ESTIMATORS OF A LOCATION PARAMETER IN THE ABSOLUTELY CONTINUOUS CASE[1]

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0. Summary. In the last decade there have been a number of papers dealing with the admissibility of translation invariant estimators of a location parameter. Blyth [2] treated sequential procedures in the case of normally or rectangularly distributed random variables. If d is estimated and θ is the actual parameter value, for Blyth, op. cit., loss was measured by $W(|d - \theta|)$ where $W(\cdot)$ was a nondecreasing function on $[0, \infty)$. In the same year Blackwell [1] treated the fixed sample size problem in the case of discrete random variables taking only a finite number of values. For Blackwell, op. cit., loss was measured by $W(d - \theta)$ where $W(\cdot)$ was assumed continuous and bounded from below but otherwise arbitrary. Blackwell showed that if the discrete random variables (which could be vector valued) took values only on the integer lattice points and if there was a unique minimax translation invariant estimator then it was admissible. Later papers by Karlin [7] and Stein [11] discuss the admissibility of Pitman's estimator for square error.

In reviewing these results we discovered that if the loss satisfied

$$(0.1) \quad 0 \leqq x < y \text{ implies } W(x) < W(y), \qquad y < x \leqq 0 \text{ implies } W(x) < W(y),$$

and if there were several minimax translation invariant estimators then no translation invariant estimator could be admissible. This and a related result constitutes Section 4.

It was logical to ask the converse question, does uniqueness imply admissibility? In the case of square error Pitman's estimator (except for changes on sets of measure zero) is necessarily unique since the loss function is strictly convex. In the case of normally distributed random variables and symmetrical loss functions as considered by Blyth, op. cit., the sample mean is the uniquely determined minimax translation invariant estimator.

In this paper we have restricted the discussion to random variables which have density functions relative to Lebesgue measure. Since it proved possible to deal with the question of admissibility for a larger class of estimators than the translation invariant estimators, we define generalized Bayes estimators as the solution of a minimization problem. Section 2 deals with the question of the existence of measurable solutions to the minimization problem. Sections 5 and 6 deal with the question, does uniqueness imply admissibility?

Section 3 deals with the question of whether generalized Bayes estimators are strongly consistent and shows that under mild restrictions this is the case.

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Section 8 is a generalization of the results of Katz [8] for minimax estimators of $\theta \varepsilon [0, \infty)$. We show how to construct such estimators whenever loss is measured by $W(d - \theta)$, $W(\cdot)$ strictly convex, non-negative, $W(0) = 0$.

Blyth, op. cit., proved admissibility only within the class of continuous risk functions. In Section 9 we remove this restriction. We then show that if the loss function $W(\cdot)$ is strictly convex and symmetrical then the sample mean based on n observations is admissible, in the nonparametric context of estimating the mean of an unknown density function, within the class of all sequential procedures having expected sample size $\leqq n$.

1. Introduction. Let $f(\cdot, \cdots, \cdot)$ be a non-negative Borel measurable function defined on Euclidian n-space such that

$$1 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n. \tag{1.1}$$

Throughout we study estimators of the parameter θ for the family

$$\{f(\cdot - \theta, \cdots, \cdot - \theta), -\infty < \theta < \infty\} \tag{1.2}$$

of density functions. We will suppose a non-negative Borel measurable function $W(\cdot)$ of a real variable is given. If θ is the actual parameter value and d is the estimated value then $W(d - \theta)$ is the measure of loss.

Throughout we discuss generalized Bayes estimators $\delta(\cdot, \cdots, \cdot)$, defined as follows. Suppose $g(\cdot)$ is a non-negative Borel measurable function. Suppose for all vectors $(x_1, \cdots, x_n)$ in Euclidian n-space that

$$\begin{aligned}\int_{-\infty}^{\infty} & W(\delta(x_1, \cdots, x_n) - \theta) f(x_1 - \theta, \cdots, x_n - \theta) g(\theta)\, d\theta \\ &= \inf_{\delta^*} \int_{-\infty}^{\infty} W(\delta^*(x_1, \cdots, x_n) - \theta) f(x_1 - \theta, \cdots, x_n - \theta) g(\theta)\, d\theta.\end{aligned} \tag{1.3}$$

Then $\delta(\cdot, \cdots, \cdot)$ will be called a generalized Bayes estimator relative to the weight function $g(\cdot)$.

If $\int_{-\infty}^{\infty} g(\theta) d\theta < \infty$ then $\delta(\cdot, \cdots, \cdot)$ is a Bayes estimator. For the weight function $g(\theta) = 1, -\infty < \theta < \infty$, and $W(x) = x^2, -\infty < x < \infty$, the estimator which results is Pitman's estimator. See Pitman [10]. This estimator is minimax within the class of all estimators, see Girshick and Savage [5]; under liberal conditions the estimator is admissible, see Stein [11].

In case $g(\theta)$ is identically one the estimator $\delta(\cdot, \cdots, \cdot)$ (if uniquely defined by (1.3)) is translation invariant, i.e.

$$\text{for all real numbers } c \text{ and vectors } (x_1, \cdots, x_n), \quad \delta(x_1 + c, \cdots, x_n + c) = \delta(x_1, \cdots, x_n) + c. \tag{1.4}$$

$\delta(\cdot, \cdots, \cdot)$ must be minimax within the class of all invariant estimators; from results of Kiefer [9], $\delta(\cdot, \cdots, \cdot)$ must be minimax within the class of all estimators. Estimators of this type were discussed by Blyth [2] in the case $f(\cdot)$ is a normal density and $W(\cdot)$ satisfies, for all x, $W(x) = W(-x)$. Blyth shows that if δ has finite risk then δ is admissible (within the class of all estimators based on a sample size n).

Location parameters for discrete random variables taking only a finite number of values were studied by Blackwell [1]. If the loss function is continuous and if the analogue of Pitman's estimator is uniquely determined then it is admissible. Blackwell gives an example in which $W(x) = |x|$, $-\infty < x < \infty$, where the analogue of Pitman's estimator is not uniquely determined and shows every translation invariant estimator must be inadmissible.

In Section 4, $W(\cdot)$ satisfying

$$\begin{aligned} &\text{if} \quad x < y \leqq 0 \quad \text{then} \quad W(x) > W(y) \geqq 0, \\ &\text{if} \quad x > y \geqq 0 \quad \text{then} \quad W(x) > W(y) \geqq 0, \\ &W(0) = 0, \end{aligned} \tag{1.5}$$

we show that a necessary condition for admissibility of a solution of (1.3) with $g(\theta) \equiv 1$ is that there be a unique solution of (1.3), a parallel with Blackwell's results. In Section 5, for a more restricted class of functions $W(\cdot)$, and for a certain class of functions $g(\cdot)$, we show uniqueness implies admissibility. We will not state here the necessary conditions.

Fox and Rubin [4] have studied loss functions of the form $W(x) = a(|x| - x) + b(|x| + x)$ and have shown that uniqueness of the solution of (1.3) for the weight function $g(\theta) \equiv 1$ implies admissibility. Their result is stronger than the results of Section 5 applied to this case.

Let

$$g(\theta) = 0, \quad \theta \leqq 0, \qquad g(\theta) = 1, \quad \theta > 0, \tag{1.6}$$

$W(\cdot)$ be strictly convex and satisfy the Condition (7.4) (see Section 7). Then the solution $\delta(\cdot, \cdots, \cdot)$ to (1.3) with this $g(\cdot)$ is a minimax estimator of $\theta \,\varepsilon\, [0, \infty)$. This includes the result of Katz [8] and a similar result of J. Sacks (unpublished). The methods of Section 5 do not give a proof of admissibility for this choice of weight function. In Section 8 we prove a general admissibility result in the special case $W(x) = x^2$. This includes the admissibility result of Katz, op. cit. In the case of minimax estimators of $\theta \,\varepsilon\, [0, \infty)$ there will in general be many different and admissible minimax estimators. See the discussion in Section 8.

