \leqq \min_{0<j<t} S(j), S(0) = u\} \\
&\quad = \{S(t) = -a, 0 \leqq \min_{0<j<t} S(j), \max_{0<j<t} S(j) \leqq N, S(0) = u\} \\
(1.2) \qquad &\quad \cup \bigcup_{1 \leqq k \leqq n} \bigcup_{0 \leqq r \leqq t} [\{S(r) = N+k, 0 \leqq \min_{0<j<r} S(j), \max_{0<j<r} S(j) \\
&\qquad \leqq N, S(0) = u\} \cap \{S(t) = -a, 0 \leqq \min_{r<j<t} S(j), S(r) = N+k\}].
\end{aligned}
$$

Since the $X(i)$ are identically distributed independent random variables,

$$
\begin{aligned}
&\Pr\{S(t) = -a, 0 \leqq \min_{r<j<t} S(j) \mid S(r) = N+k\} \\
&\quad = \Pr\{S(t-r) = -a, 0 \leqq \min_{0<j<t-r} S(j) \mid S(0) = N+k\} \\
&\qquad\qquad = g(N+k, -a, t-r).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&g(u, -a, t) = g(N, u, -a, t) \\
(1.3) \qquad &\qquad + \sum_{k=1}^{n} \sum_{r=1}^{t} h(N, u, N+k, r) g(N+k, -a, t-r)
\end{aligned}
$$

when we take conditional probabilities in (1.2). Introducing generating functions, we have

$$
\begin{aligned}
&G(u, -a, w) = G(N, u, -a, w) \\
(1.4a) \qquad &\qquad + \sum_{k=1}^{n} H(N, u, N+k, w) G(N+k, -a, w).
\end{aligned}
$$

By a similar argument

$$
\begin{aligned}
&H(u, N+k, w) = H(N, u, N+k, w) \\
(1.4b) \qquad &\qquad + \sum_{a=1}^{m} G(N, u, -a, w) H(-a, N+k, w).
\end{aligned}
$$

Upon rearrangement of (1.4a) and (1.4b) we may obtain

$$
\begin{aligned}
G(N, u, -a, w) &= G(u, -a, w) - \sum_{1}^{n} H(u, N+k, w) G(N+k, -a, w) \\
&\quad + \sum_{s=1}^{m} G(N, u, -s, w) \sum_{k=1}^{n} H(-s, N+k, w) G(N+k, -a, w),
\end{aligned}
$$

$$(1.5)\quad H(N, u, N + k, w) = H(u, N + k, w) - \sum_{s=1}^{m} G(u, -s, w)H(-s, N + k, w) + \sum_{r=1}^{n} H(N, u, N + r, w) \sum_{s=1}^{m} G(N + r, -s, w)H(-s, N + k, w).$$

From (1.3), $0 \leqq g(N, u, -a, t) \leqq g(u, -a, t)$. Similar considerations would show that $0 \leqq h(N, u, N + k, t) \leqq h(u, N + k, t)$. Hence the $G(N, u, -a, w)$ and $H(N, u, N + k, w)$ are power series without constant terms which converge for $|w| \leqq 1$.

In (1.4a) and (1.4b) let u and N be fixed and let a vary from 1 to m, k vary from 1 to n. Then there are $m + n$ equations in $m + n$ unknowns. The matrix of this system has ones on its diagonal and the other terms are either zero or power series in w, without constant term, convergent for $|w| \leqq 1$. For sufficiently small w this matrix is nonsingular so we may solve this set of equations for $G(N, u, -a, w)$ and $H(N, u, N + k, w)$.

Note that

$$\{S(t) = d, 0 \leqq \min_{0<j<t} S(j), \max_{0<j<t} S(j) \leqq N, S(0) = u\} = \bigcup_{b=0}^{N} \{S(t - 1) = b, 0 \leqq \min_{0<j<t-1} S(j), \max_{0<j<t-1} S(j) \leqq N, S(0) = u\} \cap \{X(t) = d - b\}.$$

Setting $d = N + k$, $d = j$, $0 \leqq j \leqq N$, and $d = -a$ and taking conditional probabilities, we obtain

$$h(N, u, N + k, t) = \sum_{b=1}^{N} f(N, u, b, t - 1)p(N + k - b),$$

$$f(N, u, j, t) = \sum_{b=1}^{N} f(N, u, b, t - 1)p(j - b),$$

$$g(N, u, -a, t) = \sum_{b=1}^{N} f(N, u, b, t - 1)p(-a - b).$$

For the generating functions, this implies

$$\mathfrak{F}(N, u, z, w) - z^u + \mathcal{G}(N, u, z, w) + \mathcal{H}(N, u, z, w) = w\mathfrak{F}(N, u, z, w)P(z).$$

Hence

$$(1.6)\quad \mathfrak{F}(N, u, z, w) = \frac{z^u - \mathcal{G}(N, u, z, w) - \mathcal{H}(N, u, z, w)}{1 - wP(z)} = \frac{\mathcal{R}(N, u, z, w)}{1 - wP(z)}.$$

This function has been found by G. Baxter [1]. He obtains it by a quite different approach and obtains properties of the $\mathfrak{F}(N, u, z, w)$, with varying N considered as functions of z relative to the weight function $1 - wP(z)$, with z taken on the unit circle.

