ABSTRACTS OF PAPERS

(Abstracts of papers presented at the European Regional Meeting of the Institute, Dublin, Ireland, September 3–5, 1962. Additional abstracts appeared in the June and September, 1962 issues.)

8. **A Note on Sequences of Attributes** (Preliminary report). FRANCISCO AZORIN-POCH, University of Santiago de Compostela, La Coruna, Spain.

Some situations of "intrinsic" lack of randomness in finite sequences (ordered populations or samples) of attributes are examined. A simple case consists of m classes or "levels", each with $\bar{n}$ items, and $\bar{n}/h$ periods of mh items. The extreme cases are $h = 1$ and $h = \bar{n}$, corresponding to maximum and minimum (non random) mixture. A diagram of serial association is considered as a characteristic feature of these attribute sequences. The contingency tables for successive values of: $t = (k - 1)h + a$, $(a < h)$, are obtained for $m = 2$ (cases: $k - 1 \equiv 0 \pmod{2}$) and $k - 1 \equiv 1 \pmod{2}$, and for $m = 3$ (cases: $k - 1 \equiv 0, 1, 2, \pmod{3}$). As measures of association, taking the value 1 for $t = 0$, $X^2/r(m - 1)$ (r = number of pairs), Yule's Q (in the case of two letters or classes), and Wallis and Roberts coefficients are calculated. The diagrams show as expected a succession of ones (preceded alternatively by + and − if association and dissociation are distinguished), in the extreme case $h = 1$, and diminishing values until 0 in case $h = \bar{n}$. Another approach is based on intraclass association. The usual formulae of intraclass correlation is now applied to the analysis of variance identity, for absolute frequencies of attributes instead of values of x in quantitative situations. The association refers to successive clusters of equal size in which the sequence is divided.

9. **Inequalities Applicable in Reliability Theory.** Z. W. BIRNBAUM, University of Washington.

A function $f(p)$, increasing for $0 < p < 1$, is called S-shaped when $f(0) = 0, f(1) = 1$, $f'(0) = f'(1) = 0$, $f'(p) > f(p)[1 - f(p)]/[p(1 - p)]$ for $0 < p < 1$, and $f(p) = p$ has a unique solution in (0, 1). It has been previously shown (Birnbaum, Esary and Saunders (1961) *Technometrics* **3** 55–77) that under certain general assumptions the reliability functions of multi-component structures are S-shaped. The number $S_f = 1 - \int_0^1 f(p)[1 - f(p)]/[p(1 - p)]\, dp$ is now introduced to describe the degree in which $f(p)$ is S-shaped. Inequalities are obtained which show that the closer S_f is to 1 the more S-shaped f is in an intuitive sense. Furthermore, if f and g are S-shaped, and $h = f(g)$, then $S_h > S_f$ and $S_h > S_g$. If $f(p) = \sum_{i=0}^{n} A_i p^i (1 - p)^{n-i}$, then S_f can be written explicitely in terms of the A_i.

10. **Minimax Almost Invariant Confidence Procedures and Related Optimal Ones.** RUDOLF BORGES, University of Cologne, West-Germany. (Introduced by G. Elfving)

Let the indexing set Θ of the family $(M, \mathfrak{K}, P_\vartheta)$, $\vartheta \varepsilon \Theta$, of probability fields be a group. The elements of Θ are assumed to be transformations of the sample space M onto itself which leave the family $(M, \mathfrak{K}, P_\vartheta)$, $\vartheta \varepsilon \Theta$, invariant. Furthermore, there is a σ-finite measure field $(\Theta, \mathfrak{B}, \nu)$ on the group Θ. If τ denotes the true parameter and $B \varepsilon \mathfrak{B}$ the confidence region (decision), the loss is given by $\nu(\tau^{-1}B) + c - cI(\tau, B)$, where c is some non-negative constant and $I(\tau, B) = 1$ for $\tau \varepsilon B$ and 0 otherwise. Under weak assumptions the class of all almost invariant confidence procedures is given which is minimax within the class of all almost invariant confidence procedures. Under additional topological assumptions on the group Θ it is shown from Kudō's (1955) theory that the elements of the same class are minimax within the class of all confidence procedures measurable in the product space (M, Θ). It is shown that these minimax confidence procedures are optimal in the sense of average expected size (see Pratt, 1961), and are subjective most accurate (Borges, 1962), if the measure ν is semi-invariant.

11. The Statistical Theory of Radar Interferometry. R. C. DAVIS, General Dynamics, Pomona, Calif.

In several applications of radar interferometry, one poses the following simplified problem of statistical estimation: Given continuous observations of two random time functions, $e_1(t)$ and $e_2(t)$, in a finite time interval of known duration, T, where $e_1(t) = G(\theta_0 - \theta_1)\theta_3 s_1(t) + G(\theta_0 - \theta_2)\theta_4 s_2(t) + n_1(t)$, $e_2(t) = G(\theta_0 + \theta_1)\theta_3 s_1(t + a \sin \theta_1) + G(\theta_0 + \theta_2)\theta_4 s_2(t + a \sin \theta_2) + n_2(t)$. The random functions, $s_1(t)$, $s_2(t)$, $n_1(t)$, and $n_2(t)$, are independent continuous stationary Gaussian processes with absolutely continuous spectra known to the observer. θ_0 and a_1 are known constants and $G(\theta)$ is a known function with $G(-\theta) = G(\theta)$ and $G'(\theta) < 0$ for $\theta > 0$. θ_1, θ_2, θ_3, θ_4 are unknown constants with $\theta_3 > 0$, $\theta_4 > 0$. The problem is to obtain estimates of θ_1 and θ_2 (θ_3 and θ_4 being nuisance parameters) based entirely upon observations of $e_1(t)$ and $e_2(t)$ in the time interval, $0 \leqq t \leqq T$. Maximum likelihood estimates of θ_1 and θ_2 are obtained in implicit form and approximate solutions obtained for the practical case in which $e_1(t)$ and $e_2(t)$ are random functions with band-limited spectra. The multi-parameter form of the Cramér-Rao inequality is used to obtain explicitly the covariance matrix of the M.L. estimates valid for large values of the time-bandwidth product.

12. A Method for the Selection of Tests. WALTER DE AMBROGIO, Laboratorio Ricerche Elettroniche Olivetti, Borgolombardo (Milano), Italy. (Introduced by L. Lombardi)

The decision whether it is convenient or not to execute a certain test, calls for a comparison between its cost and the informations which it supplies. In my paper the analysis of the principal factors is taken as the point of departure for the determination of the quantity of information given by each test. In order to do this: (1) One eliminates all factors which are non-significant, (2) One uses an iterative method to form an estimate of the community, (3) One arrives at the relations $X_i = a_{i1}F_1 + a_{i2}F_2 + \cdots + a_{im}F_m$ and from this $F_s = b_{s1}X_1 + b_{s2}X_2 + \cdots + b_{sn}X_n$. The product $(a_{is}b_{si})$ measures the importance of variable X_i as a "determiner" of F_s (Harman, Modern Factor Analysis). One proposes to extend the analysis to all the factors considered, keeping account of the total contribution of each factor to the variance of all the variables. The expression $Q_i = \sum_{s=1}^{m} a_{is}b_{si}\lambda_s/n$ is a measure of the quantity of information given by a certain variable.

13. Random Functions with Reciprocal Variances. B. H. DE JONGH, Amsterdam, Netherlands. (Introduced by J. Hemelrijk)

If $\Psi^2(\cdot)$ is a given probability density of a given random variable x, on the understanding that a parameter ϑ is unknown, and if there exists a function $g(\cdot)$ whose values depend on x, but not on ϑ, such that $\int g(x)\Psi^2(x, \vartheta)\, dx = \vartheta$, then under certain regularity conditions, the inequality $\operatorname{Var} g(x) \operatorname{Var} [\dot{\Psi}(x, \vartheta)/\Psi(x, \vartheta)] \geqq \frac{1}{4}$ holds. The Heisenberg inequality $\operatorname{Var} x \operatorname{Var} [\Psi'(x)/\Psi(x)] \geqq \frac{1}{4}$ is equivalent to the former inequality.

14. A Central Limit Theorem for Classes of Dependent Random Variables. FRIEDHELM EICKER, Albert-Ludwigs-Universität, Freiburg, West-Germany.

Let $x\{F\}$ be the set of all sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ of independent random variables with zero means and positive, finite variances, and having distributions out of a given set F. Let be $\xi_j = \sum_{k=-\infty}^{\infty} a_{jk}x_k$, $j = 1, 2, \cdots$, the a_{jk} being given real numbers with the property $0 < \sum_k a_{jk}^2 < \infty$ for each j. If $A_{nk} = \sum_{j=1}^{n} a_{jk}$, $\sigma_k^2 = \text{var } x_k$, let $B_n^2 = \sum_{k=-\infty}^{\infty} A_{nk}^2\sigma_k^2$ be > 0 for all n. Then the sums generating the ξ_j exist as limits in the mean, and the limiting distribution of $\zeta_n = B_n^{-1}\sum_{j=1}^{n} \xi_j = B_n^{-1}\sum_{k=-\infty}^{\infty} A_{nk}x_k$ equals $N(0, 1)$ for every sequence $\{x_k\} \varepsilon x\{F\}$ if (I) $\max_k A_{nk}^2/\sum_{k=-\infty}^{\infty} A_{nk}^2 \to 0$ for $n \to \infty$. (II) There exists for $c \geqq 0$ a bounded function $g(c)$ with $\lim_{c\to\infty} g(c) = 0$ such that $\int_{|z|\geqq c} c^2dG(c) < g(c)$ for $c \geqq 0$ and each G in F. (III) The second moments of all $G \varepsilon F$ are greater than a positive constant. If the theorem is to hold simultaneously for all sequences of $x\{F\}$ (shortly: over F) then (I)–(III) are also necessary.

An application (which generalizes related results of Moran (*Biometrika* **34** (1947) 282–283) and Diananda (*Proc. Cambridge Philos. Soc.* **49** (1953) 241–242) is made to moving average type processes $\xi_k = \sum_{j=-\infty}^{\infty} c_{j-k}x_j$, $k = 1, 2, \cdots$; $0 < \sum_{j=-\infty}^{\infty} c_j^2 < \infty$ (which are strictly stationary if, in addition, the x_j are identically distributed). Condition (I) is always fulfilled unless $\sum_{j=-\infty}^{\infty} c_j = 0$ and $T_n = \sum_{j=0}^{n} c_j \to T$ in which case the divergence of $\sum_{j=1}^{n} [(T_j - T)^2 + (\sum_{k=-j}^{j} c_k - T_j + T)^2]$ is necessary and sufficient.

15. Weighted Sums of Chi-Square Variates. JOHN GURLAND, University of Wisconsin.

The author has previously (*Ann. Math. Statist.* 1955, *Sankhyā* 1956) employed a Laguerrian expansion to approximate the distribution function of a weighted sum of chi-square variates. In the present paper it is shown that the coefficients in this expansion apart from a simple factor, are probabilities corresponding to a convolution of Negative Binomial distributions. This simplifies the required computations. A simple expression for a bound of the remainder term is obtained when the distribution is approximated by a partial sum of the Laguerre series. Its form strongly suggests the approximation will work better the farther one is out in the right hand tail of the distribution. Preliminary computations on some examples have substantiated this.

16. On a Functional Equation Arising in the Theory of Queues. WARREN M. HIRSCH, New York University.

The functional equation $\phi(b) = \int_{0-}^{\infty} \int_{0-}^{b+} \phi(b - t + x)\, dF(t)\, dG(x)$, $b \geqq 0$, where $F(x)$ and $G(x)$ are arbitrary distribution functions vanishing for $x < 0$, arises in the theory of queues. In the probabilistic setting in which this equation occurs, it can be shown that (a) $0 \leqq \phi \leqq 1$ and (b) ϕ is non-decreasing. Using these conditions and a combination of probabilistic and analytical arguments, it is possible to study the dependence of the solutions on F and G. In these arguments the monotonicity of ϕ plays a decisive role, and it is natural to wonder how the solutions behave when this assumption is relaxed. In this paper it is shown by purely analytical methods that if $\int_{0-}^{\infty} x\, dG(x) \leqq \int_{0-}^{\infty} x\, dF(x)$, $\int_{0-}^{\infty} x\, dG(x) < \infty$, and not both F and G degenerate at the same point, then there is a unique bounded solution, namely, $\phi = 0$ a.e. (Lebesgue measure). There are, however, non-zero unbounded solutions, which are exhibited. A partial result is obtained in the case

$$\int_{0-}^{\infty} x\, dG(x) = \int_{0-}^{\infty} x\, dF(x) = \infty.$$

17. Estimates of the Parameters of the Weibull Distribution (Preliminary report). GERALD J. LIEBERMAN, Stanford University.

