COMBINATORIAL RESULTS IN MULTI-DIMENSIONAL FLUCTUATION THEORY

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Combinatorial lemmas have been used quite successfully in analyzing sums of random variables. However, except for the work of Baxter [1], all the results of which the authors are aware have been restricted to the case of real numbers and real variables. In the present paper we examine one possible m-dimensional analogue of the combinatorial results contained in [2]. Other generalizations using rectilinear regions have been attempted but have been found not to lead to invariant results. Our main combinatorial result is given in Theorem 1. An application of this result to multidimensional stochastic processes is given in Theorem 2.

Let $c = (c_1, \cdots, c_n)$ be a sequence of n real numbers. Denote by $\mathcal{S}_c$ the set of all $2^n n!$ sequences which can be formed from c by permuting the numbers c_i and by assigning a $+$ or a $-$ sign to each of the c_i. If $x \varepsilon \mathcal{S}_c$, set $s_0(x) = 0$ and $s_i(x) = x_1 + \cdots + x_i$ for $i = 1, 2, \cdots, n$. We say that the sequence x has type (m, k) if exactly m of the partial sums $s_i(x)$ are greater than 0 and exactly k of the partial sums $s_i(x)$ are less than $s_n(x)$. Set $v_n(m, k; c)$ equal to the number of sequences $x \varepsilon \mathcal{S}_c$ which have type (m, k). The sequence c is said to possess *property* D if for every x in $\mathcal{S}_c$, $s_i(x) \neq 0$ for $i = 1, 2, \cdots, n$. In an earlier paper, the authors have shown (Theorem 2.1 of [2]) that $v_n(m, k; c)$ is independent of c if c possesses property D. More precisely, if c possesses property D, then

$$(1) \qquad v_n(m, k; c) = \binom{2m}{m}\binom{2k}{k} 2^{n-1-2m-2k}(n-1)!$$

for all $0 \leqq m, k \leqq n$ satisfying $m + k < n$. Furthermore, $v_n(n - m, n - k; c) = v_n(m, k; c)$.

It is shown in Corollary 3.2 of [2] that the above result (1) implies that the number of sequences $x \varepsilon \mathcal{S}_c$ which have exactly r partial sums $s_i(x)$ in the interval bounded by $s_0(x)$ and $s_n(x)$ is independent of c. In seeking an m-dimensional analogue of this result, we first observe that an interval may be regarded as a one-dimensional sphere. The event of a partial sum falling between s_0 and s_n in the one-dimensional case might then become, in the m-dimensional case, the event of a partial sum falling within the sphere which has the line segment joining

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s_0 and s_n as a diameter. With this thought in mind, we proceed formally as follows:

Let $c = (c_1, \cdots, c_n)$ be a sequence of m-dimensional vectors, $c_i = (c_{i1}, \cdots, c_{im})$. Let $s_0(x)$ denote the zero vector, and set $s_i(x) = x_1 + x_2 + \cdots + x_i$ if $i > 0$. The components of $s_i(x)$ are denoted by $s_{i1}(x)$, $s_{i2}(x)$, $\cdots$. We shall call the transformation T which takes a sequence $(x_1, x_2, \cdots, x_n)$ into the sequence $(x_2, x_3, \cdots, x_n, -x_1)$ an *inverted cyclic* transformation. Let $\mathcal{C}_c^*$ denote the set consisting of the n sequences

$$T^0(c) = c, T^1(c), \cdots, T^{n-1}(c)$$

where $c = (c_1, \cdots, c_n)$ and T is the inverted cyclic transformation defined above. For any m-tuple y, let $K(y)$ denote the m-dimensional sphere which has the line segment joining 0 and y as a diameter. Let $J_n(x)$ denote the number of partial sums $s_i(x)$ which lie in the interior of $K(s_n(x))$. The center of the sphere $K(s_n(x))$ is the point $\frac{1}{2}s_n(x)$. Define $L_n^*(x)$ to be the smallest subscript of a partial sum of greatest distance from the center. That is, $L_n^*(x)$ is the smallest value of $j = 0, 1, \cdots, n-1$ such that $|s_j(x) - \frac{1}{2}s_n(x)| = \max_{0 \leqq i \leqq n} |s_i(x) - \frac{1}{2}s_n(x)|$, where $|y|$ denotes the Euclidean length of the vector y. We say that c possesses *property C* if for every $x \in \mathcal{C}_c^*$ and every $i = 1, 2, \cdots, n-1$, $s_i(x)$ is not a boundary point of $K(s_n(x))$.

