LAJOS TAKÁCS, *Introduction to the Theory of Queues*. Oxford University Press, New York, 1962, $7.50. x +268 pp.

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The title of Lajos Takács' 1962 Oxford University Press book on queues is "Introduction to the Theory of Queues". To the reviewer this is hardly an apt title regardless of whether the book is good or bad. The book represents an intensive study of the time-dependent ($t < +\infty$) behavior of various queueing systems, most of which are non-Markov. This is a far more difficult undertaking than the study of the asymptotic (stationary) theory or merely the Markov case, which is what is customarily presented in elementary or "introductory" treatises. So to protect the unwary, who purchase books by their titles alone, the reviewer would recommend that the jacket be stamped "The Time-Dependent Theory of Queues" or something similar.

The primary stochastic variables which are studied are the standard ones: the waiting time of a customer, the queue size (i.e., length of line), and the busy period (i.e., time interval during which at least one server is busy). Wherever it is tractable the author studies the queue size both at a general time $t(0 < t < +\infty)$ and at those points in time which constitute regeneration points of the process, and similarly the waiting time at a general time t and for the nth customer. The stationary limiting probabilities are usually obtained from the time-dependent results by a Tauberian argument rather than as solutions to stationarity equations. A few other stochastic variables, which are pertinent in special situations, are studied in their appropriate contexts.

Various queueing models are dealt with, but the book is principally concerned with the single server model. Excluding the introduction, appendix, and solutions to problems, the chapter on the single server process comprises two-thirds of the length of the book. The single server queue is studied first in the completely Markov case (i.e., Poisson input-exponential service), then in the Poisson input-general service case, in the general input-exponential service, and briefly in the general input-general service case. Special forms of batch (bulk) arrival and batch service are also handled for the single server queue. In the remaining one-third of the book, the author treats the many server queue, the infinite server queue, the telephone traffic process in which arrivals are lost if the queue size exceeds a certain number, the process of servicing machines in which m machines are susceptible to randomly occurring breakdowns, and finally counter models. The degree to which each is treated depends upon the tractability of the model and the author's interests. Admittedly this list does not approach being a semi-complete catalogue of all queueing models which have been proposed and analyzed, but, as stated in the preface, such is not the author's aim.

The techniques employed in the book are primarily the ones currently used in queueing theory. They include recursion relations, generating functions, Markov chain theory, systems of differential equations, Blackwell's and Smith's renewal theorems, Tauberian arguments, Rouché's theorem, etc. Many of these techniques, if they did not originate with L. Takács, at least have been so skillfully utilized by him in queueing problems that they have become the standard tools. At one point the reader is lucky or unlucky enough to encounter a quadruple generating function. The appropriate adjective for such a quantity depends on personal taste. In the single server section it is unfortunate that the book was not written after Takács' 1962 *Annals cf Mathematical Statistics* article [1]. This article contains and illustrates a technique, based on simply obtained first-passage probabilities, which would have eliminated some of the heavier analytic proofs needed for the Poisson input-general service case.

The general readability of the book is good, although it cannot be carefully read quickly. The author is principally concerned with obtaining results, and proceeds to do so step by step, i.e., analysis-result-analysis-result-analysis . . . , etc. The book has been very carefully organized, and it continues unswervingly along its path of obtaining each successive result. Frequently the later results are based on or rely upon the earlier ones. For this reason, and also because of an index which could just as well be torn out, this book would not make a good reference source. To find and extract an isolated result, together with its proof, would be close to impossible. The notation is good, but the author is occasionally guilty of changing it. In the introduction it is decreed that ξ_n will be the queue size immediately before the nth arrival and ζ_n the queue size immediately after the nth departure. This is adhered to as far as page 46, but on page 47 and thereafter the notation is reversed. There are, in addition, a few other instances of notation reversals.

The greatest drawback to the book is its appalling rate of errors and misprints. No expression is to be trusted unless known to be correct from other sources or thoroughly checked, including all required preceding work. The reviewer checked the first one hundred and one pages and the appendix, which together constitute one half of the book, and then read the remainder without checking the more complicated expressions. The reader is continually forced to make corrections, which is surprising considering the careful style of the author's articles. Misprints (i.e., minor errors assumed to be due to poor proofreading and typesetting) and incorrect expression references abound. No proof was found to be incorrect in substance, but incorrect final formulae and missing terms were in evidence. A review is not the place for an errata sheet, but two of the errors will be listed.