The density function of the two parameter Weibull distribution can be written as $f(x) = x^{\beta-1} (\exp - x^{\beta}/\alpha)(\beta/\alpha)$ for $x > 0$, $\alpha, \beta > 0$; $= 0$ otherwise. If testing is terminated when the first r out of n items fail at times $t_1, t_2, \cdots, t_r$, respectively, the maximum likelihood equations for estimating α and β are not readily solved. A useful computational formula for $\hat{\beta}$ is given by $0 = \sum_1^{r-1} (1 + \hat{\beta} z_i) \exp - (1 + \hat{\beta} z_i) + (n - r + 1)(1 + \hat{\beta} z_r) \exp - (1 + \hat{\beta} z_r)$ where $z_i = x_i + [(1 - r)/r] \sum_1^r \ln t_k$ and $x_i = \sum_1^r \ln t_k - \ln t_i$. $\hat{\alpha}$ is then given by $\hat{\alpha} = (1/r) \sum_1^r t_k^{\hat{\beta}} + (n-r)\, t_r^{\hat{\beta}}$. An approximation for $\hat{\beta}$ is given by the solution of the equation $\exp \hat{\beta} v = (\hat{\beta} v[1 - \{(n - r)/r\}(2 + \hat{\beta} v)] + 4)/(4 - \hat{\beta} v)$ where $v = \ln t_r - \ln t_1$. As an example, for the case of $n = 2r$, this equation becomes $\hat{\beta} = 2.42/v$.

18. Characterization of the Asymptotic Distribution of a Transformed Normal Random Variable. LLOYD J. MONTZINGO, JR. and NORMAN C. SEVERO, University of Buffalo.

Let h be a function of the real variable y such that: (1) the nth derivative of h, $h^{(n)}$, is continuous in some neighborhood of $y = a$, (2) $h^{(m)}(a) = 0$ if $0 < m < n$, and (3) $h^{(n)}(a) \neq 0$. Let $y = a + bz$, b also a constant. Furthermore denote a function of b, which is $O(b^n) \neq o(b^n)$ as b tends to zero, by $O^*(b^n)$. It is shown that $\lim_{b \to 0} [h(a + bz) - \mu_0]/\sigma_0 = k_1 z^n + k_2$ for all z, where k_1, k_2 are constants, $k_1 \neq 0$, if and only if $\mu_0 = h(a) + O(b^n)$ and $\sigma_0 = O^*(b^n)$. An interesting application to random variables follows. Let Y be a normal random variable with mean μ_y and variance σ_y^2, and let $Z = (Y - \mu_y)/\sigma_y$. As $\sigma_y \to 0$, the random variable $W = (X - \mu_0)/\sigma_0$, where $X = h(Y)$, $\mu_0 = h(\mu_y) + O(\sigma_y^n)$ and $\sigma_0 = O^*(\sigma_y^n)$, has an asymptotic distribution equal to the distribution of the random variable $k_1 Z^n + k_2$. Finally, as $\sigma_y \to 0$, W is asymptotically normally distributed if and only if $n = 1$.

19. A Characterization of the Wishart Distribution. INGRAM OLKIN and HERMAN RUBIN, Stanford University and Michigan State University.

If X and Y are independent random variables having a Gamma distribution with parameters (θ, n) and (θ, m), respectively, then $X + Y$ and $X/(X + Y)$ are statistically independent. Furthermore, this independence property characterizes the Gamma distribution. (Lukacs (1955). *Ann. Math. Statist.* **26** 319.) The principal result is an extension to the Wishart distribution.

Theorem. If X and Y are $p \times p$ positive definite matrices which are independently distributed, and (1) $X + Y = WW'$ is statistically independent of $Z = W^{-1} Y W'^{-1}$, (2) the distribution of Z is invariant under the transformation $Z \to \Gamma Z \Gamma'$, where Γ is orthogonal, then X and Y have a Wishart distribution with the same scale matrix.

20. On Markov Chains the Transition Function of Which is a Finite Sum of Products of Functions of One Variable. J. TH. RUNNENBURG and F. W. STEUTEL, Mathematical Institute and Mathematical Centre, Amsterdam, Netherlands. (Invited paper)

Time-discrete stationary Markov chains with transition function $A(y \mid x) = \sum_{j=1}^r A_j(x) B_j(y)$, where r is finite and the $A_k(x)$ as well as the $B_j(y)$ are linearly independent (to give meaning to the number r), have ergodic properties which depend to a large extent on the eigenvalues of the matrix with elements $C_{jk} = \int A_k(x)\, dB_j(x)$. A number of theorems generalizing those true for time-discrete stationary Markov chains with a finite number of states are proved by making use of the methods developed for the latter chains. In the case $r = 2$, given the invariant distribution function of the Markov chain, those transition functions for which the correlation between successive variables of the chain is maximal (minimal) are derived.

21. Statistical Models and Methods in Analysis of Multiple Decrement and Transfer Schemes. ERLING SVERDRUP, University of Oslo, Norway. (Invited paper)

Models applicable, to, e.g., populations of patients in different state of health are considered. Probabilities of transfer or remaining in the same state of health are expressed by means of basic forces of death and transfer. Statistical methods for testing hypothesis and estimating these forces are developed.

22. Waldian Identities for Asymptotically Homogeneous Additive Processes. M. C. K. TWEEDIE, University of Liverpool, England.

It is well known that Wald's fundamental identity of sequential analysis, and identities which are obtainable from it by differentiation, can be proved by considering suitable martingales. As a variant of this method, some general lemmas on conditional expectations are proved in the present paper and are applied to obtain identities of these kinds in which the duration of the process is subject to certain conditions. Heterogeneous univariate or multivariate additive processes of unbounded duration lead to simple results if the cumulants of the process variable or variables are asymptotically proportional to the duration when the latter is a non-stochastic variable. Processes with this property are termed asymptotically homogeneous. The proof of the generalized fundamental identity, for such processes, requires a negative exponential bound on the duration law. The proofs of the derived identities which involve moments up to order r require conditions which increase in stringency as r increases. Martingales (or their equivalents) for derived identities of fairly low order can be found by a recursion formula or by the evaluation of the determinant of a leading principal minor of a formally infinite matrix.

23. Two Weak-Order Relations for Distribution Functions (Preliminary report). W. R. VAN ZWET, Mathematical Centre, Amsterdam, Netherlands. (Invited paper)

For the class of continuous c.d.f.'s that are strictly increasing from 0 to 1 and possess a finite absolute first moment the following weak ordering may be defined: $F(x) \prec F^*(x)$ if $F^{*-1}(F(x))$ is convex on the interval where $0 < F(x) < 1$. Here $F^{*-1}(y)$ denotes the inverse of $F^*(x)$. The gamma-distributions provide an example of this ordering: they follow one another with decreasing values of the parameter. Let $x_{i:n}$ and $x^*_{i:n}$ denote the ith order statistic of a sample of size n from $F(x)$ and $F^*(x)$ respectively. If $F(x) \prec F^*(x)$ then $F(Ex_{i:n}) \leqq F^*(Ex^*_{i:n})$ for all n and $1 \leqq i \leqq n$. Under a condition concerning differentiability of the c.d.f.'s the converse can be proved if the inequalities hold for sufficiently large n. For the subclass of symmetrical distributions a different weak ordering may be considered: $F(x) \prec' F^*(x)$ if $F^{*-1}(F(x))$ is convex on the interval where $\frac{1}{2} < F(x) < 1$. The symmetrical beta-distributions may serve as an example here. The same inequalities hold in this case for $2i > n$ whereas the theorem can be reversed in the same manner and on the same condition as mentioned above. Inequalities for $F(Ex_{i:n})$ for specific $F(x)$ can thus be obtained by comparison with distributions for which this quantity is relatively simple to calculate.

(Abstracts of papers presented at the Annual Meeting of the Institute, Minneapolis, Minnesota, September 7–10, 1962. Additional abstracts appeared in the June and September issues.)

18. Vectorial Aspects of Analysis of Variance. HOWARD W. ALEXANDER, Earlham College.

The sample space of a random experiment may be regarded as a vector space. If the experiment is of a type to which analysis of variance may be applied, then it is convenient to regard the observation vector as being resolved into components corresponding to a series of orthogonal subspaces of the fundamental vector space. These components are actually the orthogonal projections of the observation vector on the subspaces. The present paper relates these concepts to the standard procedures of analysis of variance.

19. A Definition of Subjective Probability. F. J. ANSCOMBE and R. J. AUMANN, Princeton University and Hebrew University, Jerusalem, Israel.

Most statisticians accept the notion of "physical" probability (chance) measured by frequencies, and the associated notions of independent random phenomena. For such a person, say "you", subjective probability can be defined by a slight extension of the utility theory of von Neumann and Morgenstern. Let $\mathfrak{A}$ be a set of basic prizes. If your preferences between the prizes, and between chance mixtures of the prizes, satisfy the axioms of utility theory, you have a utility for each prize or chance mixture of prizes. Let $\{h_i\}$ be the exclusive and exhaustive possible outcomes of some particular uncertain trial, such as a horse race. Let $\mathfrak{H}$ be the set of all lottery tickets, each yielding a stated prize or chance mixture of prizes from $\mathfrak{A}$ for each outcome h_i. If your preferences between the members of $\mathfrak{H}$, and between chance mixtures of these, satisfy the axioms of utility theory, you have a utility for each member of $\mathfrak{H}$. Two innocuous postulates connecting the two separate systems of preferences and utilities permit one to deduce the existence and usual properties of the probabilities that you associate with the outcomes $\{h_i\}$.

20. Bayes Decision Theory: Insensitivity to Non-Optimal Design. GORDON R. ANTELMAN, University of Chicago.

For the two-action problem on the mean of a Normal process of known variance, with linear terminal utilities, proportional sampling costs, and a Normal prior distribution of the process mean, Schlaifer conjectured that the ratio of the total expected opportunity loss for a fixed sample of size n to the total expected opportunity loss for a Bayes optimal fixed size sample of n_o is $\leqq \frac{1}{2}(n_o/n + n/n_o)$ if $n_o > 0$. It is shown that this inequality holds for this problem, for several closely related two-action problems, and for many estimation problems with quadratic terminal losses and proportional sampling costs. Another inequality of interest which holds for all of the problems considered is that at the optimum sample size, the expected terminal loss exceeds the expected cost of sampling.

For the two-action problem mentioned above, with Normality of the prior relaxed to continuity, the asymptotically optimal sample size is derived and it is noted that both inequalities hold asymptotically. For certain other terminal loss functions and sampling cost functions, generalizations of the two inequalities are shown to hold asymptotically.

21. Some Significant Theoretical Problems in Reliability. LEO A. AROIAN, Space Technology Laboratories, Redondo Beach, Calif. (Invited)

The following problems in reliability are investigated: (1) Optimum structure and allocation of reliability to subsystems. (2) Tests to show what reliability has been achieved and how this reliability may be improved. The counterpart of these problems in mathematical statistics are the problems of optimizing the probability of success of a system by proper redundancy or other techniques; the problem of confidence intervals; the problem of effective truncation of sequential life tests; the problem of simple efficient designs; and the problems of queueing theory.

Some comments are made on the role of computing machines and Monte Carlo methods in mathematical statistics.

22. **Adding Versus Deleting Predictor Variables in the Analysis of Incompletely Specified Regression Models.** T. A. BANCROFT and HAROLD J. LARSON, Iowa State University and Stanford Research Institute.

Considering only the magnitude of the two *bias* functions, a certain assumed range of tabular values for the nuisance parameters, and a range of values of the significance levels of the preliminary test of significance, it appears that a decision rule based on sequential preliminary tests of significance *deleting* predictor variables of a 'doubtful' ranked set is to be preferred to such a rule which *adds* predictor variables of the same 'doubtful' ranked set. This appears in contradiction to the 'practical' recommendation to use a decision rule based on testing for possible addition of predictor variables of a doubtful ranked set, since it provides positive information or terminates at any step beginning with the first.

23. **Goodness Criteria for 2-Sample Distribution-Free Tests** (Preliminary report). C. B. BELL, San Diego State College.