In Section 9 we consider the following non-parametric problem. Suppose $W(\cdot)$ is a strictly convex symmetric function and Ω is a class of density functions satisfying

$$\text{if} \quad f \,\varepsilon\, \Omega, \quad \int_{-\infty}^{\infty} W(x)f(x)\,dx < \infty, \quad \text{and for every real } c, \quad f(\cdot - c) \,\varepsilon\, \Omega. \tag{1.7}$$

Let $\delta(x_1, \cdots, x_n) = (1/n)\sum_{i=1}^{n} x_i$, and $\theta(f) = \int_{-\infty}^{\infty} xf(x)dx$. Within the class of all sequential procedures with expected sample size $\leqq n$ for all $f \,\varepsilon\, \Omega$, $\delta(\cdot, \cdots, \cdot)$

is an admissible estimator, provided for some $\tau > 0$,

$$1/(2\pi\tau)^{\frac{1}{2}} \exp(-(1/2\tau)x^2) \,\varepsilon\, \Omega.$$

We obtain this result by restricting the problem to one about normal random variables. By a modification of arguments due to Blyth [2] we are able to show in the normal case $\delta(\cdot, \cdots, \cdot)$ is the unique minimax estimator (except for changes on sets of measure zero). This answers a question left open by Blyth whose admissibility proof was valid only for estimators in the class of sequential procedures having continuous risk functions.

For a given weight function $g(\cdot)$ we consider the sequence of solutions $\{\delta_n(\cdot, \cdots, \cdot), n \geqq 1\}$ to (1.3) for the sample sizes $n = 1, 2, \cdots$. Under mild restrictions, and if $W(\cdot)$ is bounded or convex and satisfies

$$\inf_{|x| \geqq \epsilon} W(x) > 0 \quad \text{for all} \quad \epsilon > 0, \qquad W(0) = 0, \tag{1.8}$$

we prove strong consistency in Section 3.

In the case of a scale parameter σ, given the family $\{(1/\sigma)f(x/\sigma), 0 < \sigma < \infty\}$ of density functions, loss measured by $W((d - \sigma)/\sigma)$, the parameter may be changed to a location parameter using the substitutions $\theta = \log \sigma$, $y = \log x$, and loss $W^*(d - \theta) = W(e^{d-\theta} - 1)$. In case $W(\cdot)$ is convex, $W^*(\cdot)$ is not convex but satisfies (provided $W(0) = 0$)

$$\sup_{x<0} W^*(x) < \infty, \quad W^*(\cdot) \text{ is a convex function of } x \geqq 0, W^*(0) = 0. \tag{1.9}$$

This explains what might otherwise seem like a bizarre type of loss considered in Section 5.

Some restrictions used in Section 5 may be removed by different methods not found by us. In Section 3 the restrictions used to show certain sequences of functions are uniformly integrable are unnecessarily restrictive. One feels (see Section 8) that the estimators considered in Section 7 must be admissible for a bigger class of loss functions than square error. It remains an open question (Section 9) whether for symmetric loss functions $W(\cdot)$ not strictly convex the estimator $(1/n)\sum_{i=1}^n x_i$ is the only minimax estimator of the median based on n observations. These are, we feel, a few of the questions left open by this paper.

2. Existence of nonrandomized estimators. This section establishes notation and lemmas needed throughout. We have been advised by the referee that the types of results stated here, and their proofs, are well known. Proofs are omitted.

If $n \geqq 1$, E_n will be n-dimensional Euclidian space. Throughout x will be a real number, y a vector in E_{n-1}. $\mu(\cdot)$ will be a probability measure defined on the Borel subsets of E_{n-1}. $f(\cdot, \cdot, \cdot) \geqq 0$ on $E_1 \times E_{n-1} \times E_1$ to E_1 is jointly measurable in the three variables and

$$\begin{aligned} 1 &= \iint f(x, y, \theta)\, dx\, \mu(dy), && -\infty < \theta < \infty. \\ 1 &= \int f(x, y, \theta)\, dx, \qquad y \,\varepsilon\, E_{n-1}, && -\infty < \theta < \infty. \end{aligned} \tag{2.1}$$

Later we will specialize to

$$\text{(2.2)} \quad \text{for all } (x, y) \, \varepsilon \, E_n, \quad -\infty < \theta < \infty, \qquad f(x, y, \theta) = f(x - \theta, y, 0).$$

In this case the third variable will not be indicated.

A density function of the form (2.2) may arise as follows. Suppose $f_1(\cdot)$ is a density function defined on E_1. The joint density of n independent observations each with $f_1(\cdot)$ as density function is

$$\text{(2.3)} \quad \begin{aligned} &f_1(x_1 - \theta)f_1(x_1 - \theta + y_1) \cdots f_1(x_1 - \theta + y_{n-1}) \quad \text{with} \\ &\qquad\qquad y_i = x_{i+1} - x_1, \quad 1 \leqq i \leqq n - 1. \end{aligned}$$

We define $f(\cdot, \cdot)$ as

$$\text{(2.4)} \quad f(x, y) = \frac{f_1(x)f_1(x + y_1) \cdots f_1(x + y_{n-1})}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(x)f_1(x + y_1) \cdots f_1(x + y_{n-1}) \prod_{i=1}^{n-1} dy_i}$$

and $\mu(\cdot)$ by, for all $n - 1$ dimensional Borel sets A,

$$\text{(2.5)} \quad \mu(A) = \int \cdots \int_A \int_{-\infty}^{\infty} f_1(x)f_1(x + y_1) \cdots f_1(x + y_{n-1}) \, dx \prod_{i=1}^{n-1} dy_i .$$

The interpretation of $f(\cdot, \cdot)$ and $\mu(\cdot)$ given by (2.4) and (2.5) is required in Section 3 to prove strong consistency. In the remainder of this paper all results are valid for the general function of the form (2.2).

If a decision d is made (i.e., d is the estimated value) and if θ is the actual parameter value, loss will be measured by $W(d - \theta)$. We will assume throughout that $W(\cdot)$ satisfies the following conditions.

$$\text{(2.6)} \quad \begin{aligned} &W(x) \geqq 0, \quad -\infty < x < \infty; \qquad W(0) = 0; \\ &\text{if} \quad 0 \leqq x_1 \leqq x_2 \quad \text{then} \quad W(x_1) \leqq W(x_2); \\ &\qquad\qquad \text{if} \quad 0 \geqq x_1 \geqq x_2 \quad \text{then} \quad W(x_1) \leqq W(x_2). \end{aligned}$$

Throughout, if $\delta(\cdot, \cdot)$ is an estimator then

$$\text{(2.7)} \quad R(\delta, \theta) = \iint W(\delta(x, y) - \theta)f(x, y, \theta) \, dx \, \mu(dy)$$

defines the risk function of $R(\delta, \cdot)$ of $\delta(\cdot, \cdot)$.

Definition. Define $F(\cdot, \cdot, \cdot)$ by

$$\text{(2.8)} \quad F(c, x, y) = \int W(c - \theta)f(x, y, \theta) \, \lambda(d\theta).$$

Suppose $\lambda(\cdot)$ is a σ-finite measure defined on the Borel sets of E_1. Suppose $\delta(\cdot, \cdot)$ is a measurable function on $E_1 \times E_{n-1}$ to E_1 such that for some measurable set $N \subset E_1 \times E_{n-1}$,

$$\text{(2.9)} \quad \begin{aligned} &0 = \iint_N dx \, \mu(dy) \quad \text{and if} \quad (x, y) \not\varepsilon N, F(\delta(x, y), x, y) \\ &\qquad\qquad = \inf_c F(c, x, y) < \infty. \end{aligned}$$

Then $\delta(\cdot, \cdot)$ will be called a *generalized Bayes estimator* for $\lambda(\cdot)$. In most applications the set of values c such that $F(c, x, y) = \inf_c F(c, x, y)$ will be compact. If $\delta(x, y)$ is a generalized Bayes estimator such that for each (x, y), $\delta(x, y)$ is the largest number c satisfying $F(c, x, y) = \inf_c F(c, x, y)$ then we will call $\delta(x, y)$ *the maximal generalized Bayes estimator*. Similarly one may define *the minimal generalized Bayes estimator*. If $\lambda(\cdot)$ is a totally finite measure then $\delta(\cdot, \cdot)$ is a Bayes estimator.

If

$$(2.10) \qquad W(\infty) = \lim_{x \to \infty} W(x), \qquad W(-\infty) = \lim_{x \to -\infty} W(x),$$

and if $\min(W(-\infty), W(\infty)) < \infty$, it may be necessary to allow values of $-\infty$ or $+\infty$ in order to establish the existence of nonrandomized generalized Bayes estimators. Theorem 1 below is needed in later sections to prove the existence of *nonrandomized* estimators.

