INFINITELY DIVISIBLE DISTRIBUTIONS: RECENT RESULTS AND APPLICATIONS[1]

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1. Preliminary remarks. The theory of infinitely divisible distributions, developed primarily during the period from 1920 to 1950, has played a fundamental role in the solution of limit problems for sums of independent random variables. A full account of this theory and its applications, as it had been developed by the late 40's, were presented in the monographs of Lévy [62], Gnedenko and Kolmogorov [29] and Loève [69]. In the last ten years research in this field has been carried out along many lines. Numerous new results have been obtained and entirely new applications have been found. This paper is an attempt to give a full coverage of the results and applications obtained since 1950. Hence, we consider to be recent, results obtained since the appearance of the Russian edition of Gnedenko and Kolmogorov's book.

The scope of this paper is basically restricted to one-dimensional, real random variables. A brief mention of multi-dimensional random vectors will be made in Section 3.6, while generalizations of the theory of infinitely divisible distributions to stochastic processes and random elements in abstract spaces will be omitted.

The introduction, Section 2, contains a short presentation of basic concepts and results, to be found in the monographs mentioned above. Recent results and applications are presented, respectively, in Sections 3 and 4.

Throughout the paper the symbols r.v., r.vec., d.f., den.f., ch.f., ind., id.d., i.d., iff., nsc., and iwc. will stand, respectively, for "random variable", "random vector", "distribution function", "density function", "characteristic function", "independent", "identically distributed", "infinitely divisible", "if and only if", "necessary and sufficient conditions" and "in the sense of weak convergence".

The notation used in this paper may differ from that used in the original papers referred to.

2. Introduction. The r.v. X, or equivalently its d.f. or its ch.f., will be called i.d. if, for every positive integer n, we have $X = Y_{n1} + \cdots + Y_{nn}$ with $Y_{nk} (k = 1, \cdots, n)$ ind. and id.d. The class of i.d.r.v. will be denoted by I.

Khintchine [45a] has shown that the d.f. $F(x)$ is i.d.iff. the logarithm of its ch.f. $\varphi(t)$ is representable in the form

$$\log \varphi(t) = i\gamma t + \int_{-\infty}^{\infty} A(u, t) \frac{1 + u^2}{u^2} \, dG(u) \tag{1}$$

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where

$$(1a) \qquad A(u, t) = \exp(iut) - 1 - itu/(1 + u^2)$$

and where γ is a constant, $G(u)$ is a non-decreasing function of bounded variation, and the integrand at $u = 0$ equals $-\frac{1}{2}t^2$. The representation (1) is unique.

A formula equivalent to (1) has been given by Lévy [62], namely

$$(1b) \quad \log \varphi(t) = i\gamma t - \frac{1}{2}\sigma^2 t^2 + \int_{-\infty}^{0} A(u, t)\, dM(u) + \int_{0}^{\infty} A(u, t)\, dN(u),$$

where $A(u, t)$ is given by (1a), $M(u)$ and $N(u)$ are nondecreasing, respectively, in the intervals $(-\infty, 0)$ and $(0, +\infty)$, continuous at those and only those points, at which G is, and satisfy the relations $M(-\infty) = N(+\infty) = 0$ and, for every $\epsilon > 0$,

$$\int_{-\epsilon}^{0} u^2\, dM(u) + \int_{0}^{\epsilon} u^2\, dN(u) < \infty.$$

Formulas (1) and (1b) are generalizations of the following formula, due to Kolmogorov [50], which is valid for an i.d.r.v. X with finite variance;

$$(2) \qquad \log \varphi(t) = i\gamma t + \int_{-\infty}^{\infty} (e^{itu} - 1 - itu)(1/u^2)\, dK(u)$$

where γ is a constant equal to EX and $K(u)$ is a nondecreasing function of bounded variation with $K(-\infty) = 0$, $K(+\infty) = D^2(X)$.

Let $Y_{nk}(n = 1, 2, 3, \cdots, k = 1, 2, \cdots, k_n)$ be a double sequence of r.v. The Y_{nk} are called infinitesimal if for every $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{1 \leq k \leq k_n} P(|Y_{nk}| > \epsilon) = 0.$$

Set

$$(3) \qquad X_n = \sum_{k=1}^{k_n} Y_{nk}.$$

Khintchine [45] has shown that the class I is equivalent to the class of all limit d.f. of sequences $F_n(x)$ of the form

$$(4) \qquad F_n(x) = P(X_n - A_n < x)$$

where the Y_{nk} are ind. and infinitesimal and A_n is a sequence of constants.

Conditions for the convergence of $F_n(x)$ to a given i.d.d.f. have been given by Gnedenko [30], [31], [33].

The i.d.d.f. F is said to belong to the class L ($F \varepsilon L$), if there exists a sequence of ind.r.v. Y_k and sequences of constants $B_n > 0$ and A_n such that $F(x)$ is the limit d.f. of $F_n(x)$, where

$$(5) \qquad F_n(x) = P[(Y_1 + \cdots + Y_n)/B_n - A_n < x],$$

and the Y_k / B_n are infinitesimal.

Lévy [62] has shown that $F \varepsilon L$ iff. the functions $M(u)$ and $N(u)$ in (1b) have left and right derivatives for every u and the functions $uM'(u)(u < 0)$ and $uN'(u)(u > 0)$ are nonincreasing, where $M'(u)$ and $N'(u)$ denote either the right or the left derivative. Gnedenko and Groshev [27] have shown that if $F(x)$ has finite variance then $F \varepsilon L$ iff. the function $K(u)$ in (2) has right and left derivatives for every $u \neq 0$ and $K'(u)/u$ (left or right derivative) is nonincreasing for $u < 0$ and $u > 0$. Conditions for convergence of (5) to a nondegenerate d.f. $F \varepsilon L$ have been given by Gnedenko and Groshev [27].

The i.d.d.f. $F(x)$ is said to be stable if for all choices of $a_1 > 0$, b_1, $a_2 > 0$, b_2 there exist an $a > 0$ and b such that for every real x

$$F(a_1x + b_1) * F(a_2x + b_2) = F(ax + b) \tag{6}$$

where $*$ denotes convolution. As shown by Khintchine and Lévy [48], for a stable d.f., formula (1) is of the form

$$\log \varphi(t) = i\gamma t - c|t|^\alpha\{1 + i\beta w(t, \alpha) \operatorname{sgn} t\} \tag{7}$$

where α, β, γ, c are constants (γ any real number, $-1 \leqq \beta \leqq 1$, $0 < \alpha \leqq 2$, $c \geqq 0$), $\operatorname{sgn} t = t/|t|$ and

$$w(t, \alpha) = \begin{matrix} \tan(\frac{1}{2}\pi\alpha) & \text{if } \alpha \neq 1 \\ 2 \log |t|/\pi & \text{if } \alpha = 1 \end{matrix}.$$

An alternative formula for a suitably normalized stable d.f. has been given by Lévy [62]

$$\log \varphi(t) = -|t|^\alpha (\cos \beta - i \sin \beta \operatorname{sgn} t) \tag{7a}$$

where $\cos \beta > 0$, $|\sin \beta \cos(\frac{1}{2}\pi\alpha)| \leqq \cos \beta \sin(\frac{1}{2}\pi\alpha)$. $0 < \alpha \leqq 2$. The case $\cos \beta = 0$ corresponds to the degenerate distribution.

