FURTHER EXAMPLES OF INCONSISTENCIES IN THE FIDUCIAL ARGUMENT[1]

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1. Introduction and summary. Fisher wrote as though the fiducial argument were a well-defined form of reasoning: witness phrases like "the fiducial argument itself," (Fisher (1956), p. 120)[2] and "a genuine fiducial argument" (Fisher (1956), p. 172). However, attempts to extend the set of examples of the fiducial argument beyond the set personally approved by Fisher have often run into the difficulty that several inconsistent fiducial arguments appeared to be available for a single situation, e. g., the pair of papers by Creasy (1954) and Fieller (1954), Mauldon (1955), Tukey (1957) and Brillinger (1962).

Apparently in response to the Mauldon example, Fisher intimated that the joint fiducial distribution of several parameters should be built up "rigorously by a step by step process" (Fisher (1956), p. 172). Fisher's comment is rather confusing because the Mauldon approach does appear to be a step by step approach. Brillinger (1962) presented an artificial two-parameter example in which Fisher's type of step by step approach can be applied in inconsistent ways. In Section 2 I will show that a step by step approach leads to alternative inconsistent answers even in the basic two parameter situation of sampling from the normal distribution.

In Section 3 I will demonstrate a difference between the fiducial distribution for the means (μ_1, μ_2) of a bivariate normal distribution given by Fisher (1954) and the marginal distribution of (μ_1, μ_2) under the joint fiducial distribution of all five parameters of the bivariate normal given by Fisher (1956).

2. The mean and variance of a normal distribution. Let $N(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2 and let χ_r^2 denote the chi-square distribution with r d.f. Let M_r denote the distribution of a root mean square on r d.f., i.e., the distribution of $(Z/r)^{\frac{1}{2}}$ where Z has the χ_r^2 distribution. Let $t_r(\tau)$ denote the non-central t distribution on r d.f. with parameter τ, i.e., the distribution of $(V + \tau)/P$ where V and P are independent with the $N(0, 1)$ and M_r distributions, respectively. Finally let $G_r(t; \tau)$ denote the c.d.f. of the $t_r(\tau)$ distribution.

If $X_1, X_2, \cdots, X_N$ denote a random sample of N from $N(\mu, \sigma^2)$, then it may be agreed that any joint fiducial distribution for μ and σ^2 should depend only on the sufficient statistics, which may be taken in the form

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[2] The page numbers in the 1958 edition are zero, one or two larger than the corresponding first edition page numbers.

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$$\bar{X} = (1/N)\sum_{i=1}^{N} X_i \quad \text{and} \quad S^2 = [1/(N-1)]\sum_{i=1}^{N}(X_i - \bar{X})^2. \tag{2.1}$$

Fisher's fiducial argument (Fisher (1935, 1956)) is easily expressed in terms of the pivotal quantities

$$U = N^{\frac{1}{2}}(\bar{X} - \mu)/\sigma \quad \text{and} \quad Q = S/\sigma. \tag{2.2}$$

From a conventional viewpoint, U and Q are independent with $N(0, 1)$ and M_{N-1} distributions for any given values of μ and σ^2. The fiducial argument says that one should continue to regard U and Q as having the stated joint distribution after $\bar{X}$ and S are fixed by observation. Thus the joint fiducial distribution of μ and σ is that implied by

$$\mu = \bar{X} + SU/N^{\frac{1}{2}}Q \quad \text{and} \quad \sigma = S/Q \tag{2.3}$$

where $\bar{X}$ and S are regarded as fixed and U and Q distributed as above. Presumably a justification of this argument should seek to explain why one should continue to regard U and Q as having the distribution believed relevant before observation of $\bar{X}$ and S. However, this paper seeks only to describe the argument and not to interpret it.

Another description of the same argument is given step by step as follows (c.f. Fisher (1956) pp. 119–120):

(i) Note that $Q = S/\sigma$ is a pivotal quantity depending only on the single parameter σ. Use Q to assign a fiducial distribution to σ in the obvious way, i.e., σ is assigned the distribution of S/Q where S is fixed and Q has the M_{N-1} distribution.

(ii) Note that, given S and σ, $\bar{X} - \mu$ has the $N(0, \sigma^2/N)$ distribution and so may be regarded as a pivotal quantity, given σ, depending only on the single parameter μ. Using this pivotal quantity, μ is assigned the $N(\bar{X}, \sigma^2/N)$ distribution as a conditional fiducial distribution of μ given σ.