Extending Chapman's (1958) 1-sample results to tests of $H_0:F = G$ vs. $H_1:F > G$ (F, G continuous and strictly increasing) based on F-sample (X_i) and G-sample (Y_j), one defines a test fcn. to be (1) of structure (d) if $T = \Phi[F(X_1), \cdots, F(Y_m)]$; (2) strongly distribution-free (SDF) if $P(T \varepsilon A \mid F, G)$ depends only on GF^{-1}; (3) mono. if $T(F, G; (X_i), (Y_j)) \leqq T(F, G; (X_i), (Y_j^*))$ whenever $Y_j \leqq Y_j^*$ for all j; (4) P.O. if its power fcn. $\beta(T; \cdot)$ satisfies $\beta(T; F, H) \geqq \beta(T; F, G)$ for $H \leqq G$. All four conditions are satisfied by several 2-sample tests; e.g., t, Smirnov, Cramér-von Mises, Wilcoxon, Fisher-Yates, Van der Waerden, Epstein-Rosenbaum and Siegel-Tukey. One proves

Theorem 1. (a) (d) implies SDF; SDF implies $\beta(T; F, G) = \beta(T; U, GF^{-1})$, where U is the uniform cpf on $(0, 1)$; (b) mono. and (d) imply P.O.; P.O. implies unbiased.

Theorem 2. If T is a SDF test fcn., then for alternatives G such that $\sup(F - G) = \Delta$, power bounds are (a) $\inf \beta = \inf_{u_0} \beta[T; U, \underline{G}(u_0; \cdot)]$; and $\sup \beta = \beta(T; U, \bar{G})$, where $\underline{G}$ and $\bar{G}$ are the Birnbaum (1953) alternatives such that $\underline{G}(u_0; u) = u$ for $0 \leqq u \leqq u_0$ and $u_0 + \Delta \leqq u \leqq 1$; $= u_0$ for $u_0 < u < u_0 + \Delta$; and $\bar{G}(u) = 0$ for $0 \leqq u < \Delta$; $u - \Delta$ for $\Delta \leqq u < 1$. Further, for 2-sample SDF tests, power bounds are being computed; consistency and admissibility are being studied.

24. **On an Analog of Regression Analysis.** P. K. BHATTACHARYA, University of North Carolina. (By title)

(X, Y) has a bivariate distribution with $0 \leqq X \leqq 1$. $0 < p < 1$ is a specified number and $\phi_p(x)$ is the p-quantile of the conditional distribution of Y given $X = x$. $(X_1, Y_1), \cdots, (X_{kn}, Y_{kn})$ are independent observations on (X, Y). Let $X_{(1)} < \cdots < X_{(kn)}$ be the ordered values of $X_1, \cdots, X_{kn}$, $Y_{(i)} = Y_j$ if $X_{(i)} = X_j$. For $r = 1, \cdots, k$ and $s = 1, \cdots, n$, $X_{(\overline{r-1}\cdot n+s)} = X_{rs}$, $Y_{(\overline{r-1}\cdot n+s)} = Y_{rs}$, $Y_{r(1)} < \cdots < Y_{r(n)}$ are the ordered values of $Y_{r1}, \cdots, Y_{rn}$. Let $I_{k1} = [0, X_{1n}]$, $I_{kr} = (X_{r-1,n}, X_{rn}]$, $r = 2, \cdots, k-1$, $I_{kk} = (X_{k-1,n}, 1]$ and $f_{nk}(x) = Y_{r([np])}$ if $x \varepsilon I_{kr}$, $r = 1, \cdots, k$. Under some regularity conditions, $f_{nk} \to \phi_p$ [unif] in probability as $k \to \infty$ if $n \geqq k^\gamma$, $\gamma > 0$, and $f_{nk} \to \phi_p$ [unif] a.s. as $k \to \infty$ if $n \geqq k$. To test the hypothesis $H_0: \phi_p = \mu$, a specified function, against $H_1: \phi_p \neq \mu$, we make the transformation, $U_{rs} = 1$ if $Y_{rs} \leqq \mu(X_{rs})$ and 0 otherwise. Let $\bar{U}_r = \sum_{s=1}^{n} U_{rs}/n$, $r = 1, \cdots, k$ and $\tau_{nk} = \text{Sup}_{r=1, \cdots k}\, n^{\frac{1}{2}} |\bar{U}_r - p|/[p(1-p)]^{\frac{1}{2}}$. The limit distribution of τ_{nk} (suitably standardized) as $k \to \infty$ is obtained when $n \geqq k^{\gamma}$, $\gamma > 0$, and the test which rejects H_0 when τ_{nk} is "too large" (making use of the above distribution) is shown to be consistent. Extensions have been made to the cases when X is vector-valued and/or when several quantiles are studied simultaneously.

25. **A Sequential Test for the Two-Sample Problem.** SAUL BLUMENTHAL, University of Minnesota. (By title)

Let $X_1, X_2, \cdots$, and $Y_1, Y_2, \cdots$, represent two sequences of independent random variables. Let each X_i have cdf $F(x)$ and each Y_i have cdf $G(x)$ where $F(x)$ and $G(x)$ are absolutely continuous. The problem is to test $H_0:F(x) = G(x)$ vs. $H_1:F(x) \neq G(x)$. We propose a sequential test of the problem which is analogous to the test described by Weiss for the "goodness of fit" problem (*Ann. Math. Statist.* **32** (1961) 838–845). If $X_1, \cdots, X_{n+1}$ and $Y_1, \cdots, Y_{r(n+1)}$ (r an integer $\geqq 1$) have been observed, a region S_n of the real line $[S_n = S(X_1, \cdots, X_n, Y_1, \cdots, Y_{rn})]$ is formed and the random variable Z_n is defined as unity if X_{n+1} is in S_n and zero otherwise. The decisions to accept H_0, reject H_0 or take another observation are the same that would be made by the Wald Sequential Probability Ratio Test if $Z_1, \cdots, Z_n$ were independent, identically distributed random variables and the problem was to test $H_0:P(Z = 1) = \frac{1}{2}$ vs. $H_1:P(Z = 1) = p > \frac{1}{2}$. We show that when the error probabilities α and β of the WSPRT are small, the power curve of the proposed test is similar to that of the WSPRT and the average sample size of this test can be represented by the well known ASN approximations for the WSPRT.

26. **A Test of the Two-Sample Problem with Nuisance Location and Scale Parameters, and an Estimate of the Scale Parameter.** SAUL BLUMENTHAL, University of Minnesota.

Two related problems are considered. Given $2n$ independent observations, the first n having common c.d.f. $F(x)$ and the second n common c.d.f. $G(x)$, let $X_1 \leqq X_2 \cdots \leqq X_n$ represent the ordered values of the first n, and $Y_1 \leqq \cdots \leqq Y_n$, the ordered values of the second n. Define Z_i as $(X_{i+1} - X_i)/(Y_{i+1} - Y_i)$ $(i = 1, \cdots, n-1)$. Define $S_n(p, h) = u(p, h(n)) \sum Z_i^p$, $(0 < p \leqq 1)$ where i runs from $h_1(n)$ to $h_2(n)$ and where $u(p, h(n))$ is a normalizing constant dependent on p and on the limits of summation $h_1(n)$, $h_2(n)$. Also, define $S'_n(p, h) = u(p, h(n)) \sum (1/Z_i)^p$. The first problem is that of testing the hypothesis $H_0:F(x) = G(Ax + B)$ where $A(> 0)$, and B(real) are unspecified constants, against the alternative $H_1:F(x) \neq G(Ax + B)$. This is the two-sample problem with nuisance location and scale parameters, and can be regarded as a generalization of the usual two-sample problem. We show that the test which rejects H_0 when the product $S_n(p, h)\ S'_n(p, h)$ is "too large" is consistent against pairs of c.d.f.'s $F(x)$ and $G(x)$ satisfying certain restrictions. The type of restriction needed depends on the summation limits $h_1(n)$, $h_2(n)$. The second problem is estimation of the scale factor A assuming that $F(x) = G(Ax + B)$, $A > 0$, B real. We show that $S'_n(p, h)$ is a consistent estimator of A under the restrictions on $F(x)$ mentioned above.

27. **Maximum Likelihood Estimates of Restricted Means of Discrete and Continuous Normal Processes** (Preliminary report). H. D. BRUNK, University of California, Riverside.

Let $X_1, \cdots, X_n$ have a multivariate joint normal distribution with mean $\theta = \theta_1, \cdots, \theta_n$ and covariance matrix $\sigma^2\Gamma$, where Γ is known and σ^2 known or unknown. For the case of independent $X_1, \cdots, X_n$, the maximum likelihood estimates of θ subject to order restrictions on $\theta_1, \cdots, \theta_n$ were described by van Eeden (*Proc. Koninkl. Nederl. Akad. van Wet.-Amsterdam*, Ser. A, **60** = *Indag. Math.* **19** (1957) 128–136, 202–211), by Bartholomew (*Biometrika*, **46** (1959) 37–48); and (for sampling from populations of equal variance) by Brunk (*Ann. Math. Statist.* **26** (1955) 607–616). They are described here for arbitrary positive definite Γ. This estimation problem is an instance of the general problem discussed: that of minimizing $\int (X - \theta)B(X - \theta)\, d\mu$ where $(\Omega, \mathfrak{A}, \mu)$ is a measure space, X is a given function in $L_2 = L_2(\Omega, \mathfrak{A}, \mu)$, B is a positive definite operator on L_2, and θ ranges over the class of functions in L_2 measurable with respect to a given σ-sublattice of $\mathfrak{A}$. Another instance is the problem of maximum likelihood estimation of the mean function $\theta(t)$ of a normal process $X(t)$ with known covariance function, subject to conditions requiring that $\theta(t)$ be measurable with respect to a given σ-lattice of square-integrable functions.

28. **Subsample Order Statistics Estimates.** BENJAMIN BUCHBINDER, General Electric Missile and Space Division, Valley Forge, Penna. (By title)

A sample of $mn = N$ observations is divided into m subsamples of size n. Each group of n observations is then ordered. Subsample order statistics estimates of the mean and standard deviation (for normal and rectangular distributions) are developed, and the efficiency of each estimate is determined relative to the corresponding single sample order estimate. The computational effort required for each estimate is also determined. Whereas single sample estimates are in general more efficient than those based on subsamples, for very large samples the variance of either type of estimate is extremely small, and the computational savings effected by the use of order estimates based on subsamples is the important consideration.

Confidence intervals on the population median based on subsample order statistics were constructed, and their lengths compared with intervals based on single ordered samples. Although intervals based on single samples are shorter, for large N and best choice of m, considerable computational savings may be realized by the use of subsampling.

29. **Asymptotic Expansion for a Class of Distribution Functions.** K. C. CHANDA, Washington State University.

Investigations have been made in the past by several authors on the possibility of extending the content of the classical central limit theorem when the basic random variables are not mutually independent. Mention may be made in this connection of the work done by Hoeffding and Robbins (*Duke Math. J.* **15** (1948) 773–780), Diananda (*Proc. Cambridge Philos. Soc.* **49** (1953) 239–246) and Walker (*Proc. Cambridge Philos. Soc.* **50** (1954) 60–64). However, no attempt has been made so far to investigate whether the type of asymptotic expansions as discussed by Cramér (*Random Variables and Probability Distributions* (1937), Cambridge University Press), Berry (*Trans. Amer. Math. Soc.* **49** (1941) 122–136), Psu (*Ann. Math. Statist.* **16** (1945) 1–29) and others for the distributions of means of independent random variables could as well be extended to apply to situations where the random variables are not independent. An attempt has been made in this paper to investigate this problem somewhat systematically. The main results are the following: (i) Let $X_1, \cdots X_n$ be a sequence of stationary m-dependent random variables with $E(X_1) = 0$. If $F_n(x)$ denotes the distribution function of $\sum_{t=1}^{n} X_t/s_n$ where $s_n^2 = V(\sum_{t=1}^{n} X_t)$ and $F(x)$ is the standardized normal distribution function, then under usual regularity conditions $\operatorname{Sup}_{-\infty < x < \infty} |F_n(x) - F(x)| \leq M/n^{\frac{1}{2}}$ where M is a finite positive constant. (ii) If $\{X_t\}$ is a linear process with $E(X_t) = 0, E|X_t|^r < \infty$ for some $r \geq 3$, then $\operatorname{Sup}_{-\infty < x < \infty} |F_n(x) - F(x)| \leq M/n^{(r-2)/2}$.

30. The Choice of a Decision Procedure for Finite Decision Problems Under Complete Ignorance. J. D. CHURCH and BERNARD HARRIS, University of Nebraska.