Khintchine and Lévy [48] have shown that a non-degenerate d.f. $F(x)$ is the limit iwc. of a sequence $F_n(x)$ given by (5), with $Y_k (k = 1, 2, 3, \cdots)$ having the same d.f. $G(x)$, iff. $F(x)$ is stable. The d.f. $G(x)$ is then said to belong to the domain of attraction of the stable d.f. $F(x)$. If in formula (5) we have $B_n = n^{1/\alpha}$, where α is the exponent in (7), the attraction is said to be normal. Khintchine [44], Feller [20], Lévy [61], Gnedenko [32] and Doeblin [15] have given conditions for the d.f. $G(x)$ to belong to the domain of attraction and to the domain of normal attraction of a specific stable d.f.

Let the d.f. $F(x)$ belong to the domain of attraction of a stable law with exponent α. Then, as has been shown by Khintchine [44] and Cramér [12] for $\alpha = 2$ and by Gnedenko [34] for $0 < \alpha < 2$, for every $\delta (0 \leqq \delta < \alpha)$ the moment $E(|X|^\delta)$ exists.

The above concepts are straightforwardly generalizable to random vectors, as has been done by Lévy [62]. Let $X = (X_1, \cdots, X_p)$ and $\varphi(t) = \varphi(t_1, \cdots, t_p)$ denote, respectively, a p-dimensional r.vec. and its ch.f. Then (1) takes on the form

$$\log \varphi(t) = i(\gamma, t) + \int_{\mathfrak{R}} \left[\exp(it, u) - 1 - \frac{(it, u)}{1 + |u|^2}\right] \frac{1 + |u|^2}{|u|^2} dS(u) \tag{8}$$

where $(\cdot, \cdot)$ stands for the scalar product, γ is a constant vector, S is a completely additive set function with $S(\mathfrak{R}) < \infty$. The definition of a stable d.f. is still given by (6) with x, b_1, b_2 and b denoting vectors while a_1, a_2 and a denote positive numbers. A formula, for the multidimensional case, analogous to (7) has been given by Lévy [62] and Feldheim [19].

3. Recent results.

3.1. *Decomposition of* i.d. *laws.* The first result in this direction of research was proved in 1936 by Cramér [11] who showed that if a Gaussian d.f. can be represented as a convolution of a finite number of d.f., then these d.f. are necessarily Gaussian. In other words, the class of Gaussian distributions is factor-closed. An analogous theorem for the Poisson d.f. was proved by Raikov [81] in 1938. In his first generalizations of Cramér's and Raikov's results, Linnik [64] has stated that the identity of the d.f. of the sums $X_1 + X_2$ and $Y_1 + Y_2$ with X_1, X_2 ind., Y_1, Y_2 ind., X_1 Gaussian, X_2 Poissonian, implies $Y_1 = Y_{11} + Y_{12}$, $Y_2 = Y_{21} + Y_{22}$ with Y_{11} and Y_{21}, Gaussian, Y_{12} and Y_{22} Poissonian and Y_{11}, Y_{12}, Y_{21}, Y_{22} independent. In a series of papers [65], Linnik has obtained important results on the decomposition of i.d. laws into i.d. ones. The presentation of even his main results requires some additional notation.

Denote by I_0 the class of i.d.d.f. which can be factorized into i.d.d.f. only. Call the pair of functions $M(u)$ and $N(u)$ in formula (1b) the Poissonian spectrum. The spectrum is said to be bounded if there exists a $u_0 > 0$ such that for all u with $|u| > u_0$ we have $dM(u) = dN(u) = 0$. The spectrum is finite if (1b) is of the form

$$(9) \quad \log \varphi(t) = i\gamma t - \tfrac{1}{2}\sigma^2 t^2 + \sum_1^{m_0} \lambda_m [\exp(it\mu_m) - 1] + \sum_1^{n_0} \lambda_{-n} [\exp(-it\nu_n) - 1]$$

where $\lambda_m > 0$, $\lambda_{-n} > 0$, $\mu_m > 0$, $\nu_n > 0$, and m_0 and n_0 are finite integers. The spectrum is said to be denumerable if the sums on the right of (9) are replaced, respectively, with

$$\sum_1^{\infty} \lambda_m \left(e^{it\mu_m} - 1 - \frac{it\mu_m}{1 + \mu_m^2}\right), \qquad \sum_1^{\infty} \lambda_{-n} \left(e^{-it\nu_n} - 1 + \frac{it\nu_n}{1 + \nu_n^2}\right).$$

The spectrum is said to be rational if, for all m_1, m_2, n_1, n_2, $\mu_{m_1} : \mu_{m_2}$ and $\nu_{n_1} : \nu_{n_2}$ are rational.

Linnik's main result may now be summarized as follows: Let F be an i.d.d.f. with $\sigma > 0$ in (1b). Then in order that $F \varepsilon I_0$, it is necessary that its spectrum be finite or denumerable. The parameters μ_m and ν_n are then necessarily of the form

$$(10) \qquad \cdots k_{-2}k_{-1}\mu,\ k_{-1}\mu,\ \mu,\ \mu / k_1,\ \mu / k_1 k_2,\ \cdots$$

$$(10a) \qquad \cdots l_{-2}l_{-1}\nu,\ l_{-1}\nu,\ \nu,\ \nu / l_1,\ \nu / l_1 l_2,\ \cdots$$

where the $k - s$ and the $l - s$ are arbitrary integers larger than 1. If the spectrum is finite or denumerable and (10) and (10a) hold and if in addition, for sufficiently large $\mu_m > \mu$, $\nu_n > \nu$ the relations

$$\log \log 1 / \lambda_m > c\mu_m^{1+\alpha} \tag{11}$$

$$\log \log 1 / \lambda_{-n} > c\nu_n^{1+\alpha} \tag{12}$$

hold with $c > 0$, $\alpha > 0$, then $F \varepsilon I_0$. From this one may deduce the important corollary that if the spectrum is bounded, $F \varepsilon I_0$ iff. the spectrum is finite or denumerable, in which case the μ_m and ν_n are, respectively, of the form (10) and (10a).

A related result has been obtained by Ibragimov [38]. Namely, if K denote the class of i.d.d.f. for which $H \varepsilon I$, whenever $F \varepsilon K$ and $(F * H) \varepsilon K$, then K is the class of Gaussian d.f.

The question whether some subclasses of I are factor-closed was previously considered by Teicher [93] (see also his paper [94]).

We refer also to the expository paper by Dugué [16] and to the books by Linnik [65a] and Lukacs [72], which contain many results about the factorization of i.d.d.f.

3.2. *Some properties of* i.d. *laws*. Blum and Rosenblatt [4] have shown that the i.d.d.f. $F(x)$ is discrete iff. $G(u)$ in formula (1) is a pure jump function for which $\int_{\infty}^{\infty} (1 / u^2)\, dG(u) < \infty$; $F(x)$ is continuous iff. $\int_{\infty}^{\infty} (1 / u^2)\, dG(u) = \infty$; finally, $F(x)$ is not continuous without being discrete iff. $G(u)$ is not a pure jump function and $\int_{\infty}^{\infty} (1 / u^2)\, dG(u) < \infty$. As of now, no conditions are known for $F(x)$ to be absolutely continuous.

Shapiro [85] has proved that the moment of order $2k$ of an i.d.d.f. is finite iff. the moment of $G(u)$ of the same order is finite and he has given a formula for the semiinvariants of order $r(2 \leqq r \leqq 2k)$ of $F(x)$ expressed in terms of the moments of $G(u)$.

Chatterjee and Pakshirajan [7] and also Dwass and Teicher [17] have noticed that a non-degenerate bounded r.v. can not be i.d. The following more general result is due to Baxter and Shapiro [1]: if X is i.d. then a constant A such that $P(X > A) = 0$ exists iff. in formula (1b) $N(u) = 0$ for all $u > 0$, $\sigma^2 = 0$ and $\lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} M(u)\, du < \infty$. An analogous result holds for boundedness from below.