Clearly, (i) and (ii) together determine a joint fiducial distribution of μ and σ. Also, since the pivotal quantity in (i) is identical to Q in (2.2) and the pivotal quantity in (ii) is equivalent to U in (2.2), the step by step argument gives the same joint fiducial distribution as given by the first description.

Following the form of the step by step description above, an alternative fiducial argument will now be proposed. The alternative is achieved by regarding the parameters not as σ and μ but, equivalently, as μ/σ and σ. The roles formerly played by S and $\bar{X}$ will now be played by the equivalent sufficient statistics $\bar{X}/S$ and S. Steps (i) and (ii) now read as follows:

(i) Note that $G_{N-1}(N^{\frac{1}{2}}\bar{X}/S, N^{\frac{1}{2}}\mu/\sigma)$ is a pivotal quantity depending only on the single parameter μ/σ. Use this pivotal quantity to assign a fiducial distribution to μ/σ in the standard way, c.f. Fisher (1930) and Lindley (1958).

(ii) Note that, given $\bar{X}/S$ and μ/σ, S/σ has a distribution determined solely by $\bar{X}/S$ and μ/σ, and so may be regarded as a pivotal quantity, given μ/σ, depending on the single parameter σ. Using this pivotal quantity, one finds a conditional fiducial distribution of σ given μ/σ. The assertion that the conditional distribution of S/σ given $\bar{X}/S$ is a function of $\bar{X}/S$ and μ/σ only follows from formula (2.8) in the sequel.

The pivotal quantities here are somewhat awkward to deal with analytically in terms of density functions. Still, matters are perhaps less complex than they first appear. In the following paragraphs it will be shown that *the same distribution is assigned to μ/σ by both Fisher's argument and the alternative argument, but that the distributions assigned to σ given μ/σ are different in the two cases.*

Under Fisher's argument the distribution of μ/σ is characterized by

$$\mu/\sigma = Q\bar{X}/S + U/N^{\frac{1}{2}}, \tag{2.4}$$

in accordance with the joint distribution implied by (2.3). Under the alternative argument, since the pivotal variable $G_{N-1}(N^{\frac{1}{2}}\bar{X}/S;\, N^{\frac{1}{2}}\mu/\sigma)$ is monotone decreasing in μ/σ for given $\bar{X}/S$, the α quantile of the fiducial distribution of μ/σ is that value of μ/σ such that

$$G_{N-1}(N^{\frac{1}{2}}\bar{X}/S;\, N^{\frac{1}{2}}\mu/\sigma) = 1 - \alpha, \tag{2.5}$$

for the observed $\bar{X}$ and S. The Condition (2.5) may be expressed equivalently as follows: if U^* and Q^* are any independent random variables with $N(0, 1)$ and M_{N-1} distributions, so that $(-U^* + N^{\frac{1}{2}}\mu/\sigma)/Q^*$ has the $t_{N-1}(N^{\frac{1}{2}}\mu/\sigma)$ distribution, then, under the alternative argument, the α quantile of μ/σ is that value of μ/σ such that

$$\Pr\left(((-U^* + N^{\frac{1}{2}}\mu/\sigma)/Q^*) \leqq N^{\frac{1}{2}}\bar{X}/S\right) = 1 - \alpha \tag{2.6}$$

where $\bar{X}$ and S are fixed at their observed values. But (2.6) may be written

$$\Pr\left(Q^*\bar{X}/S + U^*/N^{\frac{1}{2}} \leqq \mu/\sigma\right) = \alpha \tag{2.7}$$

which, from (2.4), agrees with Fisher's prescription for finding the α quantile of μ/σ.

To find the conditional distribution of σ given μ/σ under Fisher's method, one must consider the distribution of σ induced by the relation $Q = S/\sigma$ where Q is assigned its conditional distribution given (2.4), i.e., given a fixed linear combination of Q and U.

Under the alternative method, one must start further back to establish $Q = S/\sigma$ as a pivotal quantity given $\bar{X}/S$ and μ/σ. Here, $\bar{X}$ and S are regarded as random given μ and σ and one seeks the conditional distribution of S/σ given $\bar{X}/S$. This problem may be put in terms of U and Q as defined by (2.2) where U and Q have their conventional $N(0, 1)$ and M_{N-1} distributions given μ and σ. In these terms one seeks the distribution of Q given $\bar{X}/S$ where

$$\bar{X}/S = \mu/\sigma Q + U/N^{\frac{1}{2}}Q. \tag{2.8}$$

It is this distribution of $Q = S/\sigma$ which is used to induce the conditional fiducial distribution of σ given μ/σ in the alternative fiducial argument.