THEOREM 1. *Assume the definitions above and that* $W(\cdot)$ *satisfies* (2.6). *Let* $N \subset E_1 \times E_{n-1}$ *be a measurable set satisfying* (2.9) *and such that*

$$(2.11) \qquad \text{if} \quad (x, y) \not\in N, \qquad \inf_c F(c, x, y) < \infty.$$

Suppose one of the following hypotheses is satisfied: (1) $W(\cdot)$ *is continuous and* $\lambda(\cdot)$ *is a* σ*-finite measure*; (2) $\lambda(\cdot)$ *is a nonatomic* σ*-finite measure. Then there exists a maximal generalized Bayes estimator and there exists a minimal generalized Bayes estimator, each satisfying* (2.9).

In certain cases in which the function $W(\cdot)$ is differentiable one might expect that

$$(2.12) \quad \left(\frac{\partial}{\partial c} \int W(c - \theta) f(x, y, \theta)\, \lambda(d\theta)\right)_{c=c_0} = \int W'(c_0 - \theta) f(x, y, \theta)\, \lambda(d\theta),$$

W' the derivative of $W(\cdot)$. We state a lemma.

LEMMA 2.1. *Suppose in addition to* (2.6) *that* $W(\cdot)$ *is convex. Let* $W'(\cdot)$ *be the right continuous derivative of* $W(\cdot)$. *Suppose* $\lambda(\cdot)$ *is a nonatomic* σ*-finite measure defined on the Borel sets of* E_1. *If for some* $(x, y) \in E_1 \times E_{n-1}$ *and real numbers* $a < b$,

$$(2.13) \qquad \int W(c - \theta) f(x, y, \theta)\, \lambda(d\theta) < \infty; \qquad a < c < b,$$

then

$$(2.14) \qquad \int |W'(c - \theta)|\, f(x, y, \theta)\, \lambda(d\theta) < \infty, \qquad a < c < b,$$

and (2.12) *holds if* $c_0 \in (a, b)$.

LEMMA 2.2. *Suppose in addition to* (2.6) *that* $W(\cdot)$ *is convex and*

$$(2.15) \qquad \lim_{|x| \to \infty} W(x) = \infty.$$

Suppose there is a measurable set $N_1 \subset E_1 \times E_{n-1}$ *satisfying* $\int\int_{N_1} dx \mu(dy) = 0$ *such that if* $(x, y) \in E_1 \times E_{n-1}$, $(x, y) \not\in N_1$, *then*

$$\int W(c - \theta)f(x, y, \theta)\, \lambda(d\theta) < \infty, \qquad -\infty < c < \infty. \tag{2.16}$$

If $\lambda(\cdot)$ *is nonatomic then every generalized Bayes estimator* $\delta(\cdot, \cdot)$ *for* $\lambda(\cdot)$ *satisfies*

$$0 = \int W'(\delta(x, y) - \theta)f(x, y, \theta)\, \lambda(d\theta) \tag{2.17}$$

except possibly for $(x, y) \in N_1$; $\int\int_{N_1} dx\mu(dy) = 0$.

Example. $W(x) = |x|$. Let $f(x, y, \theta) = f(x - \theta)$. Let $\lambda(\cdot)$ be Lebesgue measure on the Borel sets of E_1. By the preceding the maximal generalized Bayes estimator $\delta(\cdot)$ for $\lambda(\cdot)$ satisfies

$$\begin{aligned} 0 &= \int W'(\delta(x) - \theta)f(x - \theta)\, d\theta = \int_{-\infty}^{\delta(x)} f(x - \theta)\, d\theta \\ &\quad - \int_{\delta(x)}^{\infty} f(x - \theta)\, d\theta = \int_{x-\delta(x)}^{\infty} f(\theta) - \int_{-\infty}^{x-\delta(x)} f(\theta)\, d\theta. \end{aligned} \tag{2.18}$$

It follows that

$$\tfrac{1}{2} = \int_{-\infty}^{x-\delta(x)} f(\theta)\, d\theta, \qquad -\infty < x < \infty. \tag{2.19}$$

Therefore $x - \delta(x)$ is a median of $f(\cdot)$. If $f(\cdot)$ has several median values, say c and d, then $\delta_c(x) = x - c$ and $\delta_d(x) = x - d$ are both generalized Bayes estimators for $\lambda(\cdot)$. In Section 4 we show this lack of uniqueness implies $\delta_c(\cdot)$ and $\delta_d(\cdot)$ are inadmissible.

3. Strong consistency. In this section we will consider only independent identically distributed random variables with common density function $f(\cdot - \theta)$ relative to Lebesgue measure on the line. We will suppose a σ-finite measure $\lambda(\cdot)$ defined on the Borel sets of E_1 is given. We consider the sequence $\{\delta_n(\cdot), n \geqq 1\}$ of generalized Bayes estimators defined by

$$\begin{aligned} &\text{if} \quad n \geqq 1, \quad \text{all } (x_1, \cdots, x_n) \in E_n, \qquad \int W(\delta_n(x_1, \cdots, x_n) - \theta) \prod_{i=1}^n \\ &\qquad \cdot f(x_i - \theta)\lambda(d\theta) = \inf_c \int W(c - \theta) \prod_{i=1}^n f(x_i - \theta)\, \lambda(d\theta). \end{aligned} \tag{3.1}$$

We give in Theorem 2 sufficient conditions that the sequence $\{\delta_n(\cdot), n \geqq 1\}$ be a strongly consistent sequence of estimators.

We will suppose E_∞ is the set of all real valued functions defined on the integers $n \geqq 0$, E_∞ given the Cartesian product topology. We let $\mathfrak{F}$ be the σ-algebra of Borel sets of E_∞. We denote points of E_∞ by e. We suppose there is defined on $(E_\infty, \mathfrak{F})$ a σ-finite measure $\mu(\cdot)$ satisfying,

$$\begin{aligned} &\text{if } n \geqq 0 \text{ and } A_0, \cdots, A_n \text{ are Borel sets of } E_1, \quad \text{and if} \\ &\qquad E = \{e \mid e(i) \in A_i, 0 \leqq i \leqq n\}, \quad \text{then} \\ &\qquad \mu(E) = \int_{A_0} \left(\int_{A_1} \cdots \int_{A_n} \prod_{i=1}^n f(x_i - y) \prod_{i=1}^n dx_i \right) \lambda(dy). \end{aligned} \tag{3.2}$$

ASSUMPTION 1. There is an integer $M_1 \geqq 1$ such that if $n \geqq M_1$, then (3.1) defines a generalized Bayes estimator $\delta_n(\cdot)$ for $\lambda(\cdot)$.

ASSUMPTION 2. $W(\cdot)$ satisfies (2.6). If $\epsilon \neq 0$ then $W(\epsilon) > 0$.

ASSUMPTION 3. $W(\cdot)$ is a bounded function; or, $W(\cdot)$ is convex and there exists a constant $K > 0$ such that

$$\text{(3.3)} \qquad \begin{aligned} &\text{if} \quad -\infty \leqq x, y \leqq \infty \quad \text{then} \\ &W(x+y) \leqq K(W(x) + W(y)) \quad \text{and} \quad W(-x) \leqq K\,W(x) \end{aligned}$$

Define functions $X_n(\cdot)$, $n \geqq 1$ on E_∞ by

$$\text{(3.4)} \qquad \text{if} \quad n \geqq 1, \qquad X_n(e) = e(n), \quad \text{all} \quad e \,\varepsilon\, E_\infty .$$

Let $\mathfrak{F}_n \subset \mathfrak{F}$ be the least σ-algebra of sets in which $X_1(\cdot), \cdots, X_n(\cdot)$ are measurable.

ASSUMPTION 4. There exists an integer M_2 such that if $n \geqq M_2$ then $\mu(\cdot)$ restricted to $\mathfrak{F}_n$ is a σ-finite measure. There exists in $\mathfrak{F}_{M_2}$ a sequence of sets $\{A_n, n \geqq 1\}$ satisfying if $n \geqq 1$, $A_n \subset A_{n+1}$, $\mu(A_n) < \infty$, $\bigcup_{i=1}^{\infty} A_i = E_\infty$, and $\int_{A_n} W(e(0))\mu(de) < \infty$.

ASSUMPTION 5. If $W(\cdot)$ is bounded then $W(\cdot)$ is a continuous function or $\lambda(\cdot)$ is a nonatomic measure. If $W(\cdot)$ is bounded then $\lim_{x\to 0} W(x) = 0$.

Define a measurable function $\theta(\cdot)$ by $\theta(e) = e(0)$, all $e \,\varepsilon\, E_\infty$.