Lukacs and Szász [73] have shown that an analytic i.d.ch.f. can have no zeros inside its strip of convergence. It may, however, as has been shown by Lukacs [70], have zeros on the strip's boundary.

Let $\varphi(t)$ be an arbitrary ch.f. B. de Finetti [23] had proved in 1930 that $\psi(t) = \exp\{p[\varphi(t) - 1]\}$, with $p > 0$, is an i.d.d.f. Lukacs [70] has proved the same assertion for $\psi(t) = (p \dot{-} 1)/[p - \varphi(t)]$, with $p > 1$. Lukacs [72] has also shown that $-\int_0^t du \int_0^u \varphi(y)\, dy$ is the cumulant generating function of an i.d.d.f. with finite variance.

Let $F(x)$ be called a normal stratification if there exists a d.f. $H(x)$, with $H(x) \equiv 0$ for $x \leqq 0$, such that for any real x

$$F(x) = \int_0^{\infty} \Phi(x/\sigma)\, dH(\sigma)$$

where $\Phi(x)$ is the d.f. of a Gaussian r.v. X with $E(X) = 0, D^2(X) = 1$. Wintner [100a] has given nsc. for a symmetric i.d.d.f. to be a normal stratification.

3.3. *Structure of* d.f. *belonging to* L. Wintner [100] has shown that the class of symmetric L-distributions coincides with that of convolutions of the form

$$G(x) * G_{q_1}(p_1 x) * \cdots * G_{q_n}(p_n x) \tag{13}$$

and their limits, as $n \to \infty$, iwc., where $G(x)$ is the normal d.f., $p_i = p_i(n) > 0$, $q_i = q_i(n) > 0$ and $G_{q_i}(x)$ is a d.f. whose ch.f. satisfies the relation

$$\log \varphi(t) = -q_i \int_0^t (\sin^2 u / u)\, du \qquad (-\infty < t < \infty).$$

Kubik [54] has given the following characterization of the whole class L. Denote by K_0 the class of all i.d.d.f. for which $G(u)$ in formula (1) is either of the form

$$G(u) = \begin{cases} 0 & \text{for } u < A \\ a \log \dfrac{1 + A^2}{1 + u^2} & \text{for } A \leqq u \leqq 0 \\ a \log (1 + A^2) & \text{for } u > 0, \end{cases} \tag{14}$$

or of the form

$$G(u) = \begin{cases} 0 & \text{for } u < 0 \\ b \log (1 + u^2) & \text{for } 0 \leqq u \leqq B \\ b \log (1 + B^2) & \text{for } u > B, \end{cases} \tag{14a}$$

or of the form

$$G(u) = \begin{cases} 0 & \text{for } u \leqq 0 \\ c & \text{for } u > 0, \end{cases} \tag{15}$$

where $a \geqq 0, b \geqq 0, c \geqq 0$. Then the class of L-distributions coincides with that of finite convolutions of d.f. from K_0 and their limits iwc.

3.4. *Unimodality*. The d.f. $F(x)$ is said to be unimodal if there exists at least one value $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$. (This definition is due to Khintchine [46].) Applying a theorem of Lapin, stating that the class of unimodal d.f. is closed under the formation of finite convolutions, Gnedenko asserted that all $F \varepsilon L$ are unimodal. However, Chung [9] has given a counter example to Lapin's theorem, thus leaving the conjecture of unimodality of L-distributions unanswered. The question has now been clarified in a series of papers, as follows: Ibragimov and Tschernin [40] have shown that all stable d.f. are unimodal, indeed. (For symmetric stable d.f. a simple proof has recently been given by Laha [56]. See also the last paragraph of 3.5.) Wintner [100] has shown that every symmetric L-distribution is unimodal. It is interesting to note that in Wintner's proof use is made of a lemma (due to Wintner) asserting that the convolution of two symmetric unimodal distributions is unimodal. However, not all d.f. $F \varepsilon L$ are unimodal. A counterexample has been given by Ibragimov [39].

We remark finally that sufficient conditions, for the convolution of two unimodal d.f. to be unimodal, have been found by Ibragimov [37].

3.5. *Results about stable distributions*. Zolotarev [103] has shown that (7), the formula for the logarithm of a ch.f. of a stable d.f., is equivalent, for $\alpha \neq 1$, to

$$\log \varphi(t) = i\gamma t - c|t|^{\alpha} \exp\{-(i\pi / 2)K(\alpha)\beta \operatorname{sgn} t\} \tag{16}$$

where $K(\alpha) = 1 - |1 - \alpha|$ and α, β, γ and c have the same range of variation as in formula (7). In the same paper, Zolotarev has given explicit formulas for the Mellin transform and the unilateral Laplace-Stieltjes transform of a stable d.f. with either $\alpha \neq 1, \gamma = 0$ or $\alpha = 1, \beta = 0$. The unilateral Laplace transform of a stable d.f. with $\beta = \gamma = 0$ (Cauchy's stable symmetric distribution) has been given by Wintner [100].

Khintchine [47] had proved in 1938 that a non-degenerate stable d.f. has derivatives of all orders at every point while Lapin [57] had proved in 1947 that it is analytic if $\alpha \geqq 1$ and is an entire function if $\alpha > 1$. Also, it was shown (Wintner [99]) in 1941 that the density of a symmetric stable distribution with $\alpha < 1$ is analytic. Skorohod [90], has shown that the den.f. $f(x)$ of a stable d.f. with $\alpha < 1$ is necessarily of the form

$$f(x) = \begin{cases} x^{-1}g_1(x^{-\alpha}) & (x > 0) \\ x^{-1}g_2(|x|^{-\alpha}) & (x < 0), \end{cases} \tag{17}$$

where $g_1(z)$ and $g_2(z)$ are entire analytic functions. Also if $\alpha = 1$ and $\beta \neq 0, f(x)$ is analytic in some closed strip about the real axis. An expansion formula for the den.f. $f(x)$ of a non-negative r.v. having a stable law with $0 < \alpha < 1$ had been given by Humbert [36] and by Pollard [77]. For the den.f. $f(x)$ of any stable law with $\log \varphi(t)$ given by (7a), Bergström [2], [3] and Feller [22] have given the expansions:

$$f(x) = \frac{1}{\pi} \sum_{1}^{\infty} A(k, \alpha) \cos B(k, \alpha, \beta)x^{k} \tag{18}$$

for $\alpha > 1$, and

$$f(x) = -\frac{1}{\pi} \sum_{1}^{\infty} C(k, \alpha) \sin D(k, \alpha, \beta, x)(|x|^{-\alpha k}x^{-1}) \tag{18a}$$

for $0 < \alpha < 1$, where $A(k, \alpha) = (-1)^{k}\Gamma[(k + 1)/\alpha]/k!\alpha$, $B(k, \alpha, \beta) = k[(\alpha\pi + 2\beta)/(2\alpha)] + \beta/\alpha$, $C(k, \alpha) = (-1)^{k}\Gamma(\alpha k + 1)/k!$, $D(k, \alpha, \beta, x) = k[(\alpha\pi / 2) + \beta - \alpha \arg x]$ and $\arg x = \pi$ for $x < 0$. Bergström [2], [3] has also given the asymptotic formulas

$$f(x) = \frac{1}{\pi} \sum_{0}^{n} A(k, \alpha) \cos B(k, \alpha, \beta)x^{k} + O(|x|^{n+1}) \tag{19}$$