1. One error that wreaks havoc with many expressions between pages 30 and 46 is the definition of $g(\omega)$ on page 28. The reviewer speculates that in some preliminary manuscript the author must have defined $g(\omega)$ without the ω in the denominator and then changed his mind. However, only some of the resulting expressions involving $g(\omega)$ were changed. For instance,

(a) expression (19), p. 30, should be

$$[g(\omega)]^k = k \sum_{j=0}^{\infty} \binom{2j+k}{j} \frac{q^{j+k}p^j}{(2j+k)} \omega^j, \tag{19}$$

(b) the three expressions surrounding (20), p. 30, should be

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{0k}^{(n)} z^k \omega^n = \frac{\lambda g(\omega)}{[1 - g(\omega)][\mu - \lambda z\omega g(\omega)]},$$

$$\sum_{n=0}^{\infty} p_{0k}^{(n)} \omega^n = \frac{[\lambda\omega g(\omega)/\mu]^{k+1}}{\omega[1 - g(\omega)]}, \tag{20}$$

$$p_{0k}^{(n)} = \left(\frac{p}{q}\right)^{k+1} \sum_{\nu=0}^{\infty} (k + 1 + \nu) \binom{2n - k + \nu + 1}{n - k} \cdot \frac{q^{n+1+\nu} p^{n-k}}{(2n - k + \nu + 1)},$$

(c) expression (22), p. 31, should be

$$\sum_{n=1}^{\infty} f_{00}^{(n)} \omega^n = \omega g(\omega), \tag{22}$$

and

(d) $$g'(1) = \lambda/(\mu - \lambda).$$

2. On pages 40 and 41 the following changes should be made:

$$\sum_{n=1}^{\infty} \Omega_n(s)\omega^n = \frac{(\mu + s)\omega\{(\lambda - s)[1 - \omega g(\omega)]\Omega_1(s) - s\omega g(\omega)\Omega_1(\lambda[1 - \omega g(\omega)])\}}{[1 - \omega g(\omega)][(\lambda - s)(\mu + s) - \lambda\mu\omega]}, \tag{44}$$

$$\sum_{n=1}^{\infty} \Omega_n(s)\omega^n = \frac{(\lambda - s)(\mu + s)\omega\Omega_1(s) - s(\mu + s)\sum_{n=2}^{\infty} P\{\eta_n = 0\}\omega^n}{(\lambda - s)(\mu + s) - \lambda\mu\omega}, \tag{47}$$

and

$$\sum_{n=2}^{\infty} P\{\eta_n = 0\}\omega^n = \frac{\omega^2 g(\omega)\Omega_1(\lambda[1 - \omega g(\omega)])}{[1 - \omega g(\omega)]}. \tag{48}$$

There are in addition other examples, and a corrected second edition would be most welcome.

The author is somewhat remiss in acknowledging or labelling the results of others. This includes some work of Pollaczek, Khintchine, Kendall, Smith, Conolly, and others. This is not to imply that the author claims priority or is guilty of plagiarism. The results are obtained in the author's own style, and the relevant works are cited in the chapter bibliographies. However, the novice reader might not realize that some of these results are labelled with the originators' names in other publications and books.

Problems are given at the end of most sections, but they are of a curious nature. With few exceptions the problems are to obtain results which have just been derived in that section. A glance at the solutions in the back indicates that what the author has in mind is for the reader to obtain the same results by different means. Customarily, such a different proof is a more direct one which does not rely upon results built up in the preceding part of the book. The problems seem to be varied and good.

The book contains a nice appendix, in which a number of the theorems that are frequently useful in queueing theory and applied probability are neatly summarized. Each chapter has a selected bibliography which seems to be reasonably comprehensive up to the publication date. A complete bibliography of all works on queueing theory would be a book in itself.

REFERENCES

[1] TAKÁCS, L. (1962). The time dependence of a single-server queue with Poisson input and general service times. *Ann. Math. Statist.* **33** 1340–1348.