THEOREM 2. *Given Assumptions* 1 *to* 5, *if* $\{\delta_n(\cdot), n \geqq 1\}$ *are defined by* (3.1), *then for almost all* $e \,\varepsilon\, E_\infty(\mu)$,

$$\text{(3.5)} \qquad \lim_{n\to\infty} \delta_n(X_1(e), \cdots, X_n(e)) = \theta(e).$$

The remainder of this section consists in the proof of Theorem 2. Let $\mathfrak{F}_\infty \subset \mathfrak{F}$ be the least σ-algebra of sets containing $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$. We will show first that there is a real valued function $\delta^*(\cdot)$ on E_∞, measurable in $\mathfrak{F}_\infty$, such that

$$\text{(3.6)} \qquad \delta^*(e) = \theta(e) \quad \text{except for } e \text{ in a set of } \mu(\cdot) \text{ measure zero.}$$

Suppose $\phi(\cdot)$ is a bounded strictly increasing function. Then

$$\text{(3.7)} \qquad \phi^*(y) = \int_{-\infty}^{\infty} \phi(x)\, f(x-y)\, dx = \int_{-\infty}^{\infty} \phi(x-y)\, f(x)\, dx.$$

It follows that $\phi^*(\cdot)$ is a strictly increasing function. Conditional on $\theta = y$ the random variables (measurable functions) $\{X_n, x \geqq 1\}$ are independent and identically distributed with conditional density function $f(\cdot - y)$. By the strong law of large numbers with probability one,

$$\text{(3.8)} \qquad \lim_{n\to\infty} 1/n \sum_{i=1}^{n} \phi(X_i) = \phi^*(y).$$

Since this holds for each y, by Fubini's Theorem,

$$\text{(3.9)} \qquad \lim_{n\to\infty} 1/n \sum_{i=1}^{n} \phi(X_i(e)) = \phi^*(\theta(e))$$

except for e in a set of measure zero.

Since $\phi^*(\cdot)$ is strictly increasing, and continuous,

$$\lim_{n\to\infty} \phi^{*-1}(1/n \sum_{i=1}^{n} \phi(X_i)) = \theta \quad \text{a.e.} \tag{3.10}$$

The function $\phi^{*-1}(1/n\sum_{i=1}^{n} \phi(X_i(\cdot)))$ is measurable in $\mathfrak{F}_n \subset \mathfrak{F}_\infty$. Therefore the limit in (3.10) is measurable in $\mathfrak{F}_\infty$. Note that this does not assert $\theta(\cdot)$ is measurable in $\mathfrak{F}_\infty$.

Using the result just obtained we show the proof of strong consistency may be reduced to a problem in probability theory. Let M_2 and $\{A_n, n \geqq 1\}$ be as in Assumption 4. Let

$$\{\psi_{A_n}(\cdot), n \geqq 1\} \tag{3.11}$$

be the characteristic functions of the sets A_n, $n \geqq 1$ taken to be functions defined on E_{M_2}. Define constants $\{c_n, n \geqq 1\}$ by

$$\text{if} \quad n \geqq 1, \qquad c_n = \int \cdots \int \psi_{A_n}(x_1, \cdots, x_{M_2}) \prod_{i=1}^{M_2} f(x_i - y) \prod_{i=1}^{M_2} dx_i \, \lambda(dy). \tag{3.12}$$

By Assumption 4, if $n \geqq 1$, $c_n < \infty$. We may therefore define on $(E_\infty, \mathfrak{F})$ probability measures $\{\mu_n(\cdot), n \geqq 1\}$ by

$$\text{if} \quad n \geqq 1, \quad A \,\varepsilon\, \mathfrak{F}, \quad \text{then} \qquad \mu_n(A) = (1/c_n)\mu(A \cap A_n). \tag{3.13}$$

From (3.1) it follows that

$$\text{if} \quad n \geqq 1, \quad m \geqq M_2, \quad \text{then}$$

$$\begin{aligned} &\int \psi_{A_n}(x_1,\cdots, x_m)\, W(\delta_m(x_1,\cdots, x_m) - y) \prod_{i=1}^{m} f(x_1 - y)\, \lambda(dy) \\ &\quad = E(W(\delta_m - \theta) \mid \mathfrak{F}_m) \int \psi_{A_n}(x_1, \cdots, x_m) \prod_{i=1}^{m} f(x_i - y)\, \lambda(dy) \quad \text{a.e. } \mu_n(\cdot) \end{aligned} \tag{3.14}$$

and that if δ' is any function measurable in $\mathfrak{F}_m$ then

$$E(W(\delta_m - \theta) \mid \mathfrak{F}_m) \leqq E(W(\delta' - \theta) \mid \mathfrak{F}_m) \quad \text{a.e. } \mu_n, \tag{3.15}$$

this holding for all $m \geqq M_2$. To prove the theorem it is clearly sufficient to show

$$\begin{aligned} &\lim_{m\to\infty} \psi_{A_n}(X_1, \cdots, X_m) W(\delta_m(X_1, \cdots, X_m) - \theta) \\ &\qquad = 0 \quad \text{a.e. } \mu_n(\cdot), \quad n \geqq 1 \end{aligned} \tag{3.16}$$

as follows from Assumption 4. To prove (3.16) we require the existence of a function $\delta^*(\cdot)$ measurable in $\mathfrak{F}_\infty$ and equal to $\theta(\cdot)$ except on a set of measure zero. From the first part of the proof it follows that if $n \geqq 1$ this condition is satisfied for the measure $\mu_n(\cdot)$.

The subsequent argument is not affected by the simplification $M_2 = 1$. We assume this simplification. We will drop the subscript and refer simply to a probability measure $\mu(\cdot)$.

Define functions $\{\delta_n^*(\cdot), n \geqq 1\}$ by

$$\text{(3.17)} \quad \begin{array}{l} \text{if } W(\cdot) \text{ is bounded}, \quad \delta_n^* = \tan(E(\arctan \theta \mid \mathfrak{F}_n)) \quad \text{a.e. } \mu(\cdot), \quad n \geqq 1. \\ \text{If } W(\cdot) \text{ is convex}, \quad \delta_n^* = E(\theta \mid \mathfrak{F}_n) \quad \text{a.e. } \mu(\cdot). \end{array}$$

If $W(\cdot)$ is convex our Assumption 4 implies

$$\text{(3.18)} \qquad E(W(\theta)) < \infty,$$

from Assumption 2 it follows that

$$\text{(3.19)} \qquad \lim_{|x| \to \infty} W(x) = \infty.$$

Taken together (3.18) and (3.19) imply

$$\text{(3.20)} \qquad E|\theta| < \infty.$$

Therefore the definition in (3.17) is justified.

Define sets $A(n, j, \epsilon)$, $n \geqq 1, j \geqq 0, \epsilon > 0$ by

$$\text{(3.21)} \quad \begin{array}{l} A(n, j, \epsilon) \text{ is the event} \quad |\delta_{n+i} - \delta_{n+i}^*| \leqq \epsilon, \\ \qquad\qquad i = 0, \cdots, j-1 \quad \text{and} \quad |\delta_{n+j} - \delta_{n+j}^*| > \epsilon. \end{array}$$

Then if $n \geqq 1, j \geqq 0, \epsilon > 0$, $A(n, j, \epsilon) \,\varepsilon\, \mathfrak{F}_{n+j}$.