as $|x| \to 0$, for $0 < \alpha < 2$, and

$$\text{(19a)} \quad f(x) = -\frac{1}{\pi} \sum_{1}^{n} C(k, \alpha) \sin D(k, \alpha, \beta, x) \,|\, x \,|^{-\alpha k} x^{-1} + O(|\, x \,|^{-\alpha(n+1)-1)})$$

for $1 \leqq \alpha < 2$, as $|\, x \,| \to \infty$. Expansions for the den.f. $f(x)$ of a stable law have been treated by many other authors about this time. For example, Linnik [63] has given for $\alpha < 1$ an asymptotic formula as $x \to 0$ for $f(x)$ vanishing for $x < 0$, and Chung-Teh Chao [8] for $\alpha \neq 1$. Skorohod [89] has given asymptotic expressions with error estimates as $x \to \infty$ and $x \to 0$ for all cases not covered by the mentioned results. Also, he has given a complete presentation of the asymptotic formulas for all α and β. Zolotarev [101] has given an expression which allows one to use the asymptotic formulas for $\alpha \leqq 1$ to obtain asymptotic formulas for $\alpha > 1$. Some relationship satisfied by stable d.f. has been established by Zolotarev [102] and Zolotarev and Skorohod [104]. Finally, it has been shown by Medgessy [75] that the den.f. of a stable law with rational α satisfies a certain linear differential equation with constant coefficients.

A certain characterization of a stable type has been given by Ghurye [26].

For the case $\alpha = k^{-1}$ (k integral), $\beta = \pm 1$, Karlin [43a] has established the total positivity of the one-sided stable laws, that is, $F(xt^{-1/\alpha})$, where $F(x)$ is the stable d.f., as a function of $x > 0$ and $t > 0$, satisfies the determinant inequalities

$$\det |F(x_i/t_j^{1/\alpha})| \geqq 0,$$

for $n = 1, 2, 3, \cdots$, and for all $0 < x_1 < \cdots < x_n$, $0 < t_1 < \cdots < t_n$. Total positivity of order 2 implies easily unimodality. Karlin has also investigated certain regularity properties, such as the rate of decay of $f(x)$, as $x \to \infty$, the logarithmic concavity of the d.f. $F(x)$, and others. For the case $\alpha = m/n$ (m and n integral, $m < n$), $\beta = \pm 1$, Karlin has deduced an integral representation of $f(x)$, which involves an $n - 2$ fold integral of the three elementary functions e^{-u}, u^{α}, $(1 - u)^{\delta}$.

3.6. I.d.r.vec. Rvaĉeva [83] generalized the basic theorems about i.d.r.v. to the case of r.vec. In particular, she has shown that, given a sequence ξ_k, $k = 1, 2, 3, \cdots$, of p-dimensional ind. and id.d.r.vec., the d.f. $F(x_1, \cdots, x_p)$ is the limit iwc. of normalized sums of ξ_k iff. F is stable. She has also given nsc. for a d.f. to belong to the domain of attraction of a stable law with a given exponent $\alpha(0 < \alpha < 2)$. Problems similar to those treated in Rvaĉeva's paper, were also dealt with by Kinsaku Takano [95]. Some questions of factorization of i.d.r.vec. have been discussed by Dwass and Teicher [17].

4. Recent applications.

4.1. *Limit distributions for sums.* Let $Y_{nk}(n = 1, 2, \cdots, k = 1, \cdots, k_n)$ be a double sequence of ind., infinitesimal r.v. and let X_n and $F_n(x)$ be defined, respectively, by (3) and (4). Loève [68] has shown that the i.d.d.f. $F(x)$ with specific $M(u)$ and $N(u)$ in (1b) is the limit iwc. of the sequence $F_n(x)$ iff.

$$\lim_{n\to\infty} P(\min_k Y_{nk} < u) = 1 - \exp[-M(u)]$$

or 1, and $\lim_{n\to\infty} P(\max_k Y_{nk} < u) = 0$ or $\exp[N(u)]$, as $u < 0$ or $u > 0$.

Let $F_n(x)$ converge iwc. to an i.d.d.f., given by (1b) and let $h(y)$ be a function continuous at $y = 0$ with $h(0) = 0$. Of great interest is the problem of characterizing the limit distributions for sums of $h(Y_{nk})$, suitably normed, in terms of the functions $M(u)$, $N(u)$ and of the constants γ and σ in (1b). Raikov [82] and Kunisawa and Maruyama [55] have solved this problem for $h(y) = y^2$. Bochner [5], who considered random vectors $y = (y_1, \cdots, y_p)$, has obtained a solution of this problem for the functions $h(y) = O(|y|)$, $h(y) = a_1y_1 + \cdots + a_py_p + O(|y|)$, $h(y) = O(|y|^2)$, $h(y) = a_{11}y^2 + a_{12}y_{12}^2 + \cdots + a_{pp}y_p^2 + O(|y|^2)$, as $y \to 0$, and for some other functions.

A comprehensive presentation of this problem and its systematic study has been done by Loève [68], [69b]. He has obtained its solution under rather weak assumptions about the behaviour of $h(y)$ in the neighborhood of $y = 0$ and of $|y| = \infty$, in particular: for $h(y)$ increasing everywhere and of the form $h(y) = cy + O(|y|^r)$, $r > 0$, as $y \to 0$, and for $h(y)$ decreasing in $(-\infty, 0)$ increasing in $(0, +\infty)$ and of the form $h(y) = O(|y|^r)$, $r > 1$, as $y \to 0$. The reader will find many examples in Loève's paper [69b].

Some related results have been established by Shapiro [86] who considered the function h which uniformly truncates the Y_{nk}. (The function $G(u)$ of the limit i.d.d.f. is then constant outside some finite interval.) Shapiro [87] has also shown that if $F^r(x)$ $(r \geqq r_0 \geqq 1)$ is the limit, as $n \to \infty$, iwc. of a sequence of d.f. of suitably normed sums of $|Y_{nk}|^r$ and $H(x)$ is the limit iwc., as $r \to \infty$, of $F^r(x)$, then $H(x)$ is a convolution of a Poissonian and a Gaussian d.f.

Let the Y_{nk} in (3) be ind., id.d. and let them assume a finite number r of values only. It has been shown by Fisz [24] that the limit iwc. of $F_n(x)$ given by (4), is, if it exists and is non-degenerate, necessarily a convolution of at most $r - 1$ factors, where one of them may be Gaussian while the remaining are linear transformations of Poissonian r.v. The same problem, except for the assumption that the Y_{nk} are id.d., has been dealt with by Kubik [52], [53].

Gnedenko and Koroluk [28] have obtained conditions, expressed in terms of the ch.f. of the d.f. $G(y)$, for the attraction of $G(y)$ to a stable law with exponent α.

Sakovitch [84] has proved that the d.f. $G(y)$ is attracted to a stable law with exponent $\alpha(0 < \alpha \leqq 2)$ iff. the limits, as $R \to \infty$, of

$$\frac{R^2G(-R)}{\int_{-R}^{R} y^2\, dG(y)} \quad \text{and} \quad \frac{R^2[1 - G(R)]}{\int_{-R}^{R} y^2\, dG(y)}$$

exist and their sum equals $(2 - \alpha)\alpha^{-1}$; hence, giving a single condition for attraction, applicable to the entire range of α.