Consider the triplet (S, D, u) where $S = \{s_1, s_2, \cdots, s_n\}$ is a set of n states of Nature, $D = \{d_1, d_2, \cdots, d_m\}$ is a set of m strategies available to the statistician, and u is a real valued function on $D \times S$ such that $u(d_i, s_j)$ is the loss (negative utility) incurred by the statistician upon choosing d_i when Nature is in state s_j. We are concerned with determining a decision procedure for choosing a mixed strategy $\phi \varepsilon \Phi$ where Φ is the $m - 1$ dimensional probability simplex over D. Let Ξ be the $n - 1$ dimensional probability simplex over S. A mixed strategy for Nature is an element $\xi \varepsilon \Xi$. The statistician may have the information that $\xi \varepsilon \Xi_0 \subset \Xi$. If $\Xi_0 = \Xi$ the statistician is said to be in complete ignorance. A set of desirable properties is developed for decision procedures for use in the above decision problem under the hypothesis of complete ignorance. It is shown that there exists a family of decision procedures, each of which satisfies all required properties. An algorithm for determining optimal strategies is given. It is established that two very general classes of decision procedures fail to satisfy at least one of the required properties. These classes contain the minimax and minimax regret criteria, and Laplace's criterion as special cases.

31. Asymptotic Independence of (d)-Structured Order Statistics and (d)-Structured Statistics of Kolmogoroff-Smirnoff Type (Preliminary report). H. T. DAVID, Iowa State University.

Consider the function Φ associated with a statistic S_F of structure (d), as defined by Z. W. Birnbaum (*Ann. Math. Statist.* **24** (1953) 1–8). Let β be the order in n of $\sup|\Phi(x_1, \cdots, x_n) - \Phi(z_1, \cdots, z_n)|$, where the sup is taken over the region $0 \leqq x_i, z_i \leqq 1, |x_i - z_i| \leqq K/n$. Let α be such that $n^\alpha S_F(X_1, \cdots, X_n)$, $(X_1, \cdots, X_n)$ a random sample from F, possesses a continuous limit c.d.f. $\phi(s)$. Then, if $\alpha + \beta < 0$, S_F is asymptotically independent of the (d)-structured order statistic $F(X^{(n-k)})$, in the sense that $\Pr\{n^\alpha S_F \leqq s, n[1 - F(X^{(n-k)})] \leqq t\}$ tends to the product of the two respective asymptotic c.d.f.'s. All the usual statistics of structure (d) satisfy the above condition; for example, in the case of the Kolmogoroff-Smirnoff statistic itself, $\alpha = \frac{1}{2}$, and $\beta = -1$. The proof of asymptotic independence utilizes a multivariate probability transformation given by Murray Rosenblatt (*Ann. Math. Statist.* **23** (1952) 470–472).

32. Estimation of Multiple Contrasts Using a Multivariate t-Distribution. OLIVE JEAN DUNN and FRANK J. MASSEY, JR., University of California, Los Angeles.

Simultaneous confidence intervals for the means of normally distributed variates involve the evaluation, over a hypercube centered at the origin, of a multivariate analogue of the Student-t distribution. This paper summarizes some tables which for the special case when the correlations among the variates are positive and equal, and compares the lengths of the confidence intervals obtained by use of these new tables with conservative sets of intervals which use (a) tables of the studentized maximum modulus and (b) tables of the univariate student-distribution. The number of contrasts considered is from one to twenty, simultaneous confidence levels from .50 to .999 are considered.

33. Poisson Limits of Bivariate Run Distributions (Preliminary report). CAROL B. EDWARDS and H. T. DAVID, Iowa State University.

Consider n trials, each with probability of success $p(n)$. Let $R_i(n)$, $i = 1, 2$, be a run of $\alpha_i(n)$ successes, $\alpha_1(n) \neq \alpha_2(n)$. Let $N_i(n)$ be the number of runs $R_i(n)$. Then, if $E[N_i(n)] \rightarrow$

K_i, $K_i > 0$, $N_1(n)$ and $N_2(n)$ are independent Poisson asymptotically, with parameters respectively K_1 and K_2, this in contrast to the usual normal limit, which exhibits correlation. The correlated Poisson appears in the case of the joint distribution of the number $M_i(n)$ of configurations $C_i(n)$, $i = 1, 2$, where $C_i(n)$ is defined as a succession of $\alpha_i(n)$ successes immediately preceded by a succession of $\beta_i(n)$ failures. Let $\alpha_i(n)$, $\beta_i(n) \geqq 1$, and $\bar{\alpha} = \max \alpha_i(n)$, $\bar{\beta} = \max \beta_i(n) \geqq 2$. Also let $\underline{\alpha} = \min \alpha_i(n)$, $\underline{\beta} = \min \beta_i(n)$. If $E[M_i(n)] \to K_i$, $K_i > 0$, and if $\lim_{n\to\infty}[p^{\bar{\alpha}-\underline{\alpha}}q^{\bar{\beta}-\underline{\beta}}] = C$, $0 \leqq C \leqq 1$, then $M_1(n)$ and $M_2(n)$ tend in distribution to the bivariate Poisson with correlation coefficient $C^{\frac{1}{2}}$.

A useful tool has been the following bivariate extension of a formula of Fréchet. Consider $2n$ events $A_1, \cdots, A_n, B_1, \cdots, B_n$. Let (I_A, I_B) count the materializations of A's and B's. Then the mixed factorial moment of (I_A, I_B) of order (ν_1, ν_2) equals $\sum P(A_{i_1}, \cdots, A_{i_{\nu_1}}, B_{j_1}, \cdots, B_{j_{\nu_2}})$, where the summation extends over all $\binom{n}{\nu_1}\binom{n}{\nu_2}$ distinct subscript choices.

34. Basic Ideas and Methodology Underlying the Stochastic Approach to Reliability and Life Testing. Benjamin Epstein, Palo Alto, Calif. (Invited)

This paper summarizes some of the basic methodology developed during the past decade for the design and analysis of life tests.

35. The N-Response Problem. John Leroy Folks and Charles E. Antle, Oklahoma State University and Missouri School of Mines and Metallurgy.

The problem of experimentally determining a point x so that a single response depending upon x will be maximized is extended to include the situation in which there are N responses of interest. An efficient point is defined to be any point x^0 such that the responses at x^0 are not dominated by those at some other x. It is shown that under fairly weak conditions on the response functions the set of all efficient points possesses the further property that given any point x^μ not in the set, there is a member of the set whose responses dominate those at x^μ. The set of all efficient points is then called the complete set of efficient points. Means for identifying the members of this set are derived, and they are especially convenient to apply when the responses are quadratic functions. Consideration is also given to the effect of restricting the domain for x.

36. Multivariate Analysis of Variance for a Special Covariance Case. Seymour Geisser, National Institutes of Health.

In general a $k \times k$ covariance matrix of a multivariate normal distribution is specified by $k(k+1)/2$ different parameters. In certain instances the number of different parameters can be considerably reduced. We will consider here multivariate hypothesis testing for a special reduced parameter covariance situation that we shall call the uniform case of order m, i.e., $(u)_m$. Certain applications of this case to a general serial correlation model will be given when $k = 3, 4, 5$ for testing the null hypothesis as in the analysis of variances that the k means are all equal. This is in a sense analogous to the mixed model analysis of variance situation where the errors are not independent but are serially related.

37. Minimax Properties of Hotelling's and Certain Other Invariant Tests. N. C. Giri and J. Kiefer, Cornell University. (By title)

Many well-known and commonly used best invariant tests, especially ones arising in multivariate analysis (e.g., Hotelling's test T^2, the test based on the multiple correlation coefficient R^2, etc.), are known to be best invariant; however, due to the fact that the Hunt-Stein theorem does not hold for the full linear group, they have not been known to possess any minimax properties. In fact, N. C. Giri has shown (Abstract, *Ann. Math. Statist.* 1962) that Hotelling's test does not maximize the minimum power $\beta_\phi(\delta)$ among all tests ϕ of size $\alpha > 0$, on the contour $\mu'\Sigma^{-1}\mu = \delta > 0$, where μ and Σ are the unknown mean vector and covariance matrix in the usual normal model and δ is fixed. In the present paper a general theorem is proved which nevertheless yields *local* (as $\delta \to 0$) and *asymptotic* (as $\delta \to \infty$) minimax results for many of these tests. For example, in Hotelling's problem it is shown that, for every $\alpha > 0$, $\lim_{\delta\downarrow 0}[\max_\phi \beta_\phi(\delta) - \alpha]/[\beta_{T^2}(\delta) - \alpha] = 1$ and $\lim_{\delta\to\infty}\{\log \min_\phi[1 - \beta_\phi(\delta)]\}/\log[1 - \beta_{T^2}(\delta)] = 1$. The first of these yields a result which can be expressed in terms of D-optimality (local maximization of Gaussian curvature of the power function), following Isaacson, Kiefer, Lehmann. The latter may be compared with Stein's proof of the admissibility of T^2, which yields $\beta_{T^2}(\delta) - \beta_\phi(\delta) > 0$ for $\delta > C(\phi)$ (depending on ϕ) for every $\phi \neq T^2$.

38. **On Median Unbiased Estimation from Discrete Data.** W. J. Hall, University of North Carolina.

A possible interpretation of the method of median unbiased estimation is that the statistician takes a value for his estimate such that, if asked whether the true parameter value were larger or smaller than the estimated value, he would be content to flip a coin. Although median unbiased estimators are not available from discrete probability models without randomization, estimators having the above property (interpreted in the sense of classical statistical inference) are frequently available; we call them pseudo median unbiased. This method shares with that of median unbiasedness, but not ordinary unbiasedness, the attractive property that an estimator of a monotone function of a parameter is the function of the estimator of the parameter. Methods and tables for obtaining the estimates are given for binomial, Poisson and negative binomial models; the estimates coincide with randomized median unbiased estimators with the added random number replaced by one-half. The method is illustrated with some biological examples in which neither ordinary unbiasedness nor true median unbiasedness is attainable (with a non-randomized estimator).

39. **Percentage Points of the Ratio of Two Ranges and Power of the Associated Test.** H. Leon Harter, Wright-Patterson Air Force Base. (Invited)

In testing the hypothesis that the variance of two populations are equal, a test based on the ratio F' of the ranges of two samples, one from each population, is simpler than the conventional F test based on the ratio of the sample variance, and it is only slightly less powerful. In order to apply such a test, more extensive and more accurate tables of the percentage points of the ratio of the ranges of two samples from a normal population are needed, and the author has endeavored in this paper to supply them. This required the computation of auxiliary tables of the probability density function of the range and both the probability density function and the cumulative distribution function of the ratio of two ranges. Next came the computation of the table of percentage points of the ratio of two ranges. Finally, a table of the power of the F' and F tests was computed. This paper contains the tables of percentage points and power, together with a description of the method of computation and an example of their use.

40. **Tables of Non-Central Chi-Square.** George E. Haynam and F. C. Leone, Case Institute of Technology.

Three tables of the cumulative non-central chi-square have been computed. These cover a range of values of non-centrality parameter from 0 to 34 and of degrees of freedom from 1 to 100. Further, the power of the non-central chi-square distribution has been computed for selected values of alpha ranging from 0.001 to 0.1. The construction and use of these tables is discussed.

41. Controlling of Non-Recurrent Lattice Random Walks. KARL F. HINDERER, University of California and Technische Hochschule, Stuttgart, Germany.

Let $\{X_n\}$ be a sequence of independent, identically distributed, two-dimensional random vectors with integer-valued components. We assume that X_1 has a non-degenerate distribution with $E|X_1|^2 < \infty$. It is well known that the planar random walk $\{S_n\}$, $S_n = \sum_1^n X_i$, has no recurrent states if $EX_1 \neq 0$. In that case one may ask for the existence of a sequence of two-dimensional constant vectors c_n with integer components such that the 'controlled' random walk $V = \{V_n\}$, $V_n = \sum_1^n (X_i + c_i)$, possesses recurrent states. Let c_{n_k} ($n_k \uparrow \infty$) be the non-vanishing of the c_n. Denote by $2^{-1}D(a, b, c)$ the area of the triangle formed by any points a, b, c of the plane. Let M be the set of possible values of X_1 and put $g =$ gr.c.d. $\{D(a, b, 0); a, b \varepsilon M\}$, $h =$ gr.c.d. $\{D(a, b, c); a, b, c, \varepsilon M\}$.