Since (2.8) may be written

$$\mu/\sigma = Q\bar{X}/S - U/N^{\frac{1}{2}}, \tag{2.9}$$

and since U and $-U$ have the same distribution, it appears, at first sight, that the Conditions (2.4) and (2.9) lead to identical conditional distributions for Q under the two arguments. But closer inspection reveals that, although the conditional distribution of Q is being found along the same (excepting the U for $-U$ change) line in the (U, Q)-plane, the partition of the (U, Q)-plane is different in the two cases. In Fisher's case one is finding the limiting distribution of Q given

$$\mu/\sigma \leqq Q\bar{X}/S + U/N^{\frac{1}{2}} \leqq \mu/\sigma + \Delta \tag{2.10}$$

as $\Delta \rightarrow 0$. In the alternative case one is finding the limiting distribution of Q given

$$\bar{X}/S \leqq \mu/\sigma Q + U/N^{\frac{1}{2}}Q \leqq \bar{X}/S + \Delta \tag{2.11}$$

or

$$\mu/\sigma \leqq Q\bar{X}/S - U/N^{\frac{1}{2}} \leqq \mu/\sigma + Q\Delta \tag{2.12}$$

as $\Delta \rightarrow 0$. The region defined by (2.10) is a region between parallel lines, whereas that defined by (2.12) is a region between lines radiating from the point $U = N^{\frac{1}{2}}\mu/\sigma$, $Q = 0$. It follows that, apart from the normalizing constant, the conditional density of Q in the second case is Q times the conditional density of Q in the Fisher case, and so they are clearly different.

It may be of interest to display these two densities. The joint density of Q and U is

$$KQ^{N-2} \exp\{-[(N-1)/2]Q^2\} \exp\{-\tfrac{1}{2}U^2\}. \tag{2.13}$$

Proceeding as in the Fisher case, one changes to variables

$$T = Q\bar{X}/S + U/N^{\frac{1}{2}} \quad \text{and} \quad Q = Q \tag{2.14}$$

and then substitutes $T = \mu/\sigma$ to find the form of the density of Q. The result is

$$K'Q^{N-2} \exp\{-\tfrac{1}{2}[(N-1)Q^2 + N(Q\bar{X}/S - \mu/\sigma)^2]\}. \tag{2.15}$$

The only difference under the alternative argument is that the term Q^{N-2} is replaced by Q^{N-1}. It is clear, as one might expect, that for large N these distributions are but little different.

3. The parameters of the bivariate normal distribution. Fisher (1954) gave a joint fiducial distribution for the pair of means of a bivariate normal distribution. Subsequently (Fisher (1956)), he gave a joint fiducial distribution to all five parameters. Fisher's faith in the consistency of all genuine fiducial arguments was evidently so strong that he did not feel the need to discuss whether his two methods were consistent. This question has also been shrouded by analytical difficulties concerned with the Fisher (1956) method. The purpose of this section is to point out that, in the special case when the sample correlation is zero, the Fisher (1956) method becomes quite simple, but fails to agree with Fisher (1954). The section consists of three parts: (i) some preliminary notation and theory, (ii) an explanation of the Fisher (1954) idea, presented with no extra complication in a p-variate setting, and (iii) a discussion of the Fisher (1956) method.

The p-variate normal distribution will be denoted $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the associated Wishart distribution on r d.f. will be denoted $W(\boldsymbol{\Sigma}, r)$, c.f. Anderson (1958) p. 17 and p. 158. The basic sample of N from the $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution may be denoted by $p \times 1$ vectors $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_N$, but for purposes of fiducial inference one may reduce to the sufficient statistics

$$(3.1) \qquad \overline{\mathbf{X}} = \sum_{i=1}^{N} \mathbf{X}_i/N \quad \text{and} \quad \mathbf{T} = \sum_{i=1}^{N} (\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})'/(N-1),$$

where $\overline{\mathbf{X}}$ and $\mathbf{T}$ are independent with $N(\boldsymbol{\mu}, \boldsymbol{\Sigma}/N)$ and $W(\boldsymbol{\Sigma}, N-1)$ distributions. Finally, it will be convenient to have for reference two basic properties of the Wishart distribution: *if* $\mathbf{S}$ *has the* $W(\boldsymbol{\Sigma}, r)$ *distribution, then* (I), *for any* $p \times 1$ *vector* $\mathbf{a}$, *the ratio* $\mathbf{a}'\mathbf{S}\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$ *has the* χ_r^2 *distribution, and* (II), *provided* $\boldsymbol{\Sigma}$ *has rank* p *and* $r \geqq p$, *for any* $p \times 1$ *vector* $\mathbf{b}$, *the ratio* $\mathbf{b}'\boldsymbol{\Sigma}^{-1}\mathbf{b}/\mathbf{b}'\mathbf{S}^{-1}\mathbf{b}$ *has the* χ_{r-p+1}^2 *distribution.*