4.2. *Partial sums.* Let Y_j, $j = 1, 2, 3, \cdots$, denote a sequence of ind.r.v. Set

$S_k = Y_1 + \cdots + Y_k$. The so called Invariance Principle, derived by Erdös and Kac [18], allows to find the limiting distribution as $n \to \infty$, for a wide class of functionals defined on $(S_1, \cdots, S_n)$, provided the central limit theorem holds for the sequence Y_j. This was done in particular for the functionals $S_n^* = \max(S_1, \cdots, S_n)$ and $T_n = \max(|S_1|, \cdots, |S_k|)$, suitably normed. Kac and Pollard [42] have derived the limiting distribution of T_n for Y_j having the same Cauchy distribution. For Y_j id.d., with the d.f. $G(y)$ belonging to the domain of attraction of a symmetric stable d.f., Darling [14] has given the limit d.f. of S_n^*.

Set, in addition:

$$a_k = P(S_k > 0),\ S_k^+ = \max(0, S_k),\ \max_{1 \leqq k \leqq n} S_k^+ = \max(0, S_1, \cdots, S_n).$$

Spitzer [91] has proved that if $\sum_1^\infty a_k / k < \infty$, then

$$\max_{1 \leqq k \leqq n} S_k^+ \to \sup_{k \geqq 1} S_k^+ = \max_{k \geqq 1} S_k^+ < \infty,$$

with probability 1, and that $\max_{k \geqq 1} S_k^+$ has the i.d.ch.f. equal to

$$\prod_1^\infty \exp\{[\psi_k(t) - 1]k^{-1}\}$$

where $\psi_k(t)$ is the ch.f. of S_k^+.

It is well known (Lévy [61]) that if the limit d.f. $F(x)$ of $F_n(x)$, given by (4), exists, then $F(x)$ is the normal d.f.iff. $P(\max_{1 \leqq k \leqq k_n} Y_{nk} \to 0) = 1$.

The answer to the question, how the maximum term $Y_n^* = \max(Y_1, \cdots, Y_n)$ affects the S_n when the Y_j have a common d.f. belonging to the domain of attraction of a stable law with exponent $\alpha < 2$, has been given by Darling [13] who stated that the major contribution to S_n is due to Y_n^*. He has given the limiting distribution of S_n/Y_n^* (assuming $Y_j > 0$), for $0 < \alpha < 1$, and that of $(S_n - n\mu)/Y_n^*$, for $1 < \alpha < 2$, where $\mu = EY_j$.

Let X_n be given by (3) with Y_{nk} infinitesimal and independent (however, the Y_{nk} are not assumed to be id.d.). Denote by $*Y_{n1}, \cdots, *Y_{nk_n}$ the sequence $Y_{n1}, \cdots, Y_{nk_n}$ arranged in nonincreasing order: $*Y_{n1} \geqq \cdots \geqq *Y_{nk_n}$. Under the assumption that the d.f. $P(Y_{nk} < y)$ are continuous from some n on, Loève [69a] has given, for any Borel measurable function $g(*Y_{ns})$ and any fixed s, the limit d.f. of $X_n g(*Y_{ns})$ and $(X_n, *Y_{ns})$.

Let again the $Y_j, j = 1, 2, 3, \cdots$, have the same d.f. $G(y)$. Denote by $M_n(a)$ the number of S_k such that $|S_k| < a$, and by N_n the number of changes of signs of the S_k, $k = 1, 2, \cdots, n$. Chung and Kac [10] have shown that if $G(y)$ is the symmetric stable d.f. i.e., $\log \varphi(t) = -|t|^\alpha$, then: if $\alpha < 1$, $M_n(a)$ is bounded with probability 1, if $\alpha = 1$, the r.v. $\pi M_n(\alpha)(2a \log n)^{-1}$ and $N_n(2\pi \log n)^{-2}$ have the same exponential distribution with parameter equal 1; if $1 < \alpha \leqq 2$, the r.v. $2N_n\mu^{-1}n^{1/\alpha - 1}$ and $M_n(a)(2a)^{-1}n^{1/\alpha-1}$ have the same limiting distribution which d.f. is stated explicitly in their paper. Nobuyaki Ikeda [41] has shown that if $G(y)$ is a non-lattice d.f., then for the assertions of Chung and Kac to hold in the cases $\alpha = 1$ and $1 < \alpha \leqq 2$, it is sufficient to assume that $G(y)$ belongs to the domain of attraction of a symmetric stable d.f. Assuming that $G(y)$ is absolutely continuous and belongs to the domain of attraction of a symmetric stable law with exponent $\alpha(0 < \alpha \leqq 2)$, Kallianpur and Robbins [43] have found the limiting distribution for $\sum_1^n h(S_k)$, where $h(\cdot)$ is a Riemann integrable function on the real axis and equals 0 outside a certain finite interval. Under the assumption that $G(y)$ belongs to the domain of normal attraction of a symmetric stable d.f., Udagava [98] has derived the limiting distribution of the number of those S_k, $k = 1, 2, \cdots, n$, which equal zero. Assume that $G(y)$ is the stable d.f. with the parameters $\alpha(0 < \alpha < 2)$, $\beta(-1 < \beta < 1)$ and $c > 0$. Let ν denote the first index for which $S_n > 0$. Sinay [88] has shown that S_ν belongs to the domain of attraction of the stable law with exponent $\alpha' = \alpha[1 - G(0)]$ and $\beta = -1$.

Now, let Y_j, $j = 1, 2, 3, \cdots$, be a sequence of positive, continuous ind. r.v. with the common d.f. $G(y)$ such that, for some y_0, we have $1 - G(y) = h(y)y^{-\alpha}$ for all $y > y_0(0 < \alpha < 2)$, where the function $h(y)$ satisfies for every $d > 0$ the relation $h(dy)|h|(y) \to 1$, as $y \to \infty$. It is known (Doeblin [15]) that these assumptions imply that $G(y)$ belongs to the domain of attraction of a stable law with exponent α, where for $0 < \alpha < 1$, we may use in formula (5), $A_n = 0$ and B_n such that $1 - G(B_n) = n^{-1}$, while for $1 < \alpha < 2$, we may take the same B_n and use $A_n = n\mu$. Under different assumptions about the expression $h(dy)/h(y) - 1$, Lipschutz [67] has given estimates of the error term in these limit theorems. The conditions obtained show that the magnitude of the error term depends essentially on the rate of growth of $h(y)$, as $y \to \infty$. Consider a function $\varphi(n)$ such that $\varphi(n) \to \infty$ and $\varphi(n)n^{-1} \to 0$ as $n \to \infty$. Another facet concerning the behavior of X_n has been studied by Lipschutz [66] who has given conditions for the probability $P(\psi)$ to be 0 or 1, where, for $0 < \alpha < 1$, we have $P(\psi) = P(X_n < B_n/[\psi(B_n)]^{1/\alpha}$ i.o.), while for $1 < \alpha < 2$, $P(\psi) = P(X_n - n\mu < -B_n[\psi(B_n)]^{1/\alpha}$ i.o.). We remark that the analogous upper bounds had been derived earlier by Lévy [59], Marcinkiewicz [74] and Feller [21].

Let the d.f. $G(y)$ belong to the domain of attraction of a stable law $F(x)$ with $\alpha < 1$. Fortus [25] has given conditions for the relations

$$\lim_{n\to\infty} [1 - P(X_n n^{-1/\alpha} < x)]/[1 - F(x)] = 1,$$

and $\lim_{n\to\infty} P(X_n n^{-1/\alpha} < x)/F(x) = 1$ to hold uniformly with regard to x.

4.3. *Applications to the strong law of large members*. Prohorov's result, to be presented now, illustrates how broad is the range of possible applications of the theory of i.d.r.v.