Theorem 1. If V has a recurrent state and if $n_{k+1}/n_k \to 1$ ($k \to \infty$), then $\lim_k \inf |EX_1 + n_k^{-1} \sum_1^k c_{n_i}| = 0$.

Theorem 2. All points of the planar lattice are recurrent states of V if the following conditions are satisfied: (i) $\sup_k |n_k EX_1 + \sum_1^k c_{n_i}| < \infty$; (ii) $g = 1$; (iii) $\sup_k (n_{k+1} - n_k) < \infty$; (iv) there exists an integer $m > 0$ such that for $s = 1, 2, \cdots \max\{(n_{k+1} - n_k); k = s, s + 1, \cdots, s + m\} \geqq h$. Analogous results hold for linear random walks. The proof of Theorem 1 uses the strong law of large numbers. The proof of Theorem 2 is based on a result of Chung and Fuchs and a number-theoretic analysis of the set M.

42. Combinatorial Results in Multi-Dimensional Fluctuation Theory. CHARLES HOBBY and RONALD PYKE, University of Washington.

Let $\{X_t : t \geqq 0\}$ be a multi-dimensional stochastic process which has symmetric and exchangeable increments. Let $K(X_t)$ denote the sphere with the line joining X_t and the origin as a diameter. Assuming that the process is measurable and is one for which the rationals form a separating sequence, define J_T, for $T > 0$, to be the Lebesgue measure of the set $\{t \varepsilon (0, T]; X_t \varepsilon K(X_T)\}$ and define $L_T = \inf\{t \varepsilon (0, T]; \sup_{0 \leqq u \leqq t} |X_u - \frac{1}{2}X_T| = \sup_{0 \leqq u \leqq T} |X_u - \frac{1}{2}X_T|\}$. *Theorem.* If for each $t < T$, X_t does not fall on the boundary of $K(X_T)$ with probability 1, then $P[J_T \leqq x] = P[L_T \leqq x] = x/T$ for $0 \leqq x \leqq T$. The proof of this theorem is based on a simple combinatorial result for the set of partial sums of a finite number of multi-dimensional vectors.

43. Some Properties of Tukey's Test for Non-Additivity. D. HOGBEN, R. S. PINKHAM and M. B. WILK, Rutgers—The State University.

The power and sensitivity properties of Tukey's test for non-additivity in a $r \times c$ classification (*Biometrics* **5**, (1949) 232–42) are studied under an alternative multiplicative model. Assuming known error variance, a canonical form of the standardized test statistic is shown to be $T = (X + \eta Q_r Q_c)^2$ where X, Q_r and Q_c are statistically independent, X is standard normal and each Q is related to the non-central t. Properties of Q, including the moments, have been obtained previously by Hogben et al. (to be published in *Ann. Math. Statist.*). The distribution of T is approximated by that of a non-central χ^2, the approximation being exact for a 2×2 and asymptotically. Power results for several circumstances, obtained by using a scaled χ^2, which compare satisfactorily with Monte-Carlo results and a Cornish-Fisher approximation, show that the power of Tukey's test compares favorably with that of a "best" test. The test was also evaluated by means of the distribution of the significance level as characterized, partially, by sensitivity and stability, defined respectively as the mean and variance of the significance level. The latter properties are of course independent of a pre-selected "size" of test. Also, robustness to non-normality under null conditions is studied with respect to a specific class of alternative distributions.

44. **Iterated Tests of the Equality of Several Distributions.** Robert V. Hogg, University of Iowa.

Two important examples of this iterated procedure are given. The first one describes the test of the equality of the means of m independent normal distribution having common, but unknown, variance. The test is based on a sequence of Student t statistics: the first t tests the equality of the first two means, the second t tests the equality of the first three means (given that the first two means are equal), and so on. It is proved that these t statistics are mutually independent if the null hypothesis is true; hence it is easy to determine the significance level of this procedure. In addition, it is demonstrated that certain of these t statistics are mutually independent even though the null hypothesis is false; accordingly, in these cases, the probability of rejecting the null hypothesis, for a correct reason, is relatively simple to compute. Moreover, a rejection by this iterated scheme provides some reason why all m means are not equal. In the second example, the test of the equality of m distributions of the continuous type is based on a sequence of two-sample distribution-free statistics, such as the statistics proposed by Smirnov and Wilcoxon. If the distributions are equal, these statistics are proved to be mutually independent. In addition, in connection with this second example, another proof of an independence theorem of I. R. Savage is presented.

45. **Invariant Decompositions of Sums of Squares and Linear Sample Spaces.** Alan T. James, Yale University.

Let x be a column vector and G a transitive group of permutations of its n components x_i. A solution is given of the following mathematical problem. What are the possible decompositions of the sum of the squares of the x_i into quadratic forms $x'E_jx$ invariant under the permutations, with idempotent matrices E_j, $x'x = x'E_1x + x'E_2x \not\equiv \cdots$, and how can one calculate the E_j? Applications: (1) The solution is used in calculating *zonal polynomials* of latent roots in terms of which many multivariate distributions may be expanded: (2) If the x_i are regression coefficients arising from a symmetrical experimental design, the solution might be used to simplify the normal equations and invert a patterned matrix: (3) If x_i is the yield of the ith plot of an experiment whose basic design is not affected by permutations of the plots belonging to G, then a subspace, specified by a linear hypothesis for the vector of means, will usually be invariant, i.e., it will be a subspace upon which one of the above E_j projects. Hence one can characterize all possible invariant linear hypotheses.

46. **Biological Examples of Small Expected Frequencies and Chi-Squared Test.** S. K. Katti and A. N. Sastri, Florida State University and Duke University.

C. A. G. Nass (*Biometrika* 1959) has discussed the analysis of contingency tables with small expected frequencies using the chi-squared test. The method consists of evaluating two constants c and ν such that the first two moments of $c\chi^2$ are the same as those of a chi-squared distribution with ν degrees of freedom. The aim of this paper is to see how much improvement this correction makes in biological problems in which the total frequencies are small—of the order of ten, distributed over a number of cells—of the order of five. Data from 21 experiments were gathered and the results of the analyses using the ordinary chi-squared and the refinement were tabulated. It has been of interest to see that the correction increases (or decreases) the degrees of freedom and the value of the chi-squared simultaneously, thereby affecting the level of significance by a considerably smaller amount.

47. **Asymptotically Optimum Sequential Procedures. I. Inference.** J. KIEFER and J. SACKS, Cornell University and Northwestern University. (By title)

Considerations of Schwarz (1962) are extended by treating k-decision problems with or without indifference regions and with infinitely many possible states of nature, which are no longer assumed to be Koopman-Darmois (so that the shapes of regions are less explicit). As the cost c per observation approaches zero, an asymptotic lower bound on the risk is again obtained, and for any a priori distribution F we obtain a family $\{\delta_c\}$ whose risk approaches this bound uniformly on the support S of F, and which is thus asymptotically Bayes relative to all G with support S. Such k-decision sequential problems are seen to be solvable in terms of simultaneous sequential tests. For example, in testing $N(-1,1)$ against $N(1,1)$ with indifferent state $N(0,1)$, Schwarz's pentagonal bound results from three simultaneous SPRT's. For this problem a truncated (at $2|\log c|$) SPRT of $N(-1,1)$ against $N(1,1)$ is also optimum, but no triangular bounds are. (Donnelly (1957) and Anderson (1960) studied the properties of these last two.) The former two, but not the latter, asymptotically minimize the maximum expected sample size as specified bounds on error probabilities approach zero.

48. **Asymptotically Optimum Sequential Procedures. II. Designs.** J. KIEFER and J. SACKS, Cornell University and Northwestern University. (By title)

Considerations of Chernoff (1959), Albert (1961), and Bessler (unpublished) regarding sequential choice among infinitely many designs are extended to the setting of the previous abstract. The same form of conclusion regarding a family $\{\delta_c\}$ is obtained, strengthening previous results. Rather than to use procedures whose choice of design is based on maximum likelihood estimates at each stage as in the above references, the present development uses an extension of an idea first implemented by Wald (1951) in certain simpler estimation settings. A preliminary sample of size $o(|\log c|)$ but $1/o(1)$ is taken with designs chosen at the outset, and is used to "guess the true state" and thus to choose, once and for all, the designs to be used in the subsequent sequential procedure. (In Wald's case the choice was of a second sample size in a two-stage procedure.) Thus, the choice of designs is made easier in practice, and the theoretical proofs of optimality are largely referred back to the non-design considerations of the previous abstract.

49. **Linear Hypotheses With Intraclass Correlation.** H. S. KONIJN, Yale University.

For $i = 1, \cdots, n > k + 1 \geqq 1$, let $y_i = \alpha + \sum_p \beta_p x_{pi} + u_i$, where the k linearly independent vectors x_p are nonstochastic with zero sum of components, and the u_i have a joint nonsingular distribution Φ with zero means, finite and identical variances σ^2 and identical correlations ρ. (The Greek letters denote unknowns.) (1) When Φ is multinormal the likelihood does not possess a maximum. Of course, for a given ρ it does; denote the maximum likelihood estimates of α and the β_p when it is given that $\rho = 0$ by α^0, β_p^0. (2) α^0 and β_p^0 are the minimum variance unbiased linear estimates of α and β_p no matter what is ρ. (3) Confidence intervals for sets of linear combinations of α and the β_p based on Φ multinormal and $\rho = 0$ remain valid for the case in which ρ is unspecified when the set does not effectively depend on α as in the usual analysis of variance cases [Halperin (1951) *Ann. Math. Statist.* **22,** 573–580], but otherwise lose their validity as in the case of estimation of α [Walsh (1947), *Ann. Math. Statist.* **18,** 88–96] or of $\alpha + \sum\beta_p x_{p0}$ for given x_{p0}. Sometimes approximations are feasible.

50. Some Remarks on Negative Estimates of Variance in Unequal Probability Sampling. J. C. Koop, North Carolina State College. (By title)

In sampling a finite universe of N units with unequal probabilities and without replacement negative estimates of variance have been found for almost all estimators. The sign of an estimator of variance depends on the discriminant of the quadratic form in the underlying variates, and for the most part there are difficulties in arriving at meaningful interpretations for each specific case because of the complexity of the probability functions which are enmeshed in the expressions for these estimators. However, for an unbiased estimator of the population total $T = \sum_{i=1}^{N} x_i$, belonging to one of the seven possible classes (Koop, J. C. (1957). Institute of Statistics, Mimeo Series No. 296), and given by $T' = \sum_{i\epsilon s} x_i / \binom{N-1}{n-1} P_s$, where s is a sample of n units selected without replacement with total probability P_s (in regard to which $\sum_s P_s = 1$), it can be shown that a simple unbiased esimator of its variance

$$\hat{V}(T') = \left[\left(\frac{1}{\binom{N-1}{n-1} P_s} - 1\right)\sum x_i^2 + \left(\frac{1}{\binom{N-1}{n-1} P_s} - \frac{N-1}{n-1}\right)\sum_{i<j} x_i x_j\right] \Big/ \binom{N-1}{n-1} P_s$$

is always positive whenever $P_s \leqq 1 \Big/ \binom{N}{n}$. Thus all samples with high probabilities of appearing $\left(\text{in the sense of being greater than } 1 \Big/ \binom{N}{n}\right)$ will have negative estimates of variance even if the P_s are the theoretical optimum probabilities which make $V(T') = 0$.

51. Multiple Comparison Tests in Multi-Response Experiments. P. R. Krishnaiah, Univac, Blue Bell, Penna.

Let the rows of $X:n \times p$ be n independent random vectors having a p variate normal distribution with a common unknown covariance matrix Σ and means given by $E(X) = M\theta$, where $M:n \times m$ is known and $\theta:m \times p$ is unknown. In the present paper, a test based on "Step-Down Procedure" is proposed for simultaneous testing of the hypotheses $H_i:\underline{l}_i\theta = \underline{0}$, $(i = 1, 2, \cdots, K)$ where $\underline{l}_i = (l_{i1}, \cdots, l_{im})$ is a row vector of known elements subject to the restriction $\sum_{j=1}^{m} l_{ij} = 0$. This test can be applied when the variates can be arranged in descending order of importance. It is shown that the lengths of the confidence intervals associated with the above test are shorter than the lengths of the corresponding confidence intervals connected with J. Roy's test (these Annals, **29** 1177–1187). In the univariate case, the method considered in the present paper results in narrower confidence intervals than

Scheffé's method. Another test is proposed to test the hypotheses $H_1, \cdots, H_K$ simultaneously when these hypotheses can be also arranged in descending order of importance.