The Fisher (1954) method is based on the remark which is a consequence of the theory of the previous paragraph, that

$$(3.2) \qquad N^{\frac{1}{2}}(\mathbf{a}'\overline{\mathbf{X}} - \mathbf{a}'\boldsymbol{\mu})/[\mathbf{a}'\mathbf{T}\mathbf{a}/(N-1)]^{\frac{1}{2}}$$

has the central $t_{N-1}(0)$ distribution for any given $\mathbf{a}$. Thus (3.2) may be used as a pivotal variable in assigning a fiducial distribution to $\mathbf{a}'\boldsymbol{\mu}$. For any given random vector $\boldsymbol{\mu}$, the set of marginal distributions of $\mathbf{a}'\boldsymbol{\mu}$, for all $\mathbf{a}$, uniquely determine the distribution of $\boldsymbol{\mu}$. Thus, one is led to ask whether the above fiducial distributions of $\mathbf{a}'\boldsymbol{\mu}$, for all $\mathbf{a}$, are consistent with a distribution over $\boldsymbol{\mu}$. Fisher (1954) answered this question in the case $p = 2$ by writing down the density for $\boldsymbol{\mu}$ which yields the desired marginal distributions. Cornish (1961) extended this to general p.

More light may be shed on this situation by describing, in terms requiring no density functions, a joint distribution of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ which obviously provides the required distribution for $\boldsymbol{\mu}$. A distribution of $\boldsymbol{\Sigma}$ can be specified by assigning $\boldsymbol{\Sigma}^{-1}$ the $W(\mathbf{T}^{-1}, k)$ distribution for arbitrary $k \geqq p$, and a conditional distribution of $\boldsymbol{\mu}$ given $\boldsymbol{\Sigma}$ can be specified as the $N(\overline{\mathbf{X}}, (1/N)\boldsymbol{\Sigma})$ distribution. Applying property (II) of Wishart distributions to the $W(\mathbf{T}^{-1}, k)$ distribution of $\boldsymbol{\Sigma}^{-1}$ one finds that $\mathbf{a}'\mathbf{T}\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$, regarded as a random function of $\boldsymbol{\Sigma}$ for given $\mathbf{T}$, has the χ_{k-p+1}^2 distribution. Consequently

$$(3.3) \qquad N^{\frac{1}{2}}(\mathbf{a}'\overline{\mathbf{X}} - \mathbf{a}'\boldsymbol{\mu})/[\mathbf{a}'\mathbf{T}\mathbf{a}/(k-p+1)]^{\frac{1}{2}}$$

has the $t_{k-p+r}(0)$ distribution. Choosing $k = N + p - 2$ therefore provides the marginal distribution for $\mathbf{a}'\boldsymbol{\mu}$ implied by (3.2). It should be remarked that Geisser and Cornfield (1963) have also discussed the above family of distributions for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and have shown that the marginal density of $\boldsymbol{\mu}$ agrees with the Cornish (1961) generalization of Fisher's (1954) density when $k = N + p - 2$. The above joint distribution of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ which agrees with Fisher (1954) will be useful for comparisons with Fisher (1956).

Turning now to the Fisher (1956) argument for the case $p = 2$, I will show that, in the special case where the sample correlation coefficient $r = 0$, the Fisher (1956) distribution can be simply described, but that it neither agrees with any of the distributions in the previous paragraph nor does it yield the Fisher (1954) marginal distribution for $\boldsymbol{\mu}$. Some additional notation is required at this point. Let t_{ij}, t^{ij}, σ_{ij} and σ^{ij} denote the row i and column j elements of $\mathbf{T}$, $\mathbf{T}^{-1}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^{-1}$, respectively. Let $R_{N-1}(\rho)$ denote the distribution of a sample correlation coefficient on $N - 1$ d.f. when ρ is the population correlation coefficient, i.e., $R_{N-1}(\rho)$ is the distribution of $r = t_{12}/(t_{11}t_{22})^{\frac{1}{2}}$ where $\mathbf{T}$ has the $W(\boldsymbol{\Sigma}, N - 1)$ distribution and $\rho = \sigma_{12}/(\sigma_{11}\sigma_{22})^{2}$. Finally let $H_{N-1}(r; \rho)$ denote the c.d.f. of the $R_{N-1}(\rho)$ distribution.