Let Y_j, $j = 1, 2, 3, \cdots$, be a sequence of ind. symmetric r.v. and let $F_j(y) = P(Y_j < y)$. (The assumption that the Y_j are symmetric is not an essential restriction.) For $r = 0, 1, 2, \cdots$ set

$$P_r(y) = 2^{-r} \sum_{a(r)}^{b(r)} F_j(y), \tag{20}$$

where $a(r) = 2^r + 1$, $b(r) = 2^{r+1}$. Prohorov [79] has obtained the following result: The sequence Y_j obeys the strong law of large numbers, i.e., there exists a sequence of constants c_n such that as $n \to \infty$,

$$P\left[(1/n)\sum_1^n Y_j - c_n \to 0\right] = 1,$$

if for any $\epsilon > 0$ the relation

$$\sum_0^\infty P(\tau_r \geqq \epsilon) < \infty \tag{21}$$

holds where τ_r is an i.d.r.v. with ch.f. $\varphi(t)$ given by

$$\log \varphi(t) = 2^r \int_{-\infty}^{\infty} (e^{ity/2^r} - 1)\, dP_r(y). \tag{22}$$

A corollary derived from this theorem asserts that if for a certain integer $m > 0$

$$\sum_0^\infty E\tau_r^{2m} < \infty, \tag{23}$$

the sequence Y_j obeys the strong law of large numbers. We remark that this corollary contains Kolmogorov's [49] and Brunk's [6] sufficient conditions, as special cases.

4.4. *Uniform approximation of* d.f. *of sums of* r.v. Let Y_j, $j = 1, 2, 3, \cdots$, be a sequence of random variables having the same d.f. $F(y)$ and let $X_n = Y_1 + \cdots + Y_n$ and $F^{(n)}(y) = P(X_n < y)$. For any pair of d.f. $F(y)$ and $G(y)$, set

$$\rho(F, G) = \sup_{-\infty<y<\infty} |F(y) - G(y)|,$$

and let

$$\psi(n) = \sup_F \inf_{G \epsilon I} \rho(F^{(n)}, G). \tag{24}$$

Kolmogorov [51] has shown that $\psi(n) \leqq c_1 n^{-1/5}$ (c_1 and the c's below are some constants). Prohorov [80] has improved Kolmogorov's upper bound, and also has given a lower bound. Indeed, he has shown that

$$c_2 n^{-1} (\log n)^{-1} \leqq \psi(n) \leqq c_3 n^{-1/3} \log^2 n. \tag{25}$$

Tsaregradskij [96], [97] has also improved Kolmogorov's result for different special cases, including, in particular, the binomial distribution. Denote by F_p^n the binomial d.f. with the parameters n and p and let

$$\psi_B(n) = \sup_p \inf_{G \epsilon I} \rho(F_p^n, G). \tag{26}$$

Tsaregradskij has shown that $\psi_B(n) \leqq c_4 n^{-1/3}$ and that one may take $c_4 = 10$. Meshalkin [76] has given a much stronger bound, namely $\psi_B(n) \leqq c_5 n^{-2/3}$. Meshalkin also considered the problem of the uniform approximation of the multinomial distribution by d.f. of i.d.r.vec.

Studnev [92] has derived asymptotic expressions for $\inf_{G \epsilon I} \rho(F_n, G)$ where $F_n(y) = P(X_n n^{-1/2} < y)$. As one should expect, this expression depends on $F(y)$.

Kolmogorov [51] considered also the problem of uniform approximation, for the case of Y_j not identically distributed. The result, he had obtained, has been used by Hodges and LeCam [35] to obtain the following result: Let $P(Y_j = 1) = p_j$, $P(Y_j = 0) = 1 - p_j$ and $\alpha = \max(p_1, \cdots, p_n)$. Then, there exists a sequence Z_j, $j = 1, 2, 3, \cdots$, of ind. Poisson r.v. with $EZ_j = p_j$ such that

$$\sup_y |P(X_n \leqq y) - P(\sum_1^n Z_j \leqq y)| \leqq 3(\alpha)^{\frac{1}{3}}. \tag{27}$$

Of importance is the fact that the right side of (27) depends on α only. Formula (27) contains, as special cases, the classical theorem about approximating the binomial distribution (Poisson), and the generalized binomial distribution (von Mises), by the Poisson distribution. (See also LeCam [58].)

A result conceptually related has been obtained by Prohorov [78] who has shown that for an arbitrary d.f. $F(y)$ there exists a sequence $G_n(y)$ of i.d.d.f. such that $\lim_{n\to\infty} \sup_y |F^{(n)}(y) - G_n(y)| = 0$. If F contains an absolutely continuous component or if it is a step function, the result can be strengthened to

$$\lim_{n\to\infty} \mathrm{var}[F^{(n)}(y) - G_n(y)] = 0.$$

REFERENCES

[1] G. Baxter, J. M. Shapiro, "On bounded infinitely divisible random variables," *Sankhyā* **22** (1960), 253–260.

[2] H. Bergström, "On some expansions of stable distribution functions," *Arkiv för Matematik* **2** (1954), 375–378.

[3] H. Bergström, "Eine Theorie der stabilen Verteilungsfunktionen," *Arch. Mat.* **4** (1953), 380–391.

[4] J. R. Blum, M. Rosenblatt, "On the structure of infinitely divisible distributions," *Pac. Jour. Math.* **9** (1959), 1–7.

[5] S. Bochner, "Limit theorems for homogeneous stochastic processes," *Proc. Nat. Acad. Sci. USA* **40** (1954), 699–703.

[6] H. D. Brunk, "The strong law of large numbers," *Duke Math. Jour.* **15** (1948), 181–189.

[7] S. D. Chatterjee, R. P. Pakshirajan, "On the unboundedness of the infinitely divisible laws," *Sankhyā* **17** (1957), 349–350.

[8] Yeh Chow Chung, "Explicit formula for the stable law of distribution," *Acta. Math. Sinica* **3** (1953), 177–185.

[9] K. L. Chung, "Sur les lois des probabilités unimodales," *C. R. Acad. Sci. Paris* **236** (1953), 6.

[10] K. L. Chung, M. Kac, "Remarks on fluctuations of sums of independent random variables," *Mem. Am. Math. Soc.* **6** (1951).

[11] H. Cramér, "Über eine Eigenschaft der normalen Verteilungsfunktion," *Math. Zeit.* **41** (1936), 405–414.

[12] H. Cramér, "Random variables and probability distributions," *Cambridge Tracts in Mathematics*, No. 36, Cambridge, 1937.

[13] D. A. Darling, "The influence of the maximum term in the addition of independent random variables," *Trans. Am. Math. Soc.* **73** (1952), 95–107.

[14] D. A. Darling, "The maximum of sums of stable random variables," *Trans. Am. Math. Soc.* **83** (1956), 164–169.

[15] W. Doeblin, "Sur l'ensemble de puissances d'une loi de probabilité," *Studia Mathem.* **9** (1940), 71–96.

[16] D. Dugué, "Arithmétique des lois de probabilités," *Mem. Sci. Math.* No. 137, Paris, 1956.

[17] M. Dwass, H. Teicher, "On infinitely divisible random vectors," *Ann. Math. Stat.* **28** (1957), 461–470.

[18] P. Erdös, M. Kac, "On certain limit theorems of the theory of probability," *Bull. Am. Math. Soc.* **52** (1946), 292–302.

[19] B. Feldheim, "Étude de la stabilité des lois de probabilité," Doctoral Thesis, Faculté des Sci. de Paris, 1937.

[20] W. Feller, "Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung," *Math. Zeit.* I, **40** (1936), 521–559; II, **42** (1937), 301–312.

[21] W. Feller, "Fluctuation theory of recurrent events," *Trans. Am. Math. Soc.* **67** (1949), 98–119.