52. Complete Class Theorems for Unbiased Estimation. Eugene Laska, IBM Corporation, Newark, N. J.

Let $f(x, \theta)$ be a given density function continuous in θ where $\theta \varepsilon \Omega$, a compact subset of the real line. Let $\mathfrak{D}$ be the class of randomized unbiased estimators, $\delta(x)$, of θ. For squared error loss function, the class of Bayes solutions is shown to be essentially complete relative to $\mathfrak{D}$. If Ω is convex then every randomized $\delta(x)$ is inadmissible since $E\delta(x)$ has smaller risk. A theorem of Stein concerning locally best unbiased estimators is generalized to provide conditions for uniqueness (and therefore admissibility) of Bayes solutions and functional equations for their determination. If the uniqueness condition is satisfied, then the above class is a complete class of admissible unbiased estimators.

53. The Expected Intersection of a Random Sphere and a Fixed Sphere. A. G. Laurent, Wayne State University.

Let the center of a n dimensional sphere W with radius R follow a spherical normal distribution centered at the origin with standard deviation $\underline{\sigma} = 1$; let T be a n dimensional sphere with radius r, whose fixed center is at distance D from the origin. Then the expected value of $W \cap T$ is $E[W \cap T] = (2\pi Rr/D)^{n/2} D \int_0^\infty J_{n/2}(Ru)J_{n/2}(ru)J_{n/2-1}(Du)u^{-n/2} \exp - (u^2/2)\, du$, where $J_k(\cdot)$ denotes the Bessel function of order K of first kind. The paper gives also a new formula for the non central chi-square distribution. These results generalize those obtained by E. H. Smith and D. E. Stone for $n = 2$. They have direct applications to bombing problems.

54. Some Tests for Gamma Parameters With an Application to a Reliability Problem. M. M. Lentner and R. J. Buehler, Iowa State University.

If z_1 and z_2 are gamma variates with scale parameters θ_1 and θ_2, then a UMPU similar region test can be found for the hypothesis $\gamma = \gamma_0$ where $\gamma = c_1/\theta_1 + c_2/\theta_2$ (Lehmann and Scheffe). Appropriate conditional distributions are given for $(c_1, c_2) = (1, 1)$ and $(1, -1)$. Application: A series system has two dissimilar components whose expected lives are θ_1 and θ_2. When component failures are exponentially distributed, so are system failures, the mean being $\theta = (\theta_1^{-1} + \theta_2^{-1})^{-1}$. From separate estimates of θ_1 and θ_2 one can obtain confidence limits for θ.

55. Exact Power of Some Tests Based on a Generalization of Mood's Statistics. Fred C. Leone, I. M. Chakravarti and G. E. Haynam, Case Institute of Technology.

The exact power of Mood's test based on the median of "c" combined samples is developed. The power function for the median test is obtained for alternatives of translation of the exponential distribution as well as alternatives of change in location and scale of the rectangular distribution. These powers are compared with the two-sample case developed earlier. Tables of the power for selected values of a sample size are presented.

56. A Theoretical Model for Achieving Selective Biological Effects. James B. MacQueen, University of California, Los Angeles.

A biological system S is composed of k component systems $S_1, S_2, \cdots S_k$. Embedded in these systems is a system S_0 which is to be destroyed without destroying S. To accomplish this there is available a treatment space Ω. When a multiple treatment φ, an additive measure on Ω, is used, the response of the ith system, $i = 0, 1, \cdots k$, is given by $\int g_i \, d\varphi$, where g_i is the sensitivity spectrum of the ith system. The systems have critical response levels $a_0, a_1, \cdots a_k$ such that the ith system is destroyed if and only if $\int g_i \, d\varphi > a_i$. The system S can survive as a whole if all of the elements of at least one of certain essential subsets $V_1, V_2, \cdots V_M$ of the set $\{S_1, S_2, \cdots S_k\}$ are not destroyed. Whether or not there is a multiple treatment which selectively destroys S_0 can be determined by solving the M problems: Maximize $\int g_0 \, d\varphi$ subject to the constraints $\int g_{i_\delta} \, d\varphi \leqq a_{i_\delta}$ where i_δ ranges over the i for which $S_i \varepsilon V_\delta$, $\delta = 1, 2, \cdots M$. If one of these problems gives an optimal treatment φ^* for which $\int g_0 \, d\varphi^* > a_0$, a successful multiple treatment is possible. The maximization problems are easily handled as linear programming problems. Since typically such problems are approached by means of a pure treatment; i.e., a point or very small region in Ω, and this has certain advantages, it is desirable to investigate conditions under which only multiple treatments are effective. Some results on this problem are presented and a general type of critical experiment is suggested.

57. **A Generalized Branching Process.** Peter E. Ney, Cornell University.

The usual age dependent branching process (see e.g., R. Bellman and T. E. Harris, *Ann. of Math.*, **55**, No. 2; N. Levinson, *Illinois J. Math.*, **4**, No. 1) is generalized by associating a random valued characteristic with each particle. (In a cascade process the characteristic would be the energy of the particle.) Let $Z(t)$ be the sum of characteristics at time t, and $P(z, t \mid z_0)$ be the d.f. of $Z(t)$, given that the initial particle had characteristic z_0. The definition of the process suggests that $P(z, t \mid z_0)$ satisfies an integral equation (I.E.). Starting formally with the I.E. given, it is shown that it has a solution which is a d.f. and is unique among bounded solutions. Let $\mu_m(t \mid z_0) = \int_0^\infty zP(dz, t \mid z_0)$. Conditions are given for the existence of μ_m, and it is shown to satisfy a renewal I.E. Thence its asymptotic behavior for large t is determined. It is shown that under certain conditions the r.v. $Z(t)/\mu(t \mid z_0)$ converges in mean square to a r.v. $w(t \mid z_0)$. In a special case $w(t)$ is shown to possess a density function.

58. **On Reliability Inference** (Preliminary report). Edward L. Pugh, System Development Corporation, Santa Monica, Calif.

Let $F(t)$ be the c.p.f. of the time to failure of a given system. This paper considers the problem of making a statistical inference concerning the reliability $R = 1 - F(t_m)$ (where t_m denotes "mission time") from the point of view of (i) no assumption concerning $F(t)$ except continuity, (ii) assuming $F(t)$ is Weibull, and (iii) assuming $F(t)$ is exponential (a special case of the Weibull). In each case a fiducial distribution $\Phi(R; x)$ of the reliability, depending on a statistic x, is derived and is consistent with the theory of confidence intervals. In case (i), $\Phi(R; x)$ is distribution-free (independent of $F(t)$) whereas in cases (ii) and (iii) it depends on $F(t)$ as well as on the statistic x. This raises the question of the definition of a "best" fiducial distribution for a given $F(t)$, and such a definition is offered. In case (iii), fiducial distributions are derived from two statistics: the rth order statistic, t_r, and the sample mean, $\hat{\theta}$. It is shown that the random variable $\int_0^1 R \, d\Phi(R; \hat{\theta})$ is a sufficient and consistent estimate for R with less bias than the maximum likelihood estimate.

59. **On Discriminating Between Two Gamma Processes-I.** A. S. Qureishi and K. J. Nabavian, The Service Bureau Corporation, San Jose, Calif. (By title)

Given two processes, the units from which fail in accordance with the gamma distribution (in which shape parameter can only take integral values), the problem of selecting the particular process with the larger mean life is considered. Three procedures are constructed to solve this problem. They are: R_1, a non-sequential, non-replacement type of procedure; R_2, a sequential replacement type of procedure and R_3, an alternative sequential replacement type of procedure where the actual life times of failed items are considered in making a decision. Let θ_1 and θ_2 be the true values of the unknown scale parameters of the gamma distribution such that $\theta_1 > \theta_2$. It is not known which process has the parameter θ_1 and which has θ_2. Let α denote the true value of the ratio θ_1/θ_2 which must be greater than or equal to one, then the specifications of R_1 and R_2 consists of two quantities $\alpha^*(\alpha^* \geqq 1)$ and $P^*(\frac{1}{2} < P^* < 1)$ where, (1) α^* is the smallest value of α (say $\alpha = \alpha^* > 1$) that is worth detecting, and (2) $P^* > \frac{1}{2}$ is the minimum value for the probability of a correct selection whenever $\alpha \geqq \alpha^*$. For R_3, let $\theta_1^* > \theta_2^*$ be specified and it is desired to have the probability of a correct selection of at least $P^*(\frac{1}{2} < P^* < 1)$ whenever $\theta_1 \geqq \theta_1^* > \theta_2^* \geqq \theta_2$.

60. **A Method For Estimating Linear Functionals in a Time Series** (Preliminary report). PAUL H. RANDOLPH, Purdue University. (By title)

Consider a vehicle whose position $x(t)$ at times $t_1, t_2, \cdots, t_n$ is measured, giving the values $X_1, X_2, \cdots, X_n$. It is often desired to estimate some value, such as velocity, by means of an estimator $L^*[X(t_1), X(t_2), \cdots, X(t_n)]$. It is assumed that the signal is $X(t) = x(t) + n(t)$, where the $x(t)$ is the true position and $n(t)$ is a noise in the system. Often some information is available regarding the position and thus we can say that

$$x(t) = x_R(t) + x_d(t)$$

where $x_d(t)$ represents this knowledge and belongs to a class θ of functions and $x_R(t)$ is a component about which we have a statistical description. We base our choice of an estimator on a risk function $g(x_d, L^*) = P_{x_d}[|L^*(\cdot)(t) - L(x)(t)| > h]$ where h is a known constant. We choose L^* so as to minimize $\sup_{x_d} g(x_d, L^*)$. Examples are given to illustrate this procedure.

61. **On Rotation Sampling** (Preliminary report). J. N. K. RAO and JACK E. GRAHAM, Iowa State University.

When the same population is sampled on repeated occasions, it is well known that the use of rotation sampling may increase the precision of the estimators on the current occasion and of the change. Hansen et al. (*J. Amer. Statist. Assoc.*, 1955) have developed composite estimators of the population total on the current occasion and of the change (e.g., quarter-to-quarter change in a quarterly survey). However, the variance of these estimators was not investigated in detail and, moreover, the population size N on each occasion was assumed infinite. The purpose of the present paper is to develop a unified finite population theory for the composite estimators. This is accomplished by considering that the finite population is the $N!$ possible rotation patterns (assuming that N is constant from occasion to occasion) and the sample consists of one random rotation pattern from this population. The rotation pattern of the sample is formulated as follows: n_2 units stay for r occasions ($n_2 r = n$ where n is the number of units in the sample on any occasion) and leave the sample for m occasions and then come back, where m is seen to be equal to $r(N/n - 1)$. Then the general variance formulae for the composite estimators are developed and the optimum values of Q, the weight factor in the composite estimator, and the optimum values of r are determined for certain special correlation patterns of the characteristics in different occasions.

62. Some New Inequalities of Chebyshev Type. DONALD RICHTER, C. L. MALLOWS and MILTON SOBEL, Bell Telephone Laboratories, Inc. and University of Minnesota.

A number of new inequalities of Chebyshev type are derived, all of which have the property that they involve conditional expectations. A typical result is the following, valid for any real random variable X with finite second moment and any measurable set A: $P(X \text{ in } A)[E(X \mid X \text{ in } A) - E(X)]^2 \leqq [1 - P(X \text{ in } A)] \operatorname{Var} X$. Similar results are derived for the case when the single set A is replaced by a number of sets, and for the case when X and A are k-dimensional; in each case conditions for equality are given. By specializing the results, useful bounds are obtained for percentiles and for sample values. The relation between the new inequalities and previous results is carefully discussed.

63. Approximate Uncorrelating Transformations for Normal Order Statistics. SAMUEL S. SHAPIRO and MARTIN B. WILK, Rutgers—The State University. (By title)

Let V be the $n \times n$ symmetric matrix of covariances for standard normal samples of size n. Let T be the $n \times n$ lower triangular matrix with positive diagonal elements, such that $TVT' = I$. Then T is unique. If y is a vector of order statistics from $N(\mu, \sigma^2)$ then the transformation $z = Ty$ yields uncorrelated variates. The matrix T has been found to be well approximated by a "double diagonal" matrix, T^*. The non-zero elements of T^* have been tabulated for samples up to size 20. The adequacy of this approximation to T is indicated by the extent to which $I^* = T^*VT^{*\prime}$ approximates I. The diagonal terms of I^* lie between 1.000 and 1.001, the $(i, i - 1)$ terms lie between 0 and .001 and all others lie between $-.014$ and 0. This approximation has been found to be very good for use with best unbiased estimates based on order statistics (censored and uncensored). The "triple diagonal" matrix $V^{*-1} = T^{*\prime}T^*$ provides a good approximation to V^{-1}. Further the non-zero elements of V^{*-1} can be easily generated (approximately) from the first one half of the diagonal elements. The use and interest in these results lie in the following directions: in parsimony of tabulation (for normal and other distributions); in indicating that linear combinations of adjoining pairs of order statistics may, to a good approximation, be made statistically uncorrelated; in reducing substantially the arithmetic load in finding best linear unbiased estimates based on order statistics (censored and uncensored). A preliminary report of some of this work has been given in Technical Report No. N-6, Rutgers Statistics Center.

