A THEOREM ON STOPPING TIMES

BY R. M. BLUMENTHAL AND R. K. GETOOR[1]

University of Washington

Let (Ω, P^x, σ) be a time homogeneous strong Markov process with right continuous paths taking values in a locally compact space E with a countable base. The purpose of this note is to give a characterization of the Borel fields associated with stopping times for such a process.

For elaboration on the material in the next few paragraphs we refer the reader to [1] (where a slightly different sample space and notation are used). Let Δ be a point adjoined to E as the point at infinity if E is not compact and as an isolated point if E is compact. Let $\bar{E} = E \cup \Delta$ and let $\mathcal{B}$ and $\bar{\mathcal{B}}$ denote the topological Borel fields of E and $\bar{E}$ respectively. A real valued function f on E is always extended to $\bar{E}$ by the convention $f(\Delta) = 0$.

For the sample space Ω we take the set of all right continuous functions w from $[0, \infty)$ to $\bar{E}$ which also satisfy $w(t) = \Delta$ if $t \geqq \sigma(w)$, where $\sigma(w) = \inf\{t: w(t) = \Delta\}$. Given $t \geqq 0$ the mapping $w \to w(t)$ is denoted by $X(t)$ or $X(t, w)$, and $X: \Omega \to \Omega$ is the mapping $X(w)(t) = X(t, w)$. Let $\mathcal{F}^0(t)$ $(\mathcal{F}^0)$ denote the Borel field of subsets of Ω generated by the sets $X(s)^{-1}(B)$ with B in $\mathcal{B}$ and $s \leqq t$ $(s < \infty)$. If μ is a probability measure on $\mathcal{B}$ we define P^μ on $\mathcal{F}^0$ by $P^\mu(\Lambda) = \int P^x(\Lambda)\mu(dx)$ and define $\mathcal{F}$ to be the intersection over all such μ of the P^μ completions of $\mathcal{F}^0$. Define $\mathcal{F}(t)$ to be the Borel field consisting of those sets Λ such that for each probability measure μ on $\mathcal{B}$ there are sets A in $\mathcal{F}^0(t)$ and N in $\mathcal{F}$ such that $P^\mu(N) = 0$ and $(A - \Lambda) \cup (\Lambda - A) = N$. We note in passing that in some previous work we defined $\mathcal{F}(t)$ to be the intersection over all μ of the P^μ completions of $\mathcal{F}^0(t)$. In fact this gives an extension of $\mathcal{F}^0(t)$ that is a bit too restrictive; the extension given above is the one we should have used.

A function $T: \Omega \to [0, \infty]$ is called a stopping time if for every $t > 0$ the set $\{T < t\}$ is in $\mathcal{F}(t)$. The Borel field $\mathcal{F}(T)$ is then defined to be

$$\{\Lambda \,\varepsilon\, \mathcal{F}: \Lambda \cap \{T < t\} \,\varepsilon\, \mathcal{F}(t) \quad \text{for all } t\}.$$

The strong Markov property implies that $\mathcal{F}(t) = \bigcap_{t<s} \mathcal{F}(s)$ so that notation is consistent with what one gets by regarding a constant function as a stopping time.

Suppose now that T is a stopping time, and define the mapping $Y: \Omega \to \Omega$ by $Y(w)(t) = Y(t, w) = X(\min(t, T(w)), w)$. For each t in $[0, \infty)$ the mapping $w \to Y(t, w)$ is measurable relative to $\mathcal{B}$ and $\mathcal{F}(T)$ so that if $\mathcal{G}^0(T)$ denotes the Borel field generated by the sets $Y(t)^{-1}(B)$ with B in $\mathcal{B}$ and t in $[0, \infty)$ we have $\mathcal{G}^0(T) \subset \mathcal{F}(T)$. Consequently if $\mathcal{G}(T)$ consists of those sets which for each

Received 14 January 1964.

[1] This research was supported in part by the National Science Foundation, NSF-GP1737.

finite measure μ on $\mathcal{B}$ differ from a set in $\mathcal{G}^0(T)$ by an $\mathcal{F}$ set of P^μ measure 0 then $\mathcal{G}(T) \subset \mathcal{F}(T)$ also. Our theorem is as follows.