When $r = 0$, so that $\mathbf{T}^{-1}$ is diagonal, the class of $W(\mathbf{T}^{-1}, k)$ distributions for $\boldsymbol{\Sigma}^{-1}$ becomes especially simple. Indeed, for given k, the distribution of $\boldsymbol{\Sigma}^{-1}$ may be characterized by the assertion that σ^{11}/t^{11}, σ^{22}/t^{22} and $\sigma^{12}/(\sigma^{11}\sigma^{22})^{\frac{1}{2}}$ are independent with χ_k^2, χ_k^2 and $R_k(0)$ distributions. I will now show that, if $r = 0$, *Fisher's* (1956) *distribution for* $\boldsymbol{\Sigma}$ *is characterized by the assertion that* σ^{11}/t^{11}, σ^{22}/t^{22} *and* $\sigma^{12}/(\sigma^{11}\sigma^{22})^{\frac{1}{2}}$ *are independent with* χ_{N-1}^2, χ_{N-1}^2 *and* $R_N(0)$ *distributions.* It will then be clear that, as has also been shown by Geisser and Cornfield (1963) using a different argument, that Fisher's (1956) distribution for $\boldsymbol{\Sigma}^{-1}$ differs from any of the $W(\mathbf{T}^{-1}, k)$ family. Finally I will demonstrate a difference in the marginal distributions for $\boldsymbol{\mu}$.

Fisher (1956) p. 170 denotes my quantities σ^{11}/t^{11}, σ^{22}/t^{22} and $\sigma^{12}/(\sigma^{11}\sigma^{22})^{\frac{1}{2}}$ by u^2, v^2 and $-\rho$. In Fisher's step by step argument, ρ is first assigned a fiducial distribution given r, and then, given ρ and r, u and v are used as pivotal quantities to determine the remainder of the fiducial distribution of $\boldsymbol{\Sigma}$. It is obvious, however, from Fisher's formulas (220) or (223) that, when $r = 0$, u^2 and v^2 are assigned independent χ_{N-1}^2 distributions regardless of ρ and so independent of ρ. It remains, therefore, only to check that Fisher assigns the $R_N(0)$ distribution to $-\rho$.

This last step concerns the original example of Fisher (1930). The pivotal quantity $H_{N-1}(r; \rho)$ is used to assign the fiducial distribution to ρ, so that α quantile of the fiducial distribution of ρ is that value of ρ such that

$$H_{N-1}(r; \rho) = 1 - \alpha \tag{3.4}$$

for the observed r. Now, if U and V have independent $N(0, 1)$ distributions, then U and $\rho U + (1 - \rho^2)^{\frac{1}{2}}V$ are bivariate normally distributed with correlation coefficient ρ. Consequently, if $U_1, U_2, \cdots, U_{N-1}, V_1, V_2, \cdots, V_{N-1}$ have independent $N(0, 1)$ distributions, then

$$(3.5)\qquad W = \frac{\sum_{i=1}^{N-1} U_i(\rho U_i + (1-\rho^2)^{\frac{1}{2}} V_i)}{\left[\sum_{i=1}^{N-1} U_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{N-1} (\rho U_i + (1-\rho^2)^{\frac{1}{2}} V_i)^2\right]^{\frac{1}{2}}}$$

has the $R_{N-1}(\rho)$ distribution. Thus, Condition (3.4) may be written $\Pr(W \leqq r) = 1 - \alpha$ which becomes, when $r = 0$,

$$(3.6)\qquad \Pr\left(\sum_{i=1}^{N-1} U_i(\rho U_i + (1-\rho^2)^{\frac{1}{2}} V_i) \leqq 0\right) = 1 - \alpha$$

or

$$(3.7)\qquad \Pr\left(\sum_{i=1}^{N-1} U_i V_i \Big/ \sum_{i=1}^{N-1} U_i^2 \leqq \rho/(1-\rho^2)^{\frac{1}{2}}\right) = \alpha.$$