64. On the Inadmissibility of Some Standard Estimates in the Presence of Prior Information. MORRIS SKIBINSKY and LOUIS J. COTE, University of California, Berkeley, and Purdue University.

Suppose Θ, X are random variables such that $P(X = x \mid \Theta = \theta) = f(x,\theta)$, $x = 0, 1, \cdots, n$, $0 \leqq \theta \leqq 1$, where $f(x, \theta)$ is the value at x of the binomial frequency function with parameters n, θ, and Θ is distributed for some α, δ, $0 < \alpha < 1$, $0 < \delta \leqq \frac{1}{2}$ so as to be in $|\delta, 1 - \delta|$ with probability $\geqq 1 - \alpha$. A maximum likelihood method is proposed for predicting Θ from X. Each member in a class of predictors for Θ suggested by the method is shown to be uniformly better over the class of distributions above described than the standard $X \mid n$, relative to the squared difference loss function, provided only that $\alpha > 0$ is sufficiently small. Numerical computations give precise values of α for selected n, δ. A second analogous example concerns prediction for the random mean of a normal distribution with known variance, for which similar results are obtained.

65. A New Approach to Factorial Experiments. J. N. SRIVASTAVA, University of North Carolina.

In this paper, a new approach to factorial experiments has been introduced. It is somewhat parallel to Hotelling's method of principal component analysis. The new approach combines most of the good properties of the classical and response surface approaches, and is largely nonparametric. A new criterion in the estimation of response surfaces, namely, the principle of contiguity (due to S. N. Roy) has been explained, and the new approach is shown to satisfy it. Preliminary theory has been developed for two cases (i) when all the factors are structured, (ii) when all the factors are completely unstructured.

66. Use of Some Extraneous Information in the Estimation of Regression Coefficients. YUKIO SUZUKI, Institute of Statistical Mathematics, Tokyo, Japan.

Consider the normal regression model: $y = \sum_{i=1}^{p} \beta_i x_i + \epsilon$. In this paper, we treat the problem of the estimation of regression coefficients $\{\beta_i\}$ $(i = 1, 2, \cdots, p)$ in the situation that some extraneous information is given and from it a joint probability measure of a certain subset of regression coefficients, say $\{\beta_i\}$ $(i = 1, 2, \cdots, k)$, can be derived. This probability measure can be regarded as a prior information concerned regression coefficients to be estimated. Thus by means of Bayes' theorem, for a sample obtained by observation on the regression model, there corresponds a posterior distribution of $\{\beta_i\}$ $(i = 1, 2, \cdots, p)$. The unbiased estimate with minimum variance of our problem, is the mean vector of the posterior distribution corresponding to the sample obtained. Of course, this estimate is more or less different from the one obtained by using only the sample information. Also, several analogous problems are discussed here.

67. On Characterization and on Complete Solution of Two-Person Zero-Sum Games (Preliminary report). JOSEPH V. TALACKO, Marquette University.

Every Two-Person Zero-Sum Game G, equivalent to a quadruple $G = (T, S; ((a_{ij})); 1_n, -1_m; 0)$, has, as a complete solution, a finite set of distinct resolvents $R = \{S, T; ((\bar{a}^{ij})); \alpha, \beta; v\}$. The sets S, T are indices; the $((a_{ij}))$ is the original pay-off matrix; and 1_n and 1_m are vectors of units of dimensions m, n. The scalar v is the value of the game; $((\bar{a}^{ij})) = \bar{A}$ is a set of resolvent [pseudoinverses] matrices, where $\bar{H} \subseteq \bar{A}$ submatrices, called characteristic kernels, have the property that every $\bar{H}_k = ((h^{ij}))$;

$$\sum_i h^{ij} = \sum_j h^{ij} = 0.$$

Characteristic kernels are invariant with respect to the value of the game. Every distinct resolvent is identified unmistakably by the distinct characteristic kernel and by the pair of strategies. A single algorithm called *The Symmetric Method* solves mechanically every Two-Person Zero-Sum Game without adjustments of the given pay-off matrix, even if the value of the game is zero; the large problems, degenerate games.

68. Reduction of Wasted Stringency in Ranking and Counting Procedures Through the Use of Supplementary Criteria (Preliminary report). JOHN W. TUKEY, Princeton University.

Procedures centered on ranks and counts habitually give rise to distributions offering limited choices of actual significance level. Randomization offers only an impractical solution. Separation of configurations giving borderline values to the initial criterion into "significant" and "non-significant" on the basis of a supplementary criterion cannot only greatly reduce wastage of stringency but can have additional value. Since supplementary computation will only be necessary for borderline configurations, the increase in average effort is small. Use of the sum of squares of ranks (or the highest [lowest] rank) as a supplementary criterion for Wilcoxon and signed-rank procedures, and of sums of "next-cell" counts for Duckworth and quadrant-sum procedures are straightforward applications. Where the observations occur in time order, bear rationally-assigned serial numbers, or have rationally-alphabeticizable names or titles, the supplementary criterion may well be an indicator of trend, with more apparently trending configurations assigned to non-significance. Sign-test procedures, other binomial significance and confidence procedures, and the "exact test" for 2×2 tables can all be treated in this way. Computation of supplemented tables for Wilcoxon and signed-rank procedures has begun.

69. **Minimax Estimate of an Inverse-Binomial Parameter.** M. T. WASAN, Queen's University, Ontario, Canada.

Let X be a random variable with a family of Probability functions $P\{X = x\} = pq^{x-1}x = 1, 2, \cdots, \infty$ where $0 < p \leqq 1$ and a loss function is squared error. We want to find a minimax estimate of p. Let us consider the following estimate in order to have constant risk, i.e., $p = 3/4$ when $x = 1$ and $p = 1/4$ when $x \geqq 2$, then risk, i.e., expected loss is 1/16. We try a following apriori distribution of $p:p = 1/4$ with probability 2/3 and $p = 1$ with probability 1/3. It can be readily checked that the constant risk estimate of p is Bayes estimate corresponding to the apriori distribution. Thus by (*) Theorem 2.1 of Hodges, J. L., Jr. and Lehmann, E. L., Some Problems in Minimax Point Estimation, *Ann. Math. Statist.* **21,** 182–197, it is a minimax estimate of p.

Now let X be a r.v. with a family of probability functions $p[X = x] = qp^{x-1}x = 1, 2, \cdots \infty$ and $0 \leqq p < 1$ and a loss function is squared error. It can be easily seen that $p = 1/4$ when $x = 1$ and $p = 3/4$ when $x \geqq 2$ is the constant risk estimate. We try a following apriori distribution of $p:p = 0$ with probability 1/3 and $p = 3/4$ with probability 2/3. The constant risk estimate is Bayes estimate corresponding to the apriori distribution. Hence from (*) it follows that it is a minimax estimate of p.

70. **Sample Size Determination for Tolerance Regions for the Exponential Distribution.** DAVID L. WEEKS and LEE J. BAIN, Oklahoma State University.

γ probability and β expectation tolerance regions are derived based on four commonly used estimators of the parameter in the exponential distribution. These estimators are functions of the sample mean, observations from a censored sample (with and without replacement), and the order statistics. The sample size required to obtain γ probability tolerance regions of the form $\text{Prob}[\beta < P_{L(\hat{\theta}_i)} < \beta + \epsilon] = \gamma$ based on each of the four estimators is determined where $P_{L(\hat{\theta}_i)}$ is the content of the tolerance region based on limits which are a function of $\hat{\theta}_i$. Also $0 < \beta < 1$, $0 < \epsilon < 1 - \beta$, and $0 < \gamma < 1$.

The variances of the contents based on each estimator is also derived for the β expectation tolerance region.

71. **A Generalized Single Server, Poisson Input, Queueing Process** (Preliminary report). PETER D. WELCH, IBM Corporation and Columbia University. (By title)

The following generalization of the $M/G/1$ queueing process is considered. If a customer arrives when the server is busy, his service time has an arbitrary distribution function

$G_b(x)$; while if he arrives when the server is idle, his service time has a different arbitrary distribution function $G_e(x)$. Results are obtained which characterize the transient and asymptotic distributions of the queue size, waiting time, and virtual waiting time.

72. **Pre-Emptive Resume Priority Queues** (Preliminary report). PETER D. WELCH, IBM Corporation and Columbia University.

The following queueing problem is considered. The input is the superposition of r independent Poisson processes, each process corresponding to a priority level. Associated with each priority level, there is a distinct arbitrary service time distribution function $G_k(x)$; $k = 1, \cdots, r$. A single server operates under a pre-emptive resume priority service discipline. For this process, Miller (1960, *Ann. Math. Statist.* **31,** 86–103) has characterized the virtual waiting time for the kth priority level, $k = 1, \cdots, r$. We obtain results which characterize the transient and asymptotic behavior of the actual waiting time and of certain queue sizes. Further, using an argument different from his, Miller's results on the virtual waiting time are obtained.

73. **Estimation of Parameters of the Weibull Distribution.** JOHN S. WHITE, General Motors Research Laboratories, Warren, Michigan. (Invited)

The Weibull distribution, $F(x) = 1 - \exp[-(x/\theta)^{\beta}]$, is frequently used for the analysis of fatigue life. Several techniques for estimation of the parameters $\beta =$ "slope" and $\theta =$ "characteristic life" are presented in this paper. The simplest approach is to plot the ordered failure times $(x_1, \cdots, x_r)$, $r \leqq n$, of a random sample of size n on Weibull probability graph paper and visually fit a straight line to the plot. A second approach is to fit the line to the plot using least squares. Thirdly, the parameters may be estimated by the method of maximum likelihood. These three methods are discussed in detail and Monte Carlo results comparing least squares and maximum likelihood estimation are presented.

74. **On Two-Stage Non-Parametric Estimation.** ELIZABETH Y. H. YEN, University of Minnesota.

Let X and Y be two independent random variables with distribution functions F and G respectively. An estimable parameter $\theta(F, G)$ is to be estimated with a fixed sample size N and with U-statistics as estimators. Two kinds of two-stage estimators are defined—U', a function of the second stage observations only, and U'', a function of all N observations. Let U^o be the unbiased one-stage estimator. When the "nuisance parameters" concerning (F, G) are known, the lower bound of the variance (or the asymptotic variance for large N) of U^o can be minimized by a suitable allocation of observations to X and Y. If the risk is mean squared error, then denote the risks of U^o, U', and U'' by $R(U^o)$, $R(U')$, and $R(U'')$ respectively. It is shown that the ratios $R(U')/R(U^o)$ and $R(U'')/R(U^o)$ can, by the proper allocation of the observations, be made to tend to unity as N tends to infinity. The asymptotic distributions of U' and U'' are shown to be normal. Using these estimators as test statistics the relative efficiencies (Pitman's criterion) of U' and U'' with respect to U^o are unity. The same technique can be extended to estimable parameters of more than two independent random variables.

75. **On Conditions for Equality of Best and Simple Linear Least Squares Estimators.** GEORGE ZYSKIND, Iowa State University. (Invited)

Consider the model $y = X\beta + e$ where X is a known $n \times p$ matrix of rank r and the $n \times 1$ vector of errors, e, satisfies $E(e) = 0$, $E(ee') = \sigma^2 V$, where V is nonsingular. Any one of the following conditions is both necessary and sufficient for the equality of every simple least square estimator with the corresponding best linear unbiased estimator.

(1) The column space of the matrix X is an invariant subspace of the matrix V.

(2) A matrix Q exists satisfying the relation $VX = XQ$.

(3) A subset of r eigenvectors of V exists forming a basis for the column space of X.

(4) A full rank reparametrization exists so that $E(y) = X\beta = W\theta$, where every column of the $n \times r$ matrix W is an eigenvector of V.