It follows from (3.15) that

$$\text{(3.22)} \quad \begin{aligned} &\int_{A(n,j,\epsilon)} W(\delta_{n+j}(e) - \theta(e))\, \mu(de) \\ &= \int_{A(n,j,\epsilon)} E(W(\delta_{n+j} - \theta) \mid \mathfrak{F}_{n+j})(e)\, \mu(de) \\ &\leqq \int_{A(n,j,\epsilon)} E(W(\delta_{n+j}^* - \theta) \mid \mathfrak{F}_{n+j})(e)\, \mu(de) \\ &= \int_{A(n,j,\epsilon)} W(\delta_{n+j}^*(e) - \theta(e))\, \mu(de). \end{aligned}$$

If $W(\cdot)$ is convex then using (3.3), Jensen's inequality and the fact the sets $A(n, 0, \epsilon), A(n, 1, \epsilon), \cdots$ are pairwise disjoint it follows that

$$\text{(3.23)} \quad \begin{aligned} &\sum_{j=0}^{\infty} \int_{A(n,j,\epsilon)} W(\delta_{n+j}^*(e) - \theta(e))\, \mu(de) \\ &\leqq (K^2 + K) \int W(\theta(e))\, \mu(de) < \infty, \qquad n \geqq 1. \end{aligned}$$

If $W(\cdot)$ is bounded then

$$\text{(3.24)} \quad \sum_{j=0}^{\infty} \int_{A(n,j,\epsilon)} W(\delta_{n+j}^*(e) - \theta(e))\, \mu(de) \leqq \sup_x W(x), \qquad n \geqq 1.$$

From the definition of the functions $\{\delta_n^*(\cdot), n \geqq 1\}$, the martingale theorems, Doob [3], and the relation $\theta = E(\theta \mid \mathfrak{F}_\infty)$ except on a set of measure zero, it follows that

$$\lim_{n\to\infty} \delta_n^* = \theta \quad \text{a.e. } \mu(\cdot). \tag{3.25}$$

Define for $n \geqq 1$, $\epsilon > 0$ sets $B(n, \epsilon)$ by

$$B(n, \epsilon) \text{ is the event } \quad \sup_{m \geqq n} |\delta_m^* - \theta| < \epsilon/2. \tag{3.26}$$

Then

$$\text{if} \quad \epsilon > 0, \qquad \lim_{n\to\infty} \mu(B(n, \epsilon)) = 1. \tag{3.27}$$

On the event $B(n, \epsilon) \cap A(n, j, \epsilon)$,

$$|\delta_{n+j} - \delta_{n+j}^*| > \epsilon \quad \text{and} \quad |\delta_{n+j}^* - \theta| < \epsilon/2. \tag{3.28}$$

It follows that

$$|\delta_{n+j} - \theta| > \epsilon/2. \tag{3.29}$$

Then by (3.22)

$$\begin{aligned} &\min\,(W(\epsilon/2), W(-\epsilon/2))\,\mu(B(n, \epsilon) \cap A(n, j, \epsilon)) \\ &\qquad \leqq \int_{B(n, \epsilon) \cap A(n, j, \epsilon)} W(\delta_{n+j}(e) - \theta(e))\,\mu(de) \\ &\qquad \leqq \int_{A(n, j, \epsilon)} W(\delta_{n+j}^*(e) - \theta(e))\,\mu(de). \end{aligned} \tag{3.30}$$

Define sets $A(n, \epsilon)$ for $n \geqq 1$ and $\epsilon > 0$ by

$$A(n, \epsilon) = \bigcup_{j=0}^{\infty} A(n, j, \epsilon). \tag{3.31}$$

$A(n, \epsilon)$ is the event that $\sup_{m \geqq n} |\delta_m - \delta_m^*| > \epsilon$.

From (3.30),

$$\begin{aligned} &\min\,(W(\epsilon/2), W(-\epsilon/2))\,\mu(B(n, \epsilon) \cap A(n, \epsilon)) \\ &\qquad \leqq \sum_{j=0}^{\infty} \int_{A(n, j, \epsilon)} W(\delta_{n+j}^*(e) - \theta(e))\,\mu(de). \end{aligned} \tag{3.32}$$

The proof will be completed by showing the right side of (3.32) tends to zero as $n \to \infty$. For since $\min(W(\epsilon/2), W(-\epsilon/2)) > 0$ and since (3.27) holds if $\epsilon > 0$ it will follow that if $\epsilon > 0$, $\lim_{n\to\infty} \mu(A(n, \epsilon)) = 0$. That is to say,

$$\text{if} \quad \epsilon > 0, \qquad \mu(\{e \mid \limsup_{n\to\infty} |\delta_n(e) - \delta_n^*(e)| > \epsilon\}) = 0. \tag{3.33}$$

From (3.25) the conclusion of Theorem 2 will follow. To prove the right side of (3.32) tends to zero we will prove uniform integrability. In view of (3.32), (3.24) it suffices to show

$$0 = \lim_{\Delta\to 0} \sup_{\substack{\mu(B) \leqq \Delta \\ n \geqq 1}} \int_B \sum_{j=0}^{\infty} \psi(n, j, \epsilon, e)\, W(\delta_{n+j}^*(e) - \theta(e))\,\mu(de) \tag{3.34}$$

where if $n \geqq 1, j \geqq 0, \epsilon > 0, \psi(n, j, \epsilon, \cdot)$ is the characteristic function of $A(n, j, \epsilon)$. If $W(\cdot)$ is bounded (3.34) follows at once. If $W(\cdot)$ is convex then by Jensen's inequality,

$$\begin{aligned} W(\delta^*_{n+j} - \theta) &\leqq K(W(E(\theta \mid \mathfrak{F}_{n+j})) + W(-\theta)) \\ &\leqq KE(W(\theta) \mid \mathfrak{F}_{n+j}) + K^2 W(\theta). \end{aligned} \tag{3.35}$$

It suffices to show (3.34) holds for the functions $E(W(\theta) \mid \mathfrak{F}_{n+j})$, $n \geqq 1, j \geqq 0$. Let $\psi_B(\cdot)$ be the characteristic function of the set B.

$$\begin{aligned} &\int \sum_{j=0}^{\infty} \psi_B(e)\, \psi(n, j, \epsilon, e)\, E(W(\theta) \mid \mathfrak{F}_{n+j})(e)\, \mu(de) \\ &\qquad = \int \sum_{j=0}^{\infty} E(\psi_B \mid \mathfrak{F}_{n+j})\,(e)\, \psi(n, j, \epsilon, e)\, W(\theta(e))\, \mu(de). \end{aligned} \tag{3.36}$$

Since the sets $A(n, j, \epsilon)$, $j \geqq 0$ are pairwise disjoint it follows that

$$\sum_{j=0}^{\infty} E(\psi_B \mid \mathfrak{F}_{n+j}) \psi(n, j, \epsilon, \cdot) \leqq 1 \quad \text{a.e. } \mu(\cdot) \tag{3.37}$$

and

$$\int \sum_{j=0}^{\infty} E(\psi_B \mid \mathfrak{F}_{n+j})\, \psi(n, j, \epsilon, e)\, \mu(de) \leqq \mu(B). \tag{3.38}$$

Let

$$c = \int W(\theta(e))\, \mu(de). \tag{3.39}$$

By the Neyman-Pearson lemma, among all functions ϕ on E_∞ satisfying

$$0 \leqq \phi \leqq 1, \qquad \int \phi(e)\, \mu(de) \leqq \Delta, \tag{3.40}$$

a function $\phi^*(\cdot)$ such that

$$\int \phi^*(e)\, W(\theta(e))\, \mu(de) = \sup_\phi \int \phi(e)\, W(\theta(e))\, \mu(de) \tag{3.41}$$

is given as follows.

$$\begin{aligned} &\text{If} \quad \alpha > 0 \quad \text{let } E_1(\alpha) \text{ be the event} \quad W(\theta) > \alpha\, c; \\ &\qquad\qquad\quad\ \text{let } E_2(\alpha) \text{ be the event} \quad W(\theta) < \alpha\, c; \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{let } E_3(\alpha) \text{ be the event} \quad W(\theta) = \alpha\, c. \end{aligned} \tag{3.42}$$

Choose $\alpha = \alpha_0$ to satisfy

$$\mu(E_1(\alpha_0)) \leqq \Delta, \qquad \lim_{\alpha \to \alpha_0} \mu(E_1(\alpha)) \geqq \Delta. \tag{3.43}$$

Let $\beta \geqq 0$ satisfy

$$\Delta = \mu(E_1(\alpha_0)) + \beta\mu(E_3(\alpha_0)). \tag{3.44}$$

Define $\phi^*(\cdot)$ by

$$\begin{aligned} \phi^*(e) &= 1, & e &\varepsilon E_1(\alpha_0); \\ \phi^*(e) &= \beta, & e &\varepsilon E_3(\alpha_0); \\ \phi^*(e) &= 0, & e &\varepsilon E_2(\alpha_0). \end{aligned} \tag{3.45}$$

It follows from (3.37), (3.38), (3.41) and (3.45) that

$$\begin{aligned} \sup_{\mu(B) \leqq \Delta} \int_B \sum_{j=0}^{\infty} \psi(n, j, \epsilon, e)\, W(\theta(e))\, \mu(de) \\ \leqq \int_{\{e \mid W(\theta(e)) \geqq \alpha_0 c\}} W(\theta(e))\, \mu(de). \end{aligned} \tag{3.46}$$

From (3.46) it follows that (3.34) must hold.