THEOREM. $\mathcal{G}(T) = \mathcal{F}(T)$.

This fact must be common knowledge, and its proof is very easy. But it is a useful characterization of $\mathcal{F}(T)$ and we have not been able to find a proof of it in the literature. In the following paragraphs we will collect some standard observations and then write down the calculation which leads to this result.

LEMMA 1. *If T is $\mathcal{G}(T)$ measurable then $\mathcal{G}(T) = \mathcal{F}(T)$.*

PROOF. Let μ be a probability measure on $\mathcal{B}$ and let f and λ with or without subscripts denote respectively a bounded $\mathcal{B}$ measurable function and a strictly positive constant. Suppose $g:\Omega \to (-\infty, \infty)$ is a finite product of the form

$$g(w) = \prod_k \int_0^\infty \exp(-\lambda_k t) f_k(X(t, w))dt. \tag{1}$$

If we write each integral in the product as $\int_0^T + \int_T^\infty$ then g may be written as a finite sum of products where each summand has the form

$$\left(\prod_i \int_0^T \exp(-\lambda_i t) f_i(X(t))dt\right)\left(\prod_j \int_T^\infty \exp(-\lambda_j t) f_j(X(t))dt\right). \tag{2}$$

Let us now assume that T is $\mathcal{G}(T)$ measurable. Then the product over i in (2) is $\mathcal{G}(T)$ measurable. On the other hand if Λ is in $\mathcal{F}(T)$ the strong Markov property yields

$$E^\mu\left(\prod_j \int_T^\infty \exp(-\lambda_j t) f_j(X(t))dt; \Lambda\right) = E^\mu\left(\varphi(X(T))\exp(-T\sum_j \lambda_j); \Lambda\right)$$

where $\varphi(x) = E^x \prod_j \int_0^\infty \exp(-\lambda_j t) f_j(X(t))dt$. Of course $\varphi(X(T))\exp(-T\sum\lambda_j)$ is $\mathcal{G}(T)$ measurable, so we have just showed that

$$E^\mu(g \mid \mathcal{G}(T)) = E^\mu(g \mid \mathcal{F}(T)) \tag{3}$$

whenever g is of the form (1). It follows by standard reasoning that (3) remains true whenever g is any bounded $\mathcal{F}$ measurable function, and since μ is arbitrary $\mathcal{G}(T) = \mathcal{F}(T)$ is established.

LEMMA 2. *Let T be a countably valued stopping time whose finite values are $a_1 < a_2 < \cdots$ and suppose that $\{T \leqq a_i\}$ is in $\mathcal{F}^0(a_i)$ for all i. Then T is $\mathcal{G}^0(T)$ measurable.*

PROOF. Let $Y:\Omega \to \Omega$ be the mapping defined earlier in terms of X and T and note that if Λ is a set in $\mathcal{F}^0(a)$ then $Y^{-1}(\Lambda) \cap \{T \geqq a\}$ is equal to $\Lambda \cap \{T \geqq a\}$. Now $\{T = a_1\} = \Lambda_1$ is in $\mathcal{F}^0(a_1)$ and so $\Lambda_1 = Y^{-1}(\Lambda_1)$. Let $\Lambda_k = \{T = a_k\}$ and suppose as an induction hypothesis that $\Lambda_k = Y^{-1}(\Lambda_k)$ for all $k \leqq n$. Now $Y^{-1}(\Lambda_{n+1}) \cap \{T \geqq a_{n+1}\} = \Lambda_{n+1} \cap \{T \geqq a_{n+1}\} = \Lambda_{n+1}$, but the induction hypothesis implies that $Y^{-1}(\Lambda_{n+1})$ is disjoint from $\{T \leqq a_n\}$ and so $Y^{-1}(\Lambda_{n+1}) = \Lambda_{n+1}$, completing the proof.

It is easy to see that Lemma 2 remains valid if we alter the last hypothesis to read $\{T \leqq a_i\}$ is in $\mathcal{F}(a_i)$ and the conclusion to assert that T is $\mathcal{G}(T)$ measurable.

