NOTE ON THE BERRY-ESSEEN THEOREM

By Melvin L. Katz

Institute for Defense Analyses and University of Chicago

1. Introduction. Consider a sequence of independent and identically distributed random variables $\{X_1, X_2, \cdots\}$ such that $EX = 0$ and $EX^2 = 1$. In this case it is well known that

$$A_n = \sup_x \left| P\{S_n/\sqrt{n} \leqq x\} - \int_{-\infty}^{x} (2\pi)^{\frac{1}{2}} e^{-t^2/2}\, dt \right| \to 0$$

as $n \to \infty$, where $S_n = \sum_{k=1}^{n} X_k$, and it is a problem of interest to determine the rate of convergence of A_n to zero. If in addition it is assumed that $E|X|^3 < \infty$, then Berry and Esseen have independently shown that $A_n \leqq$ constant $E|X|^3/\sqrt{n}$.

The purpose of this note is to show that it follows immediately from the Berry-Esseen results that bounds on A_n are obtainable under much less restrictive conditions than the existence of the third moment.

2. The result. Denote by G the class of functions $g(x)$ defined on the real line satisfying the following conditions:

(a) $g(x)$ is non-negative, even, non-decreasing on $[0, \infty)$, and

$$\lim_{x\to\infty} g(x) = \infty.$$

(b) $x/g(x)$ is defined for all x and non-decreasing on $[0, \infty)$.

The result of this note may now be stated as follows:

Theorem.[1] *Suppose $g(x) \varepsilon G$ and $\{X_1, X_2, \cdots\}$ is a sequence of independent, identically distributed random variables such that $EX = 0$ and $EX^2 = 1$. If $EX^2g(X) < \infty$ then there exists an absolute constant C, such that*

$$\sup_x \left| P\{S_n/\sqrt{n} \leqq x\} - \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt \right| \leqq \frac{CEX^2g(X)}{g(\sqrt{n})}.$$

Proof. For $k = 1, 2, \cdots, n$, let $X_{k,n}$ denote X_k truncated at $\sqrt{n}$, $\mu_n = EX_{k,n}$, $\sigma_n^2 = EX_{k,n}^2 - (EX_{k,n})^2$, and $S_{n,n} = \sum_{k=1}^{n} X_{k,n}$. It follows from elementary calculations that

$$0 \leqq (1 - \sigma_n^2) \leqq 2EX^2g(X)/g(\sqrt{n}) \tag{1}$$

and

$$\sqrt{n}|\mu_n| \leqq EX^2g(X)/g(\sqrt{n}). \tag{2}$$

If $\sigma_n \leqq \frac{1}{2}$ then from (1) one has $1 \leqq \frac{8}{3}EX^2g(X)/g(\sqrt{n})$ and hence in this case the theorem is trivially true with $C = \frac{8}{3}$. Thus in the rest of the proof it will be assumed that $\sigma_n > \frac{1}{2}$.

Received October 18, 1962; revised March 1, 1963.

[1] I would like to thank the referee for greatly improving the theorem by showing that C could be chosen as a constant independent of $g(x)$.

To apply the Berry-Esseen results it is necessary to work with random variables having finite third moments; this is accomplished through the following inequality:

$$P\{S_{n,n}/\sqrt{n} \leqq x\} - nP\{|X| > \sqrt{n}\} \leqq P\{S_n/\sqrt{n} \leqq x\} \leqq P\{S_{n,n}/\sqrt{n} \leqq x\} + nP\{|X| > \sqrt{n}\}. \tag{3}$$

From (3) it follows that

$$\begin{aligned} P\{S_n/\sqrt{n} \leqq x\} - \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt & \\ \leqq \sup_x \Big| P\{(S_{n,n} - n\mu_n)/\sigma_n \sqrt{n} \leqq x/\sigma_n - n^{\frac{1}{2}}\mu_n/\sigma_n\} & \\ - \int_{-\infty}^{x/\sigma_n - n^{\frac{1}{2}}\mu_n/\sigma_n} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt \Big| + \sup_x \Big| \int_{x}^{x/\sigma_n - n^{\frac{1}{2}}\mu_n/\sigma_n} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt \Big| & \\ + nP\{|X| > \sqrt{n}\}. & \end{aligned}$$

The proof proceeds by bounding the terms on the right hand side of inequality (4). From the Berry-Esseen result it follows that the first term on the right side of (4) is bounded by $C_1 E|X_{k,n} - \mu_n|^3/\sqrt{n}$, where C_1 is an absolute constant. Further

$$\begin{aligned} C_1 E\,|X_{k,n} - \mu_n|^3/\sqrt{n} &\leqq 4C_1(E\,|X_{k,n}|^3 + |\mu_n|^3)/\sqrt{n} \\ &\leqq 8C_1 E\,|X_{k,n}|^3/\sqrt{n} \leqq \frac{8C_1}{\sqrt{n}} \int_{|x| \leqq \sqrt{n}} \frac{x^2 g(x)}{g(x)/|x|} F(dx) \\ &\leqq 8C_1 EX^2 g(X)/g(\sqrt{n}), \end{aligned} \tag{5}$$

where $F(x)$ is the distribution function of X_k.

The second term on the right hand side of (4) can be bounded as follows:

$$\begin{aligned} \sup_x \Big| \int_{x}^{x/\sigma_n - n^{\frac{1}{2}}\mu_n/\sigma_n} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt \Big| &\leqq (2\pi)^{-\frac{1}{2}}\{(1 - \sigma_n)/\sigma_n + 2(n)^{\frac{1}{2}}\,|\mu_n|/\sigma_n\} \\ &\leqq (2\pi)^{-\frac{1}{2}}\{8EX^2 g(X)/g(\sqrt{n})\}, \end{aligned}$$

where the last inequality follows immediately from (1) and (2).

Finally since $g(x)$ is even and non-decreasing on $[0, \infty)$ $nP\{|X| > \sqrt{n}\} \leqq EX^2 g(X)/g(\sqrt{n})$. Thus

$$P\{S_n/\sqrt{n} \leqq x\} - \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt \leqq CEX^2 g(X)/g(\sqrt{n}) \tag{6}$$

with $C = 1 + 8C_1 + 8/(2\pi)^{\frac{1}{2}}$. It is clear from the proof that $-CEX^2 g(X)/g(\sqrt{n})$ is a lower bound for the left hand side of (6) and this completes the proof.

REFERENCES

[1] Esseen, C. G. (1943). On the Liapounoff limit of error in the theory of probability. *Ark. Mat. Astr. Fys.* **29 A** No. 9.

[2] Loève, M. (1960). *Probability Theory,* (2nd ed.). Van Nostrand, New York.