Having established uniform integrability the proof is completed by showing

$$\lim_{n \to \infty} \sum_{j=0}^{\infty} \psi(n, j, \epsilon, \cdot) W(\delta_{n+j}^* - \theta) = 0 \quad \text{a.e. } \mu(\cdot). \tag{3.47}$$

But

$$\sum_{j=0}^{\infty} \psi(n, j, \epsilon, \cdot) W(\delta_{n+j}^* - \theta) \leqq \sup_{j \geqq 0} W(\delta_{n+j}^* - \theta). \tag{3.48}$$

Since $W(\cdot)$ is assumed to be continuous at zero, (3.47) follows from (3.48) and (3.25). That completes the proof of Theorem 2.

4. Nonuniqueness and inadmissibility. In this section we will continue using the notation of Section 2 with one modification. x will be a real variable, y an $n - 1$ dimensional vector variable, $\mu(\cdot)$ a probability measure on the Borel sets of E_{n-1}. We will discuss the family of density functions

$$f(\cdot, \cdot, \theta) = f(\cdot - \theta, \cdot), \qquad -\infty < \theta < \infty. \tag{4.1}$$

Let D_I be the family of estimators of the form $x + \delta(y)$. In the terminology of the introduction $x + \delta(y)$ is a translation invariant estimator. The risk of such an estimator is

$$\begin{aligned} \iint W(x + \delta(y) - \theta) f(x - \theta, y)\, dx\, \mu(dy) \\ = \iint W(x + \delta(y)) f(x, y)\, dx\, \mu(dy). \end{aligned} \tag{4.2}$$

It follows that an element of D_I is minimax in D_I if and only if the element is a generalized Bayes solution for the weight function $g(\theta) = 1, -\infty < \theta < \infty$.

Therefore if $x + \delta(y)$ is minimax in D_I,

$$\text{for almost all } y, \quad \int W(\delta(y) + \theta) f(\theta, y)\, d\theta \tag{4.3}$$
$$= \inf_c \int W(c + \theta) f(\theta, y)\, d\theta.$$

From this it follows at once that if $x + \delta_1(y)$ and $x + \delta_2(y)$ are two estimators, each minimax in D_I then the estimators

$$x + \min(\delta_1(y), \delta_2(y)) \quad \text{and} \quad x + \max(\delta_1(y), \delta_2(y)) \tag{4.4}$$

are minimax in D_I.

We now state and prove the following theorem. We assume throughout the sequel D_I has at least one estimator having finite risk.

THEOREM 3. *Suppose $W(\cdot)$ satisfies*

$$\text{if} \quad 0 \leqq x < y,\ W(x) < W(y); \qquad \text{if} \quad y < x \leqq 0,\ W(x) < W(y). \tag{4.5}$$

If $x + \delta_1(y)$ and $x + \delta_2(y)$ are two estimators each minimax in D_I define

$$A = \{y / \delta_1(y) \neq \delta_2(y)\}; \tag{4.6}$$

$$\text{let } \eta \text{ be any real number}; \tag{4.7}$$

$$\begin{aligned} \delta(x, y) &= x + \max(\delta_1(y), \delta_2(y)), && x \leqq \eta, \\ \delta(x, y) &= x + \min(\delta_1(y), \delta_2(y)), && x > \eta. \end{aligned} \tag{4.8}$$

If

$$\int\int_A f(x, y)\, dx\, \mu(dy) > 0 \tag{4.9}$$

then no estimator in D_I is admissible. The estimator (4.8) *is better than every element of D_I.*

PROOF. In view of (4.4) we may assume without loss of generality that $\delta_1(y) \leqq \delta_2(y)$ for all y. By construction of $\delta_1(\cdot)$ and $\delta_2(\cdot)$, for almost all $y(\mu)$ such that

$$\int_{-\infty}^{\infty} f(x, y)\, dx < \infty, \tag{4.10}$$

$x + \delta_1(y)$ and $x + \delta_2(y)$ are minimax within the class of invariant estimators for the family

$$\{f(\cdot - \theta, y),\ -\infty < \theta < \infty\} \tag{4.11}$$

of density functions. Consequently for almost all $y(\mu)$,

$$R(y) = \int_{-\infty}^{\infty} W(x + \delta_1(y)) f(x, y)\, dx = \int_{-\infty}^{\infty} W(x + \delta_2(y)) f(x, y)\, dx. \tag{4.12}$$

We now calculate

$$R(y, \theta) = \int_{-\infty}^{\infty} W(\delta(x, y) - \theta)f(x - \theta, y)\, dx. \tag{4.13}$$

We have

$$\begin{aligned} R(y, \theta) &= \int_{-\infty}^{\eta} W(x + \delta_2(y) - \theta)f(x - \theta, y)\, dx \\ &\quad + \int_{\eta}^{\infty} W(x + \delta_1(y) - \theta)f(x - \theta, y)\, dx \\ &= R(y) + \int_{-\infty}^{\eta} (W(x + \delta_2(y) - \theta) - W(x + \delta_1(y) - \theta))f(x - \theta, y)\, dx \\ &= R(y) + \int_{-\infty}^{\eta+\theta} (W(x + \delta_2(y)) - W(x + \delta_1(y))f(x, y)\, dx. \end{aligned} \tag{4.14}$$

We show $W(x + \delta_2(y)) - W(x + \delta_1(y))$ as a function of x can change sign at most once. If $x \leqq -\delta_2(y) \leqq -\delta_1(y)$ then $x + \delta_1(y) \leqq x + \delta_2(y) \leqq 0$. From (4.5) it follows that $W(x + \delta_2(y)) - W(x + \delta_1(y)) \leqq 0$. If $-\delta_2(y) \leqq -\delta_1(y) \leqq x$ then $0 \leqq x + \delta_1(y) \leqq x + \delta_2(y)$. From (4.5) it follows that $W(x + \delta_2(y)) - W(x + \delta_1(y)) \geqq 0$. If $-\delta_2(y) \leqq x \leqq -\delta_1(y)$ then $x + \delta_1(y) \leqq 0 \leqq x + \delta_2(y)$. Therefore in this interval $-W(x + \delta_1(y))$ is a nondecreasing function of x and $W(x + \delta_2(y))$ is a nondecreasing function x. These statements taken together show

$$F(x, y) = W(x + \delta_2(y)) - W(x + \delta_1(y)) \tag{4.15}$$

can change sign at most once, $-\infty < x < \infty$;

$$\lim_{x\to-\infty} F(x, y) \leqq 0; \qquad \lim_{x\to+\infty} F(x, y) \geqq 0.$$

Since

$$\int_{-\infty}^{\infty} F(x, y)f(x, y)\, dx = 0 \tag{4.16}$$

it follows that

$$\int_{-\infty}^{\eta+\theta} F(x, y)f(x, y)\, dx \leqq 0, \qquad -\infty < \theta < \infty. \tag{4.17}$$

If $\delta_1(y) < \delta_2(y)$ and given (4.5) the above argument shows that $F(x, y) = 0$ for at most one x. Therefore

$$\text{if } \delta_1(y) < \delta_2(y), \quad -\infty < \theta < \infty, \qquad \int_{-\infty}^{\eta+\theta} F(x, y)f(x, y)\, dx < 0; \tag{4.18}$$

$$\text{if} \quad \delta_1(y) < \delta_2(y), \quad -\infty < \theta < \infty; \qquad R(y, \theta) < R(y). \tag{4.19}$$

On the hypothesis (4.9),

$$\int R(y, \theta)\, \mu(dy) < \int R(y)\, \mu(dy). \tag{4.20}$$

Theorem 3 now follows.

We consider in the following corollary the situation if there is a nontrivial randomized invariant estimator which is minimax within the class of all randomized invariant estimators. A randomized estimator is a family of probability measures $\lambda_{x,y}(\cdot)$ defined on the real Borel sets, one measure for each (x, y). The property of invariance says

$$\text{for all Borel sets } A, \quad \text{all real } x, \quad \lambda_{x,y}(A) = \lambda_{0,y}(A - x). \tag{4.21}$$

Corollary 3.1. *Assume the hypotheses of Theorem 3 and in addition*

$$W(-\infty) = W(\infty). \tag{4.22}$$

Suppose $\lambda_{x,y}(\cdot)$ is a randomized invariant estimator minimax within the class of randomized invariant estimators. Suppose on a set of y's having positive μ measure the support of $\lambda_{0,y}(\cdot)$ contains more than one point. Then no randomized invariant estimator is admissible.