[22] W. Feller, "On a generalization of Marcel Riesz' potentials and the semigroup generated by them," *Festskrift tillägnad Marcel Riesz,* Lund, 1952.

[23] B. de Finetti, "Le funzioni caratteristiche di legge instantanea," *Rend. Accad. Lincei*, Ser. 6, **12** (1930), 278–282.

[24] M. Fisz, "The limiting distributions of sums of arbitrary independent and identically distributed r-point random variables," *Studia Mathematica* **14** (1953), 111–123.

[25] R. L. Fortus, "A uniform limit theorem for distributions attracted to a stable law with exponent less than one" (in Russian), *Teor. Ver. Prim.* **2** (1957), 486–487.

[26] S. G. Ghurye, "A remark on stable laws," *Skand. Aktuarietidskrift* (1958), 68–70.

[27] B. V. Gnedenko, A. V. Groshev, "On the convergence of distribution laws of normalized sums of independent random variables" (in Russian), *Rec. Math.* [*Mat. Sbornik*] (N.S.) **6** (48), (1939), 521–541.

[28] B. V. Gnedenko, V. S. Korolyuk, "Some remarks on the theory of domains of attraction of stable distributions" (in Russian), *Doklady Akad. Nauk Ukr. SSR.* No. 4 (1950), 275–278.

[29] B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*" (in Russian, Gostehizdat, Moscow), 1949, English translation, Addison-Wesley, Cambridge, 1954.

[30] B. V. Gnedenko, "Über die Konvergenz der Verteilungsgesetze von Summen von einander unabhängiger Summanden," *C. R. (Doklady) Acad. Sci. URSS* (N.S.) **18** (1938), 231–234.

[31] B. V. Gnedenko, "On the theory of limit theorems for sums of independent random variables" (in Russian), *Izv. Akad. Nauk USSR Ser. Mat.* (1939), 181–232, 643–647.

[32] B. V. Gnedenko, "On the theory of domains of attractions of stable laws" (in Russian), *Uchenye Zapiski Moskov Gos. Univ. Matematika* **30** (1939), 61–81.

[33] B. V. Gnedenko, "Limit theorems for sums of independent random variables" (in Russian), *Usp. Mat. Nauk.* **10** (1944), 115–165. English translation, Transl. No. 45, *Amer. Math. Soc.*, New York.

[34] B. V. Gnedenko, "On some properties of limit distributions for normalized sums" (in Russian), *Ukr. Mat. Zhurnal* No. 1 (1949), 3–8.

[35] J. L. Hodges and L. LeCam, "The Poisson approximation to the Poisson binomial distribution," *Ann. Math. Stat.* **31** (1960), 737–740.

[36] P. Humbert, "Nouvelles correspondances symboliques," *Bull. Soc. Math. France* **69** (1945), 121–129.

[37] I. A. Ibragimov, "On the composition of unimodal distributions" (in Russian), *Teor. Ver. Prim.* **1** (1956), 283–288.

[38] I. A. Ibragimov, "A theorem in the theory of infinitely divisible laws" (in Russian), *Teor. Ver. Prim.* **1** (1956), 485–489.

[39] I. A. Ibragimov, "A remark on the class L of probability distributions" (in Russian), *Teor. Ver. Prim.* **1** (1957), 121–124.

[40] I. A. Ibragimov, K. E. Tschernin, "On the unimodality of stable laws" (in Russian), *Teor. Ver. Prim.* **4** (1959), 453–456.

[41] NOBUYAKI IKEDA, "Fluctuation of sums of independent random variables," *Mem. Fac. Sci. Kyusyu Univ.* Ser. A **10** (1956), 10–28.
[42] M. KAC, H. POLLARD, "The distribution of the partial sums of independent random variables," *Canad. J. Math.* **2** (1950), 375–384.
[43] G. KALLIANPUR, H. ROBBINS, "The sequence of sums of independent random variables," *Duke Math. Jour.* **21** (1954), 285–307.
[43a] S. KARLIN, "Total positivity, absorption probabilities and applications," Technical Report No. 28, Applied Math. and Stat. Lab., Stanford University, 1961.
[44] A. YA KHINTCHINE, "Sul dominio di attrazione della leggi di Gauss," *Giorn. Ist. Ital. Att.* **6** (1935), 378–393.
[45] A. YA KHINTCHINE, "Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze," *Rec. Math. (Mat. Sbornik)* (N.S.) **2** (44) (1937), 79–120.
[45a] A. YA KHINTCHINE, "Deduction nouvelle d'une formule de M. Paul Lévy," *Bull. Univ. d'État Moskou,* Sér Internat. Sec. A **1** No. 1 (1937), 1–5.
[46] A. YA KHINTCHINE, "On unimodal distributions" (in Russian), *Nauchn. Issl. Inst. Mat. Mech. Tomsk. Gos. Univ.* **2** No. 2 (1938), 1–7.
[47] A. YA KHINTCHINE, *Limit Theorems for Sums of Independent Random Variables* (in Russian), Gonti Pub., Moscow-Leningrad, 1938.
[48] A. YA KHINTCHINE, P. LÉVY, "Sur les lois stables," *C. R. Acad. Sci. Paris* **202** (1936), 374–376.
[49] A. N. KOLMOGOROV, "Sur la loi des grandes nombres," *C. R. Acad. Sci. Paris* **91** (1930), 910–911.
[50] A. N. KOLMOGOROV, "Sulla forma generale di una processo stocastico omogeno," *Atti. Acad. Naz. Lincei. Rend. Cl. Ser. Fis. Mat. Nat.* (6) **15** (1932), 805–808, 866–869.
[51] A. N. KOLMOGOROV, "Two uniform limit theorems for sums of independent summands" (in Russian), *Teor. Ver. Prim.* **1** (1956), 426–436.
[52] L. KUBIK, "The limiting distributions of cumulative sums of independent two-valued random variables," *Studia Mathematica* **18** (1959), 295–309, "Remarks to the paper of L. Kubik, "The limiting distributions . . ." *Studia Mathematica* **19** (1960), 249.
[53] L. KUBIK, "O rozkladach granicznych sum r-punktowych zmienych losowych," *Roczniki PTM*, Ser. I *Prace Matematyczne* **4** (1960), 111–118.
[54] L. KUBIK, "Some remarks on the class L of probability distributions," *Bull. Polish Acad. Sci.* **9** (1961), 339–340.
[55] K. KUNISAWA, G. MARUYAMA, "Some properties of infinitely divisible laws," *Rep. Stat. Appl. Res. Union Jap. Sci. Eng.* **1** No. 3 (1951), 22–27.
[56] R. G. LAHA, "On a class of unimodal distributions," *Proc. Am. Math. Soc.* **12** (1961), 181–184.
[57] A. L. LAPIN, "On some properties of stable laws" (in Russian), Dissertation, 1947.
[58] L. LECAM, "An approximation theorem for the Poisson binomial distribution," *Pac. Jour. Math.* **10** (1960), 1181–1197.
[59] P. LÉVY, "Sur les séries dont les termes sont des variables eventuelles indépendantes," *Studia Mathem.* **3** (1931), 119–155.
[60] P. LÉVY, "Sur les intégrales dont les éléments sont des variables aléatoires indépendantes," *Ann. R. Scuola Norm. Sup. Pisa* (2) **3** (1934), 337–366; **4** (1935), 217–218.
[61] P. LÉVY, "Propriétés asymptotiques des sommes de variables aléatoires indépendantes ou enchâinées," *J. Math. Pures Appl.* (9) **14** (1935), 347–402.
[62] P. LÉVY, *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.
[63] YU. V. LINNIK, "On stable probability laws with exponent less than one" (in Russian), *Dok. Akad. Nauk USSR* **94** (1954), 619–621.
[64] YU. V. LINNIK, "On factorizing the composition of a Gaussian and a Poissonian law" (in Russian), *Teor. Ver. Prim.* **2** (1957), 34–59.
[65] YU. V. LINNIK, "General theorems on the factorization of infinitely divisible laws"

(in Russian), *Teor. Ver. Prim.* I, **3** (1958), 3–40; II, **4** (1959), 55–85; III, **4** (1959), 150–171.