Let T be a stopping time and define T_n by $T_n = (k + 1)/2^n$ if $k/2^n \leqq T < (k + 1)/2^n$, $k = 0, 1, \cdots$, and $T_n = \infty$ if $T = \infty$.

LEMMA 3. $\mathfrak{F}(T) = \bigcap_n \mathcal{G}(T_n)$.

PROOF. It is obvious that $\mathfrak{F}(T) = \bigcap_n \mathfrak{F}(T_n)$. But Lemma 2, or rather the sentence following its proof, asserts that T_n is $\mathcal{G}(T_n)$ measurable, and so by Lemma 1, $\mathcal{G}(T_n) = \mathfrak{F}(T_n)$.

In the rest of the proof T will be a stopping time, T_n will be defined as above, and $Y_n:\Omega \to \Omega$ will be defined as was Y earlier, but using T_n instead of T. We will set

$$S(w) = \inf\{t: Y(t, w) = Y(r, w) \quad \text{for all } r \geqq t\}.$$

Clearly $S \leqq T$, S is $\mathcal{G}^0(T)$ measurable and $X(S) = X(T)$ whenever T is finite.

Let H be the set of holding points for our process. If $\{A_n\}$ is a collection of open sets with compact closures forming a base for the topology of E then H consists of Δ together with $\{x: \sup_n I_{A_n}(x) E^x(e^{-R_n}) < 1\}$ where R_n is the time the process first hits the complement of $\bar{A}_n$. Consequently H is in $\bar{\mathfrak{B}}$. Given numbers $r < s$, let

$$\Lambda(r, s) = \{w: X(r, w) \varepsilon H, X(t, w) = X(r, w) \quad \text{for all } t \varepsilon [r, s]\}.$$

Routine calculations show that for any probability measure μ on $\mathfrak{B}$ we have

$$\text{(a)} \qquad P^\mu(\Lambda(r, s)) = 0,$$

$$\text{(b)} \quad P^\mu(Y \neq Y_n, X(T) \varepsilon H, T < \infty) \leqq P^\mu(T < \infty, X(T) \varepsilon H, X(t + T) \neq X(T) \quad \text{for some } t \leqq 2^{-n}) \to 0 \text{ as } n \to \infty, \quad \text{and}$$

$$\text{(c)} \qquad P^\mu(S < T, X(T) \varepsilon H, T < \infty) \leqq \sum P^\mu(\Lambda(r, s)) = 0$$

where the sum in (c) is over all rationals r, s with $r < s$.

If t is any positive number then $I_{\{T \leqq t\}} = P^\mu(T < t \mid \mathcal{G}(T_n)) = P^\mu(T < t, X(T) \varepsilon H \mid \mathcal{G}(T_n)) + P^\mu(T < t, X(T) \not\varepsilon H \mid \mathcal{G}(T_n))$. Because of (c) the second summand just above is the indicator function of $\{S < t, X(S) \not\varepsilon H\}$, which is $\mathcal{G}^0(T)$ measurable. We may write the first summand as $f \cdot \varphi_n(Y_n)$ where f is the indicator function of $\{X(S) \varepsilon H, S < \infty\}$ and $\varphi_n:\Omega \to [0, 1]$ is $\mathfrak{F}^0$ measurable and is such that $\varphi_n(Y_n)$ is a version of $P^\mu(T < t \mid \mathcal{G}(T_n))$. Now $f \cdot \varphi_n(Y)$ is $\mathcal{G}^0(T)$ measurable, and

$$E^\mu(|\varphi_n(Y_n) - \varphi_n(Y)| f) \leqq P^\mu(X(T) \varepsilon H, T < \infty, Y_n \neq Y)$$

which according to (b) approaches 0 as $n \to \infty$. Thus regardless of μ we have $I_{\{T<t\}}$ displayed as a limit in the mean with respect to P^μ of $\mathcal{G}^0(T)$ measurable functions. Hence T is $\mathcal{G}(T)$ measurable, and from Lemma 1 it follows that $\mathcal{G}(T) = \mathfrak{F}(T)$.

REFERENCE

[1] BLUMENTHAL, R. M., GETOOR, R. K., and MCKEAN, H. P., JR. (1962). Markov processes with identical hitting distributions. *Illinois J. Math.* **6** 402–420.