Proof. The risk function of $\lambda_{x,y}(\cdot)$ is

$$\begin{aligned}\iiint W(x + z - \theta)\lambda_{0,y}(dz)\, f(x - \theta, y)\, dx\, \mu(dy) \\ = \iiint W(x + z) f(x, y)\, dx\, \lambda_{0,y}(dz)\, \mu(dy).\end{aligned} \tag{4.23}$$

Therefore if $\lambda_{x,y}(\cdot)$ is minimax within the class of invariant estimators then for almost all y (μ) for almost all z in the support of $\lambda_{0,y}(\cdot)$,

$$\int W(x + z) f(x, y)\, dx = \inf_c \int W(x + c) f(x, y)\, dx. \tag{4.24}$$

For by Theorem 1 of Section 2 and using (4.22) there is an almost everywhere finite valued estimator $x + \delta(y)$ satisfying, for almost all $y(\mu)$,

$$\int W(x + \delta(y)) f(x, y)\, dx = \inf_c \int W(x + c) f(x, y)\, dx. \tag{4.25}$$

Since $\int W(x + z) f(x, y) dx$ is a continuous function of z, and since the support of $\lambda_{0,y}(\cdot)$ is compact ($\pm\infty$ included) it follows that for almost all $y(\mu)$, for all z in the support of $\lambda_{0,y}(\cdot)$, (4.24) holds. This implies, using (4.22), that for almost all $y(\mu)$ the support of $\lambda_{0,y}(\cdot)$ is bounded.

Again by Theorem 1 of Section 2 and using (4.22) there is a minimal solution $x + \delta_1(y)$ and a maximal solution $x + \delta_2(y)$ each of which are solutions of (4.25) and for almost all $y(\mu)$ each estimator is finite valued. Consequently if on a set of y's of positive μ measure the support of $\lambda_{0,y}(\cdot)$ contains two points then on a set of y's of positive μ measure, $\delta_1(y) < \delta_2(y)$. The Corollary now follows from Theorem 3.

COROLLARY 3.2. *If* $W(x) = |x|$, $-\infty < x < \infty$, *and if* $f(\cdot)$ *is a density function such that for numbers* $c_1 < c_2$,

$$\int_{-\infty}^{c_1} f(x)\,dx = \int_{-\infty}^{c_2} f(x)\,dx = 1/2, \tag{4.26}$$

then every invariant estimator based on a single observation is inadmissible.

(See the example at the end of Section 2.)

The following theorem is the final result of this section.

THEOREM 4. *Suppose* $W(\cdot)$ *is a continuous and even function satisfying* (4.5). *If* $W(\cdot)$ *is not strictly convex there exists a density function* $f(\cdot)$ *which is an even function having compact support and a real number* $\epsilon \neq 0$ *such that*

$$\int_{-\infty}^{\infty} W(x+\epsilon)f(x)\,dx = \inf_c \int_{-\infty}^{\infty} W(x+c)f(x)\,dx. \tag{4.27}$$

From Theorem 4 it follows for the constructed density function that no invariant estimator can be admissible (single observation).

To prove Theorem 4 we first examine the case $W(\cdot)$ is not convex. There must exist numbers x_1^*, x_2^*, x_0 satisfying $0 \leqq x_1^* < x_0 < x_2^*$ and such that if $0 < \alpha_0 < 1$, $x_0 = \alpha_0 x_1^* + (1-\alpha_0)x_2^*$ then

$$W(\alpha_0 x_1^* + (1-\alpha_0)x_2^*) - \alpha_0 W(x_1^*) - (1-\alpha_0)W(x_2^*) > 0. \tag{4.28}$$

Define a function $F(\cdot)$ by

$$F(\alpha) = W(\alpha x_1^* + (1-\alpha)x_2^*) - \alpha W(x_1^*) - (1-\alpha)W(x_2^*). \tag{4.29}$$

Then $F(\cdot)$ is a continuous function and is positive at $\alpha = \alpha_0$. Define numbers α_1, α_2 as follows.

$$\begin{aligned} \alpha_1 &= \sup\{\alpha \mid \alpha<\alpha_0,\ \alpha \geqq 0,\ F(\alpha) = 0\}; \\ \alpha_2 &= \inf\{\alpha \mid \alpha > \alpha_0,\ \alpha \leqq 1,\ F(\alpha) = 0\}. \end{aligned} \tag{4.30}$$

Define numbers x_1, x_2 as follows.

$$\alpha_1 x_1^* + (1-\alpha_1)x_2^* = x_1; \qquad \alpha_2 x_1^* + (1-\alpha_2)x_2^* = x_2. \tag{4.31}$$

Then

$$\text{if} \quad 0 < \alpha < 1, \qquad W(\alpha x_1 + (1-\alpha)x_2) - \alpha W(x_1) - (1-\alpha)W(x_2) > 0. \tag{4.32}$$

Define numbers ϵ^*, x_3 by

$$\epsilon^* = (x_2 - x_1)/2; \qquad x_3 = (x_1 + x_2)/2. \tag{4.33}$$

Suppose $f_\beta(\cdot)$ is a density function satisfying

$$\text{if } -\infty < x < \infty, \quad f_\beta(x) = f_\beta(-x); \qquad 1/2 = \int_{x_3-\beta}^{x_3+\beta} f_\beta(x)\,dx. \tag{4.34}$$

Then

$$\lim_{\beta\to 0} \int_{-\infty}^{\infty} W(x)f_\beta(x)\, dx = 1/2\, W(-x_3) + 1/2W(x_3)$$

$$(4.35)\quad \begin{aligned} &= W(x_3) = W(((x_3 + \epsilon^*) + (x_3 - \epsilon^*))/2) > (W(x_3 + \epsilon^*) \\ &\quad + W(x_3 - \epsilon^*))1/2 = (W(x_3 + \epsilon^*) + W(-x_3 + \epsilon^*))1/2 \\ &\qquad = \lim_{\beta\to 0} \int_{-\infty}^{\infty} W(x + \epsilon^*)f_\beta(x)\, dx. \end{aligned}$$

Therefore we may find $\beta > 0$ so small that

$$(4.36)\quad \int_{-\infty}^{\infty} W(x + \epsilon^*)f_\beta(x)\, dx < \int_{-\infty}^{\infty} W(x)f_\beta(x)\, dx \leqq \lim_{x\to\infty} W(x).$$

By continuity $\int_{-\infty}^{\infty} W(x + c)f_\beta(x)dx$ assumes its minimum for some finite value of $c \neq 0$. That completes the proof of Theorem 4 in the case $W(\cdot)$ is not a convex function.

In the case $W(\cdot)$ is a convex function but is not strictly convex there must exist numbers x_1, x_2 such that $0 < x_1 < x_2$ satisfying

$$(4.37)\quad \text{if}\quad 0 \leqq \alpha \leqq 1 \quad\text{then}\quad W(\alpha x_1 + (1 - \alpha)x_2) = \alpha W(x_1) + (1 - \alpha)W(x_2).$$

Then

$$(4.38)\quad \text{if}\quad x_1 < x < x_2, \quad\text{the derivative}\quad W'(x) = (W(x_2) - W(x_1))/(x_2 - x_1).$$

Since $W(\cdot)$ is an even function,

$$(4.39)\quad \text{if}\quad -x_2 < -x < -x_1 \quad\text{then}\quad W'(-x) = -W'(x).$$

Let $x_0 = (x_1 + x_2)/2$ and $\beta > 0$ be so small that

$$(4.40)\quad x_1 < x_0 - \beta < x_0 + \beta < x_2 .$$

Let $f_\beta(\cdot)$ be an even density function satisfying (4.34). If $\epsilon > 0$ satisfies

$$(4.41)\quad x_1 < x_0 - \beta - \epsilon < x_0 + \beta + \epsilon < x_2$$

then

$$(4.42)\quad 0 = \int_{-\infty}^{\infty} W'(x + \epsilon)f_\beta(x)\, dx.$$

Theorem 4 now follows by the corollary of Section 2.

5. Admissibility. In this section, in order to treat the case of estimators based on $n \geqq 1$ independent observations we continue using the notation of Section 4. We will suppose in this section and in Section 6 that $W(\cdot)$ satisfies (2.6) and the following assumptions.

(5.1) For all real numbers c, $\iint_{-\infty}^{\infty} W(x + c)f(x, y)\, dx\, \mu(dy) < \infty$.