Now the conditional distribution of

$$(3.8)\qquad Z = \sum_{i=1}^{N-1} U_i V_i \Big/ \left[\sum_{i=1}^{N-1} U_i^2\right]^{\frac{1}{2}}$$

given $U_1, U_2, \cdots, U_{N-1}$ is $N(0, 1)$, so that (3.7) may be written

$$(3.9)\qquad \Pr\left(Z \Big/ \left[\sum_{i=1}^{N-1} U_i^2\right]^{\frac{1}{2}} \leqq \rho/(1-\rho^2)^{\frac{1}{2}}\right) = \alpha$$

where $Z, U_1, U_2, \cdots, U_{N-1}$ have independent $N(0, 1)$ distributions. Finally, (3.9) may be altered to

$$(3.10)\qquad \Pr\left(Z \Big/ \left[Z^2 + \sum_{i=1}^{N-1} U_i^2\right]^{\frac{1}{2}} \leqq \rho\right) = \alpha.$$

But $Z/[Z^2 + \sum_1^{N-1} U_i^2]^{\frac{1}{2}}$ has the $R_N(0)$ distribution, which gives the desired result.

To show that the marginal distribution of $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ for Fisher (1956) is inconsistent with Fisher (1954), I will characterize the fiducial distributions of μ_1, in the two cases. In either case one may consider the quantity

$$(3.11)\qquad \frac{N^{\frac{1}{2}}(\bar{X}_1 - \mu_1)}{t_{11}^{\frac{1}{2}}} = \frac{N^{\frac{1}{2}}(\bar{X}_1 - \mu_1)}{\sigma_{11}^{\frac{1}{2}}} \cdot \left[\frac{t^{11}}{\sigma^{11}}\right]^{\frac{1}{2}} \cdot \frac{1}{(1-\rho^2)^{\frac{1}{2}}}$$

as determining the fiducial distribution of μ_1. In the Fisher (1956) approach the three factors on the right side of (3.11) are independent with $N(0, 1)$, inverse square root of χ^2_{N-1} and inverse square root of beta $((N-1)/2, \frac{1}{2})$ distributions. On the other hand, it was shown above that the Fisher (1954) approach can be reached through assigning the $W(\mathbf{T}^{-1}, N)$ distribution to $\mathbf{\Sigma}^{-1}$, and then the three factors on the right side of (3.11) are independent with $N(0, 1)$, inverse square root of χ^2_N and inverse square root of beta $((N-1)/2, \frac{1}{2})$

distributions. These are clearly different. It is curious that both approaches yield the same marginal distribution for ρ, but the differences for σ^{11} and σ^{22} carry over into differences for μ_1 and μ_2.

Although the foregoing proves the inconsistency rigorously only for the isolated case $r = 0$, the continuity properties of the mappings involved leave little doubt that the inconsistency holds in a neighborhood of $r = 0$. Surely the presumption, until proved otherwise, is that the inconsistency is general.

REFERENCES

BRILLINGER, DAVID R. (1962). Examples bearing on the definition of fiducial probability with a bibliography. *Ann. Math. Statist.* **33** 1349–1355.

CORNISH, E. A. (1961). The simultaneous fiducial distribution of the location parameters of the multivariate normal distribution. Technical Paper No. 8, CSIRO Division of Math. Statist., Melbourne.

CREASY, MONICA A. (1954). Limits for the ratio of means. *J. Roy. Statist. Soc. Ser. B* **16** 186–194.

FIELLER, E. C. (1954). Some problems in interval estimation. *J. Roy. Statist. Soc. Ser. B* **16** 175–185.

FISHER, R. A. (1930). Inverse probability. *Proc. Cambridge Philos. Soc.* **26** 528–535.

FISHER, R. A. (1935). The fiducial argument in statistical inference. *Ann. Eugenics* **6** 391–398.

FISHER, R. A. (1954). Discussion of the symposium on interval estimation. *J. Roy. Statist. Soc. Ser. B* **16** 212–213.

FISHER, R. A. (1956). *Statistical Methods and Scientific Inference.* Hafner, New York (Second edition, 1958).

GEISSER, SEYMOUR and CORNFIELD, JEROME (1963). Posterior distributions for multivariate normal parameters. To appear in *J. Roy. Statist. Soc. Ser. B.*

LINDLEY, V. V. (1958). Fiducial distributions and Bayes' theorem. *J. Roy. Statist. Soc. Ser. B* **20** 102–107.

MAULDON, J. G. (1955). Pivotal quantities for Wishart's and related distributions, and a paradox in fiducial theory. *J. Roy. Statist. Soc. Ser. B* **17** 79–85.

TUKEY, J. W. (1957). Some examples with fiducial relevance. *Ann. Math. Statist.* **28** 687–695.