(5) The matrix V is expressible in the form

$$V = \begin{pmatrix} O_1' \\ O_2' \end{pmatrix} \Lambda(O_1, O_2),$$

where the matrix $O = (O_1, O_2)$ is orthogonal and O_1 is *any* orthonormal basis of the column space of X, O_2 is *any* orthonormal basis of the orthogonal complement of the column space of X, and Λ is any diagonal matrix with positive diagonal elements.

(6) The covariance matrix V can be diagonalized by an orthogonal matrix specified as in (5).

(7) If P denotes the orthogonal matrix projection operator on the column space of X then $VP = PV$, i.e., the matrices V and P commute.

(8) The matrix VP is symmetric.

It should be noted that the simple least square estimator of $E(y) = X\beta$ is $X\tilde{\beta} = Py$. Since for standard situations, such as those of the common experimental designs, the vector Py is known it follows that for those situations the projection operator P can be obtained immediately. It is then a simple matter to check condition (8), i.e., the symmetry (or the lack of it) of the matrix VP. An examination of induced covariance structures in classification models is made by appealing to the above stated properties.

(Abstracts not connected with any meeting of the Institute.)

1. Asymptotic Power and Asymptotic Relative Efficiency of Mood's Test for Incomplete Block Designs. V. P. Bhapkar, University of Poona, India.

In a previous paper (V. P. Bhapkar, *Ann. Math. Statist.* **32** (1961), 846–863) the author generalized to incomplete block situations the M-test proposed by Mood for the hypothesis of equality of "treatment" effects in a Randomized Blocks design. In this paper the asymptotic power of this generalized test is obtained for translation-type alternatives and the asymptotic efficiency is computed relative to the F-test and Friedman's χ_r^2-test. This includes, then, as a special case results obtained by Sathe (*Ann. Math. Statist.* **32** (1961), 631) for the complete block situation with odd number of treatments. It has been shown that the asymptotic efficiency of M-test relative to the F-test is the same for any block design with the same number, say k, of plots per block, and moreover, depends only on k and the distribution. A similar statement is shown to hold relative to the χ_r^2-test for BIBD and is conjectured for PBIBD and any block design with the same number of plots per block.

2. Multivariate Tests of Hypotheses with Incomplete Data. R. P. Bhargava, Stanford University and Forest Research Institute. Dehra Dun, India.

Consider a sample from a multivariate normal population in which some of the observations are missing. These missing observations form a certain pattern which we call a monotone sample. The present paper deals with a discussion of the maximum likelihood estimates of the parameters for a monotone sample which are then used in providing likelihood ratio tests for many multivariate problems, e.g., Hotelling's T^2, Wilk's tests, general linear hypotheses, etc. In each case the results generalize the usual statistics when there are no missing observations. In addition some linear hypotheses which are more general than the usual general hypothesis are defined and the test statistics obtained. The null distributions of the likelihood ratio statistics are obtained implicitly (i.e., as products of powers of beta random variables, etc.) as well as asymptotically using Box's method [1949].

3. Note on a Conditional Property of Student's *t*. R. J. Buehler and A. P. Feddersen, Iowa State University.

For one degree of freedom the 50% fiducial interval based on Student's t is $x_{\min} \leqq \mu \leqq x_{\max}$. In the subset $3|x_1 - x_2| \geqq 2|x_1 + x_2|$ it is shown that for all (μ, σ), $x_{\min} \leqq \mu \leqq x_{\max}$ occurs with probability exceeding 51.8%. The choice of subset is based on Stein's 1961 Wald Lectures.

4. Monotonicity of a Family of Test Procedures for MANOVA and for the Test of Independence Between Two Sets of Variates. Somesh Das Gupta, University of North Carolina.

It is known that under certain invariance restrictions and with the usual normality assumption the test procedures for (i) testing multivariate linear hypothesis, or for (ii) testing independence between two sets of variates, depend on the latent roots of a random matrix; the power function of such a test involves certain non-centrality parameters which can be expressed as the latent roots of a certain matrix and can be regarded as measures of deviations from the hypothesis to be tested. It is shown that for each one of the above cases (i) and (ii), there exists a fairly large class of test procedures, characterized by a set of symmetric functions of the roots, such that the power function of any test belonging to this class increases monotonically as each non-centrality parameter, separately, increases. The likelihood-ratio test and the generalized T^2-test of Hotelling belong to this class and thus the power functions of these two tests for each one of the above testing problems have the monotonicity property. However, Roy's largest-root test does not belong to this class but the monotonicity of the power function of this test has been shown by Roy and Mikhail (*Ann. Math. Statist.*, **32** (1961) 1145–1151).

5. Some Special Problems in Classification. Somesh Das Gupta, Columbia University.

Consider the problem of classifying an experimental unit into one of k univariate normal populations such that the unknown population parameters satisfy the assumptions made in the usual ANOVA model. This situation arises when the k populations are identified with the k cells of a statistical design and random samples are available from each of these cells. It is shown that for this problem, the maximum likelihood (ML) classification rule (the rule which assigns the experimental unit to that population for which the maximum likelihood is the largest in the set of k possible maximum likelihoods) is an admissible rule with simple loss function. This result is generalized to the multivariate case when the common dispersion matrix is known and the unknown mean vectors are restricted as in the usual MANOVA model. Secondly, a class of admissible rules is derived for the problem of classification into one of k multi-normal populations $\mathfrak{N}(\mathbf{y}_i, \Sigma_i)$, $(i = 1, \cdots, k)$, when the Σ_i's are known and it is further known that the experimental unit to be classified comes from one of these populations; it is also shown that for this problem, the ML rule is ϵ-admissible.

6. On Some Statistical Inferences for Weibull Laws (Preliminary report). Satya D. Dubey, Procter and Gamble Co., Cincinnati, Ohio.

Using moment estimators for the location, scale and shape parameters of the 3-parameter Weibull law as trial values, their maximum likelihood (M.L.) estimators are computed by means of the Newton-Raphson method. The large sample covariance matrix of the M.L. estimators for these three parameters is derived which exists when the shape parameter is larger than two. This result is used to generate B.A.N. estimators or the approximate M.L. estimators for the parameters, which are asymptotically jointly efficient. The asymptotic properties of M.L. estimators are used to test hypotheses and construct asymptotically smallest confidence regions for Weibull parameters. Special cases of Weibull laws are discussed. The first four cumulants of the derivative of the logarithmic likelihood function have been derived for all the 1-parameter Weibull case which are used to indicate the effects of skewness and kurtosis of M.L. estimators in case of small samples. They involve polygamma functions in some cases. The Cornish-Fisher expansion has been employed to obtain closer approximations to the approximate confidence interval for the location parameter of the 1-parameter Weibull law. Finally, lower confidence limit for several reliability functions and upper confidence limit for several intensity functions (failure rates) of Weibull laws have been derived.

7. On Amounts of Information in σ-fields. S. G. Ghurye, University of Minnesota.

Another proof of the following theorem proved by Kallianpur (On the amount of information contained in a σ-field, *Contributions to Probability and Statistics*, Stanford University Press, 1960) is given: Let P, Q be probability measures on a measurable space $(S, \mathfrak{F})$, such that P is absolutely continuous with respect to Q. For any finite $\mathfrak{F}$-measurable partition $\underline{A} = [A_1, A_2, \cdots, A_n]$ of S, let $I(\underline{A}) = \sum_1^n P(A_i) \log [P(A_i)/Q(A_i)]$, and let $I(\mathfrak{F})$ denote the supremum of $I(\underline{A})$ over all $\underline{A}$. Then $I(\mathfrak{F}) = \int_S \log (dP/dQ)\, dP$.

Other results concerning $I(\mathfrak{F})$ are proved; e.g., (1) If $\mathcal{G}$ is a sufficient sub-field of $\mathfrak{F}$ for the family $[P, Q]$, then $I(\mathfrak{F}) = I(\mathcal{G})$; but the converse is not true. (2) If $\mathfrak{F}_i$, $i = 1, 2$, are sub-σ-fields of $\mathfrak{F}$ which are independent under both P and Q, and $\mathcal{G}$ is the σ-field generated by $\mathfrak{F}_1 \cup \mathfrak{F}_2$, then $I(\mathcal{G}) = I(\mathfrak{F}_1) + I(\mathfrak{F}_2)$.

8. Effects of Inequality of Variance-Covariance Matrices in Multivariate Analysis of Variance. I. The Case of Two Samples. K. Ito and W. J. Schull, University of Michigan.

Current interest in the multivariate analysis of variance (MANOVA) must inevitably lead to a scrutiny of the robustness of this test. Among the assumptions underlying MANOVA is the equality of variance-covariance matrices. The purpose of this paper is to explore the consequences of violation of this assumption. In the two-sample case when the samples are both large, of equal or near equal size, and when the relationship of one variance-covariance matrix, Σ_1, to the other, Σ_2, is such that the latent roots, λ_i, of $\Sigma_1\Sigma_2^{-1}$ are not distinct, within the range $0.5 \leqq \lambda \leqq 2.0$, the effect on the confidence level and power of the test of the inequality of Σ_1 to Σ_2 is not pronounced, but increases with p, the number of variates. When the latent roots of $\Sigma_1\Sigma_2^{-1}$ are distinct, then, for $p = 2$, two cases must be considered. If both roots are less than unity, the effect on the confidence level and the power of the test is somewhat more marked than when the roots are not distinct. If one root is greater than unity and the other not, the effect is less than in the case where the roots are not distinct.

9. Asymptotic Power of Certain Test Criteria Based on First and Second Differences. A. R. Kamat, Gokhale Institute of Politics and Economics, Poona, India.

In a recent paper (Kamat, A. R. and Sathe, Y. S. (1962). *Ann. Math. Statist.* **33** 186–200) the asymptotic power of the six ratio criteria w^2, w_2^2, u^2, W, W_2. and U (based on the first and second differences of observations) was discussed for the alternative of serial correlation between successive observations. In the present paper a similar technique is used to determine the relative asymptotic efficiencies of the same six criteria for certain other alternatives of serial correlation and for alternatives of linear, quadratic and sinusoidal trends. It is found that there are two different patterns of relative asymptotic efficiencies. For certain alternatives of serial correlation the six criteria have efficiencies in the order w^2, w_2^2, W_2, W, u^2, U. For certain other alternatives of serial correlation, however, and for all alternatives of trend the criteria are not all comparable. They separate into two groups (i) w^2, W, w_2^2, W_2 and (ii) u^2, U and the order of their efficiency in each group is the order in which they are written above.

10. On a Theorem of Halmos Concerning Unbiased Estimation of Moments.

HENDRIK S. KONIJN, Yale University.

In "The Theory of Unbiased Estimation" (*Ann. Math. Statist.* 1946) Halmos investigated which functions over certain classes of distributions of points on the line admit unbiased estimates and which are all possible unbiased estimates of any such given function, with particular attention to moments. He mentions the desirability of discussing this for the class of normal distributions. That is done in the present paper. The results differ considerably from those for the class discussed by Halmos.

11. A New Formula for Sample Sizes for Population Tolerance Limits.

G. P. STECK, Sandia Corporation, Albuquerque, New Mexico.

Let $x_r(b)$ denote the solution of the equation $I_b(x - r + 1, r) = \alpha$, where $I_x(p, q)$ is Karl Pearson's notation for the incomplete beta function. Let $L_r(b) = x_r(b)/x_1(b)$, where $x_1(b) = \log \alpha/\log b$. It can be shown that $L_r(1) = \chi_{2r}^2(\alpha)/(-2 \log \alpha)$ and $L_r'(1) = (r - 1)/(2 \log \alpha)$, where $\chi_{2r}^2(\alpha)$ is the upper 100 α per cent point of a chi-square distribution with $2r$ degrees of freedom. Consequently, the Taylor expansion of L_r to two terms is $L_r(b) \cong [\chi_{2r}^2(\alpha) + (1 - b)(r - 1)]/(-2 \log \alpha)$ and, therefore, $x_r(b) = x_1(b)L_r(b) \cong [\chi_{2r}^2(\alpha) + (1 - b)(r - 1)]/(-2 \log b)$. Writing

$$-\log b = -\log \{1 - (1 - b)\} \cong 1 - b + \tfrac{1}{2}(1 - b)^2 \cong 2(1 - b)/(1 + b)$$

for the first term and writing $-\log b \cong 1 - b$ for the second term, one obtains $x_r(b) \cong \frac{1}{4}\chi_{2r}^2(\alpha)(1 + b)/(1 - b) - (r - 1)/2$, which is the empirically determined approximation given by Scheffé and Tukey (*Ann. Math. Statist.*, **15** (1944) 217).