[65a] YU. V. LINNIK, *The Factorization of Probability Laws* (in Russian), Izdatelstvo, Leningradskogo Univ., 1960.

[66] M. LIPSCHUTZ, "On strong bounds for sums of independent random variables which tend to a stable distribution," *Trans. Am. Math. Soc.* **81** (1956), 135–154.

[67] M. LIPSCHUTZ, "On the magnitude of the error in the approach to stable distributions," *Ind. Mathem.* I, II, **18** (1956), 281–294.

[68] M. LOÈVE, "Relations entre lois limites," *C. R. Acad. Sci. Paris* **239** (1954), 1585–1587.

[69] M. LOÈVE, *Probability Theory*, Van Nostrand, New York, 1955.

[69a] M. LOÈVE, "Ranking limit problem," 177–194, *Proc. Third Berkeley Symp. Stat. Prob.* **2** 1956.

[69b] M. LOÈVE, "A l'interiéur du problème central," *Publ. Inst. Stat. Univ. de Paris* **6** (1957), 313–325.

[70] E. LUKACS, "Remarks concerning characteristic functions," *Ann. Math. Stat.* **28** (1957), 717–723.

[71] E. LUKACS, *Characteristic Functions*, No. 5 of *Griffin's Statistical Monographs and Courses*, New York, 1960.

[72] E. LUKACS, "On a transformation of characteristic functions," *Portugal. Math.* **16** (1957), 31–35.

[73] E. LUKACS AND O. SZÁSZ, "On analytic characteristic functions," *Pac. Jour. Math.* **2** (1952), 615–625.

[74] J. MARCINKIEWICZ, "Quelques theorèmes sur la théorie des probabilités," *Bull. Seminaire Math. Univ. Wilno* **2** (1939), 22.

[75] P. MEDGESSY, "Partial differential equations for stable density functions and their applications," *Tud. Akad. Mat. Kut. Int. Közl.* **1** (1956), 489–518.

[76] L. D. MESHALKIN, "On approximation of multinomial distributions by infinitely divisible laws" (in Russian), *Teor. Ver. Prim.* **5** (1960), 114–124.

[77] H. POLLARD, "The representation of e^{-x^λ} as a Laplace integral," *Bull. Am. Math. Soc.* **52** (1946), 908–910.

[78] YU. V. PROHOROV, "On sums of identically distributed random variables" (in Russian), *Dok. Akad. Nauk. USSR*, **105** (1955), 645–647.

[79] YU. A. PROHOROV, "Strong stability of sums and infinitely divisible laws" (in Russian), *Teor. Ver. Prim.* **3** (1958), 153–165.

[80] YU. V. PROHOROV, "On a uniform limit theorem of A. N. Kolmogorov" (in Russian), *Teor. Ver. Prim.* **5** (1960), 103–113.

[81] D. A. RAIKOV, "On the decomposition of Gauss's and Poisson's laws" (in Russian), *Izv. Akad. Nauk USSR, Ser. Mat.* **1** (1938), 91–124.

[82] D. A. RAIKOV, "On a connection between the central limit theorem in the theory of probability and the law of large numbers" (in Russian), *Izv. Akad. Nauk USSR, Ser. Mat.* **1** (1938), 323–338.

[83] E. L. RVAĈEVA, "On domains of attraction of multidimensional distributions" (in Russian), *Uĉenye Zapiski Lvovskogo Gosud. Univ., ser. Mekh-Mat.*, No. 6 **29** (1954), 5–44.

[84] G. P. SAKOVITCH, "A single form for the conditions for attraction to stable laws" (in Russian), *Teor. Ver. Prim.* 1 (1956), 357–361.

[85] J. M. SHAPIRO, "A condition for existence of moments of infinitely divisible distributions, *Can. Jour. Math.* **8** (1956), 69–71.

[86] J. M. SHAPIRO, "Sums of independent truncated random variables," *Ann. Math. Stat.* **28** (1957), 754–761.

[87] J. M. SHAPIRO, "Sums of powers of independent random variables," *Ann. Math. Stat.* **29** (1958), 515–522.

[88] YA. G. SINAY, "On the distribution of the first positive sum for a sequence of independent random variables" (in Russian), *Teor. Ver. Prim.* **2** (1957), 126–135.

[89] A. V. Skorohod, "Asymptotic formulas for stable distribution laws" (in Russian), *Dokl. Akad. Nauk USSR* **98** (1954), 731–734.

[90] A. V. Skorohod, "On a theorem concerning stable distributions" (in Russian), *Usp. Mat. Nauk* **9** No. 2 (1954), 189–190.

[91] F. Spitzer, "A combinatorial lemma and its application to probability theory," *Trans. Am. Math. Soc.* **82** (1956), 323–339.

[92] Yu. P. Studnev, "On the question of approximating the distributions of sums by infintely divisible laws" (in Russian), *Teor. Ver. Prim.* **5** (1960), 465–469.

[93] H. Teicher, "On the factorization of distributions," *Ann. Math. Stat.* **25** (1954), 769–774.

[94] H. Teicher, "On the convolution of distributions," *Ann. Math. Stat.* **25** (1954), 775–778.

[95] Kinsaku Takano, "On some limit theorems of probability distributions," *Ann. Inst. Stat. Mat. Tokyo* **6** (1954), 37–113.

[96] I. P. Tsaregradskij, "On the uniform approximation of the binomial distribution by infinitely divisible laws" (in Russian), *Teor. Ver. Prim.* **3** (1958), 470–474.

[97] I. P. Tsaregradskij, "Approximation of distributions of sums by infinitely divisible distributions" (in Russian), *Teor. Ver. Prim.* **4** (1959), 243–246.

[98] M. Udagawa, "On some limit theorems for the sums of identically distributed random variables," *Kodai Math. Sem. Rep.* **8** (1956), 86–92.

[99] A. Wintner, "The singularities of Cauchy's distributions," *Duke Math. Jour.* **8** (1941), 678–681.

[100] A. Wintner, "Cauchy's stable distributions and an 'explicit formula' of Mellin," *Amer. Jour. Math.* **78** (1956), 819–861.

[100a] A. Wintner, "Indefinitely divisible symmetric laws and normal stratifications," *Publ. Inst. Stat. Univ. de Paris* **6** (1957), 327–336.

[101] V. M. Zolotarev, "Expressing of the density of a stable distribution with exponent α greater than one by means of a density width exponent $1/\alpha$" (in Russian), *Dok. Akad. Nauk USSR* **98** (1954), 735–738.

[102] V. M. Zolotarev, "On analytic properties of stable distribution laws" (in Russian), *Vestnik Leningr. Univ.* **1** (1956), 49–52.

[103] V. M. Zolotarev, "Mellin-Stieltjes transforms in probability theory" (in Russian), *Teor. Ver. Prim.* **2** (1957), 444–469.

[104] V. M. Zolotarev, A. V. Skorohod, "On some analytic properties of stable distribution functions" (in Russian), *Usp. Mat. Nauk* **10** No. 4 (1955), 181.