II. The distribution of the queue during a busy period. Define the random variable $S^+(t)$ inductively: $S^+(0) = S(0)$, $S^+(t) = \min[S^+(t-1), N] + X(t)$. Let $S^N(t) = \min[S^+(t), N]$. Let

$$f^+(N, u, j, t) = \Pr\{S^+(t) = j, 0 \leqq \min_{0<k<t} S^+(k) \mid S^+(0) = u\}$$

$$F^+(N, u, z, t) = \sum_{j=0}^{N} f^+(N, u, j, t) z^j,$$

$$\mathfrak{F}^+(N, u, z, t) = \sum_{t \geqq 0} F^+(N, u, z, t) w^t.$$

$$G^N(N, u, -a, w) = \sum_{t>0} \Pr\{S^N(t) = -a, 0 \leqq \min_{0<k<t} S^N(k) \mid S(0) = u\} w^t$$

$$\mathfrak{F}^N(N, u, z, w) = \sum_{j \geqq 0, t \geqq 0} \Pr\{S^N(t) = j, 0 \leqq \min_{0<k<t} S^N(k) \mid S(0) = u\} z^j w^t$$

$$M(u, w) = \sum_{t=1}^{\infty} \sum_{k=1}^{n} h(N, u, N+k, t) w^t \text{ and}$$

$$Q(u, w) = \sum_{t>0} \Pr\{S^+(t) > N, 0 \leqq \min_{0<k<t} S^+(k) \mid S(0) = u\} w^t.$$

By a standard decomposition

$$\{S^+(t) > N, 0 \leqq \min_{0<k<t} S^+(k), S^+(0) = u\}$$

$$= \{S^+(t) > N, 0 \leqq \min_{0<k<t} S^+(k), \max_{0<k<t} S^+(k) \leqq N, S^+(0) = u\}$$

$$\cup \bigcup_{t>s>0} [\{S^+(t) > N, 0 \leqq \min_{s<k<t} S^+(k), \max_{s<k<t} S^+(k) \leqq N, S^+(s) > N\}$$

$$\cap \{S^+(s) > N, 0 \leqq \min_{0<k<s} S(k), S^+(0) = u\}].$$

For $j > N$, $\{S^+(t) > N, 0 \leqq \min_{s<k<t} S(k), \max_{s<k<t} S^+(k) \leqq N, S^+(s) = j\}$ imposes the same restriction on $X(s+1), \cdots, X(t)$ as does

$$\{S(t) > N, 0 \leqq \min_{s<k<t} S(k), \max_{s<k<t} S(k) \leqq N, S(s) = j\}.$$

Hence we obtain $Q(u, w) = M(u, w) + Q(u, w)M(N, w)$ upon taking conditional probabilities through the above decomposition and introducing generating functions, so

$$Q(u, w) = M(u, w)/\{1 - M(N, w)\}. \tag{2.1}$$

For $j > N$, $\{S^+(t) = k, 0 \leqq \min_{s<r<t} S^+(r), \max_{s<r<t} S^+(r) \leqq N, S^+(s) = j\}$ imposes the same restrictions on $X(s+1), \cdots, X(t)$ as does

$$\{S(t) = k, 0 \leqq \min_{s<r<t} S(r), \max_{s<r<t} S(r) \leqq N, S(s) = j\}.$$

Hence, taking conditional probabilities through the decomposition

$$\{S^+(t) = k, 0 \leqq \min_{0<r<t} S^+(r), S^+(0) = u\}$$

$$= \{S^+(t) = k, 0 \leqq \min_{0<r<t} S(r), \max_{0<r<t} S(r) \leqq N, S^+(0) = u\}$$

$$\cup \bigcup_{0<s<t} \{S^+(t) = k, 0 \leqq \min_{s<r<t} S^+(r), \max_{s<r<t} S^+(r) \leqq N, S^+(s) > N\}$$

$$\cap \{S^+(s) > N, 0 \leqq \min_{0<r<s} S^+(r), S^+(0) = u\}$$

yields

$$f^+(N, u, k, t) = f(N, u, k, t)$$

$$+ \sum_{0<s<t} f(N, N, k, s - t) \Pr\{S^+(s) > N, 0 \leqq \min_{0<r<s} S^+(r) \mid S^+(0) = u\}.$$

The introduction of generating functions yields

$$\mathfrak{F}^+(N, u, z, w) = \mathfrak{F}(N, u, z, w) + Q(u, w)[\mathfrak{F}(N, N, z, w) - 1]. \tag{2.2}$$