THEOREM 1. *If c possesses property C, then for each $k = 0, 1, \cdots, n-1$, there is* (i) *exactly one element $x \in \mathcal{C}_c^*$ for which $J_n(x) = k$, and* (ii) *exactly one element $x \in \mathcal{C}_c^*$ for which $L_n^*(x) = k$.*

PROOF. Let $x = T^j(c)$ be an arbitrary element of $\mathcal{C}_c^*$. Since

$$x = (c_{j+1}, c_{j+2}, \cdots, c_n, -c_1, \cdots, -c_j),$$

we see that

$$\begin{aligned} s_i(x) - \tfrac{1}{2}s_n(x) &= s_{i+j}(c) - \tfrac{1}{2}s_n(c) && \text{if } i+j \leqq n \\ &= \tfrac{1}{2}s_n(c) - s_{i+j-n}(c) && \text{if } i+j > n \end{aligned} \tag{2}$$

As i ranges over the integers $1, 2, \cdots, n$, $|s_i(x) - \frac{1}{2}s_n(x)|$ ranges over the set of values $\{|s_r(c) - \frac{1}{2}s_n(c)| : r = 1, 2, \cdots, n\}$. It follows from property C that all of these values are distinct. Thus for a given integer $0 \leqq k \leqq n-1$, we may define $j = j(k)$ to be that subscript for which $|\frac{1}{2}s_n(c) - s_i(c)| < |\frac{1}{2}s_n(c) - s_j(c)|$ for exactly k values of $i = 0, 1, \cdots, n-1$. If $x = T^j(c)$, then $s_i(x)$ is in the interior of $K(s_n(x))$ if and only if $|s_i(x) - \frac{1}{2}s_n(x)| < |\frac{1}{2}s_n(x)|$, since $\frac{1}{2}s_n(x)$ is the center of the sphere $K(s_n(x))$. But $|\frac{1}{2}s_n(x)| = |\frac{1}{2}s_n(c) - s_j(c)|$. Hence by the definition of $j = j(k)$, $J_n(x) = k$. Since there is a unique $j(k)$ for each k, the proof of (i) is complete. To prove (ii), observe that from (2) it follows that

$$\begin{aligned} L_n^*(T^j(c)) &= L_n^*(c) - j && \text{if } 0 \leqq j \leqq L_n^*(c) \\ &= n + L_n^*(c) - j && \text{if } L_n^*(c) < j < n \end{aligned}$$

which implies that as j runs over the integers $0, 1, \cdots, n-1$, so does $L_n^*(T^j(c))$. This completes the proof of the theorem.

The one-dimensional case of part (ii) of the above theorem was proved in Corollary 1 of [3].

Theorem 1 has an immediate application to multidimensional stochastic processes. We say that the random vector $Y = (Y_1, Y_2, \cdots, Y_n)$ is (i) *exchangeable* if its distribution function (d.f.) is invariant under all permutations of the coordinate variables and (ii) *symmetric* if its d.f. is invariant under all sign attachments to the coordinate variables. Let $\{X_t: t \geqq 0\}$ be an m-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{A}, P)$, which has symmetric and exchangeable increments. That is, for every choice of real numbers $h, t_1, t_2, \cdots, t_k$ and every integer $k > 0$, satisfying $0 \leqq t_1 < t_1 + h \leqq t_2 < t_2 + h \leqq \cdots < t_k + h$, the sequence of random vectors,

$$(X_{t_1+h} - X_{t_1}, \cdots, X_{t_k+h} - X_{t_k}),$$

is symmetric and exchangeable. Assume further that the process is measurable, separable, and is such that the rationals form a separating sequence. Define the random variable J_T to be the Lebesgue measure of the set

$$\{t \varepsilon (0, T]; X_t \varepsilon K(X_T)\},$$

and the random variable

$$L_T = \inf \{t \varepsilon (0, T]; \sup_{0 \leqq u \leqq t} |X_u - \tfrac{1}{2}X_T| = \sup_{0 \leqq u \leqq T} |X_u - \tfrac{1}{2}X_T|\}.$$

Then, in view of Theorem 1, a direct approximation argument yields

THEOREM 2. *If the increments of the process $\{X_t: t \geqq 0\}$ possess property C with probability* 1, *then under the above assumptions, J_T and L_T are uniformly distributed over* $(0, T)$. *That is, $P[J_T \leqq x] = P[L_T \leqq x] = x/T$ for $0 \leqq x \leqq T$.*

Of particular interest is the fact that the random variables J_T and L_T are therefore distribution-free over the subclass of all m-dimensional stable processes, including Brownian motion. Theorem 1 also applies to the following model which prompted the investigations of this paper. Let n fair spinners, of possibly different radii, be given. At random, choose an ordering of these spinners. Consider the random walk obtained by spinning each of the ordered spinners in succession, placing the center of each succeeding spinner at the indicated point on the circumference of the previous spinner. Now consider the problem of obtaining the distribution of the number of outcomes which fall within the circle having the line joining the origin and the final outcome as diameter. It is an easy consequence of Theorem 1 that this distribution is uniform over the integers $0, 1, \cdots, n - 1$.

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