Since $\Pr(S^N(t) = j) = \Pr(S^+(t) = j), j < N$, and $\Pr(S^N(t) = N) = \Pr\{S^+(t) = N\} + \Pr\{S^+(t) > N\}$ so,

$$\begin{aligned} \mathfrak{F}^N(N, u, z, w) &= \mathfrak{F}^+(N, u, z, w) + Q(u, w), \\ \mathfrak{F}^N(N, u, z, w) &= \mathfrak{F}(N, u, z, w) + Q(u, w)\mathfrak{F}(N, N, z, w). \end{aligned} \tag{2.3}$$

A similar argument yields

$$G^N(N, u, -a, w) = G(N, u, -a, w) + Q(u, w)G(N, N, -a, w). \tag{2.4}$$

III. The distribution of the transient queue. The random variable $S^I(t)$ corresponding to the number of customers in the queue at time t is defined inductively: $S^I(0) = S(0)$, $S^I(t) = \max\{\min[S^I(t-1) + X(t), N], 0\}$. Let

$$f^I(N, u, j, t) = \Pr\{S^I(t) = j \mid S^I(0) = u\},$$

$$\mathfrak{F}^I(N, u, z, w) = \sum_{j=0}^{N} \sum_{t=0}^{\infty} f^I(N, u, j, t) z^j w^t,$$

$$\tau(u, w) = \sum_{t>0} \Pr\{S^I(t-1) + X(t) < 0,$$

$$0 \leqq \min_{0<j<t} (S^I(j-1) + X(j)) \mid S^I(0) = u\} w^t,$$

and $T(u, w) = \sum_{t>0} \Pr\{S^I(t-1) + X(t) < 0 \mid S^I(0) = u\} w^t$. Arguments similar to those used in finding $M(u, w)$ and $Q(u, w)$ yield

$$\tau(u, w) = \sum_{a=1}^{k} G^*(N, u, -a, w), \qquad T(u, w) = \tau(u, w)/(1 - \tau(0, w). \tag{3.1}$$

Arguments similar to those used in deriving (2.3) yield

$$\mathfrak{F}^I(N, u, z, w) = \mathfrak{F}^N(N, u, z, w) + T(u, w)\mathfrak{F}^N(N, 0, z, w). \tag{3.2}$$

IV. The steady state queue. We assume that the number of persons in the queue has a limiting distribution as time becomes infinite. Define $H^I(z)$ to be its generating function. Then $H^I(z) = \lim_{t\to\infty} F^I(N, u, z, t)$ where

$$\mathfrak{F}^I(N, u, z, w) = \sum_{t\geq 0} F^I(N, u, z, t)w^t.$$

Standard arguments (as in [2]) yield $H^I(z) = \lim_{w\uparrow 1} (1 - w)\mathfrak{F}^I(N, u, z, w)$. Since $p(-m) > 0$, the queue is certain to be empty occasionally, so

$$\lim_{w\uparrow 1} (1 - w)\mathfrak{F}^N(N, u, z, w) = 0.$$

From (3.2) then,

$$\begin{aligned} H^I(z) &= \lim_{w\uparrow 1} (1 - w)T(n, w)\mathfrak{F}^N(N, 0, z, w) \\ &= \lim_{w\uparrow 1} (1 - w)[\tau(u, w)/(1 - \tau(0, w))]\mathfrak{F}^N(N, 0, z, w) \\ &= \mathfrak{F}^N(N, 0, z, 1)/[(d/dw)\tau(0, w)]_{w=1}. \end{aligned}$$

The application of (2.2) and (1.6) yields

$$\mathfrak{F}^N(N, 0, z, 1) = [\mathfrak{R}(N, 0, z, 1) + Q(0, 1)\mathfrak{R}(N, N, z, 1)]/(1 - P(z)).$$

Let $D(u, w) = \sum_{a=1}^m G(N, u, -a, w)$. Apply (2.3) and set $u = 0$ to obtain $\tau(0, w) = D(0, w) + Q(0, w)D(N, w)$. Then

$$\tau'(0, 1) = D'(0, 1) + Q'(0, 1)D(N, w) + Q(0, 1)D'(N, 1).$$

Since

$$Q(0, w) - M(0, w)/(1 - M(N, w)),$$
$$(1 - M(N, 1))Q'(0, 1) = R'(0, 1) + M'(N, 1)Q(0, 1).$$

Since $M(N, 1) = \sum H(N, N, N + k, 1), M(N, 1) + D(N, 1) = 1$. Combining this with the above, we have

$$\begin{aligned} \tau'(0, 1) &= D'(0, 1) + M'(0, 1) + Q(0, 1)[D'(N, 1) + M'(N, 1)] \\ &= (d/dw)[\mathfrak{R}(N, 0, 1, w) + Q(0, 1)\mathfrak{R}(N, N, 1, w)]|_{w=1}. \end{aligned}$$

Hence

$$H^I(z) = \frac{\mathfrak{R}(N, 0, z, 1) + Q(0, 1)\mathfrak{R}(N, N, z, 1)}{(d/dw)[\mathfrak{R}(N, 0, 1, w) + Q(0, 1)\mathfrak{R}(N, N, 1, w)]|_{w=1}} \frac{1}{(1 - P(z))}.$$

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