Except on a set of μ measure zero, the function

(5.2) $\int_{-\infty}^{\infty} W(x + c)f(x, y)\, dx$ assumes its minimum at the single point $c = \delta_0(y)$.

(5.3) $W(\cdot)$ is bounded, or uniformly continuous, or convex, or satisfies $\sup_{x \leqq 0} W(x) < \infty$, $W(\cdot)$ is convex in $x > 0$.

There is a real number p, $1 < p \leqq 2$ such that

$\iint |x|^p\, W(x)f(x, y)\, dx\, \mu(dy) < \infty$. If $W(\cdot)$ is convex,

(5.4) $$\iint |W'(d + x)|\, |x|^p f(x, y)\, dx\, \mu(dy) < \infty$$

for all real numbers d. If $W(\cdot)$ is partly bounded, partly convex then

$$\int \mu(dy) \int_0^{\infty} |W'(d + x)| \cdot |x|^p f(x, y)\, dx < \infty \quad \text{for all} \quad d \geqq 0.$$

In this section and in Section 6 we suppose a weight function $g(\cdot)$ is given which satisfies the following conditions.

(5.5) $g(\cdot)$ is a bounded function.

(5.6) $\alpha = \lim_{\theta \to -\infty} g(\theta)$ exists and $0 < \alpha$; $\beta = \lim_{\theta \to \infty} g(\theta)$ exists and $0 < \beta$.

Except on a set of μ measure zero the function

(5.7) $\int_{-\infty}^{\infty} W(c - x + \theta)f(\theta, y)g(x - \theta)\, d\theta$ assumes its minimum value at a single finite point $c = \delta(x, y)$.

We will define here almost admissibility. An estimator $\delta^*(\cdot, \cdot)$ will be called almost admissible relative to the weight function $g(\cdot)$ if given any other estimate $\eta(\cdot, \cdot)$ as good as $\delta^*(\cdot, \cdot)$ then

(5.8) $$\int_{-\infty}^{\infty} (R(\delta^*, \theta) - R(\eta, \theta))g(\theta)\, d\theta = 0.$$

The main result of this section is as follows.

THEOREM 5. *Assume* (5.1) *to* (5.7). *If* $W(\cdot)$ *is bounded then* $\delta(\cdot, \cdot)$ *is almost admissible. If* $\mu(\cdot)$ *puts total mass on a single point then* $\delta(\cdot, \cdot)$ *is almost admissible.*

If $W(\cdot)$ *is not bounded and the support of* $\mu(\cdot)$ *contains more than one point we* *will suppose* $f(\cdot, \cdot)$ *vanishes off a compact set of Euclidian n-space. Then* $\delta(\cdot, \cdot)$ *is almost admissible.*

The methods we use here do not give information about almost admissibility of $\delta(\cdot, \cdot)$ if the weight function $g(\cdot)$ is unbounded. In Section 8 we prove a different admissibility theorem where $W(x) = x^2$ for all x. The restriction on $g(\cdot)$ used in Section 8 is that there exist numbers $k > 0$, $0 < \beta < 1$ such that for all x, $g(x) \leqq k(1 + |x|^\beta)$. This shows that at least for some loss functions some unbounded weight functions $g(\cdot)$ give rise to almost admissible estimators. That it will not be possible to prove admissibility for arbitrary weight functions is shown by the example

$$g(\theta) = e^\theta, \qquad W(x) = W(-x) \quad \text{for all } x. \tag{5.9}$$

Then if $\delta(\cdot, \cdot)$ is a generalized Bayes solution relative to $g(\cdot)$,

$$\begin{aligned} &\int_{-\infty}^{\infty} W(\delta(x, y) - \theta)f(x - \theta, y)e^\theta \, d\theta \\ &\quad = \int_{-\infty}^{\infty} W(\delta(x, y) - x + \theta)f(\theta, y)e^{x-\theta} \, d\theta \\ &\quad = e^x \int_{-\infty}^{\infty} W(\delta(x, y) - x - \theta)f(-\theta, y)e^\theta \, d\theta. \end{aligned} \tag{5.10}$$

From the definition of $\delta(\cdot, \cdot)$, for almost all y,

$$x - \delta(x, y) = \delta(0, y), \qquad -\infty < x < \infty. \tag{5.11}$$

Let $\delta_0(\cdot)$ be defined as in (5.2).

In view of Theorem 3, Section 4, since $\delta(\cdot, \cdot)$ defined by (5.11) is an invariant estimator, $\delta(\cdot, \cdot)$ is inadmissible unless $\delta(0, y) = \delta_0(y)$ almost everywhere (μ), where $x + \delta_0(y)$ is the generalized Bayes solution relative to the weight function $g(\theta) \equiv 1$. If the loss $W(\cdot)$ is strictly convex then

$$0 = \int_{-\infty}^{\infty} W'(x + \delta_0(y) - \theta)f(x - \theta, y) \, d\theta. \tag{5.12}$$

It is clear that for almost all $y(\mu)$,

$$0 < \int_{-\infty}^{\infty} W'(x - \delta_0(y) - \theta)f(x - \theta, y)e^\theta \, d\theta. \tag{5.13}$$

Therefore for almost all $y(\mu)$, $\delta(0, y) > \delta_0(y)$, and $\delta(\cdot, \cdot)$ cannot be admissible.

Passing from the conclusion of almost admissibility to the conclusion of admissibility requires additional arguments. Define a set K_g by

$$K_g = \left\{\theta \mid \text{all } \epsilon > 0, \int_{\theta-\epsilon}^{\theta+\epsilon} g(x) \, dx > 0\right\}. \tag{5.14}$$

The set K_g is called the support of $g(\cdot)$ and has the property that the intersection of K_g with any bounded closed interval is a compact set.

One would like to conclude from (5.8) that

$$\text{(5.15)} \qquad \text{if} \quad \theta \,\varepsilon\, K_g \quad \text{then} \quad R(\eta, \theta) = R(\delta^*, \theta).$$

In the case $W(\cdot)$ is bounded it follows that $R(\eta, \cdot)$ is a continuous function. (5.15) then follows. In case $W(\cdot)$ is strictly convex (5.15) may be proven as follows. Let

$$\text{(5.16)} \qquad A = \{(x, y) \mid \eta(x, y) \neq \delta^*(x, y)\}.$$

The function $\int\int_A f(x - \theta, y)dx\mu(dy)$ is a continuous function of θ. If $\theta \,\varepsilon\, K_g$ and $\int\int_A f(x - \theta, y)dx\mu(dy) > 0$ then the same holds on a set in K_g having positive Lebesgue measure. Define a new estimator $\eta'(\cdot, \cdot)$ by

$$\text{(5.17)} \qquad \eta'(x, y) = (1/2)(\eta(x, y) + \delta^*(x, y)) \quad \text{all } (x, y).$$

From the strict convexity of $W(\cdot)$ it follows that

$$\text{(5.18)} \qquad \text{if} \quad \theta \,\varepsilon\, K_g, \iint_A f(x - \theta, y)\, dx\, \mu(dy) > 0 \quad \text{then}$$
$$R(\eta', \theta) < 1/2(R(\eta, \theta) + R(\delta^*, \theta)) \leqq R(\delta^*, \theta).$$

Therefore

$$\text{(5.18)} \qquad \text{if for some} \quad \theta_0 \,\varepsilon\, K_g, \quad \iint_A f(x - \theta_0, y) > 0, \quad \text{then}$$
$$\int_{-\infty}^{\infty} (R(\delta^*, \theta) - R(\eta', \theta))g(\theta) > 0.$$

This contradicts the almost admissibility of $\delta^*(\cdot, \cdot)$.

In the remaining cases covered by Theorem 5 we do not have results about whether almost admissibility implies admissibility. This question has been settled affirmatively by Fox and Rubin [4] for the special type of loss function they consider and for the weight function $g(\theta) \equiv 1$.

To prove Theorem 5 we construct a sequence $\{\delta_n(\cdot, \cdot), n \geqq 1\}$ of nonrandomized Bayes estimators. Define $q(\cdot)$, $\{q_n(\cdot), n \geqq 1\}$ by

$$\text{(5.20)} \qquad \text{if } p \text{ is the number in (5.4)}, \qquad q(\theta) = 1/(1 + |\theta|^p),$$
$$-\infty < \theta < \infty, \quad \text{and, if} \quad n \geqq 1, \quad q_n(\theta) = q(\theta/n).$$

If $n \geqq 1$, $\delta_n(\cdot, \cdot)$ is defined to be the maximal Bayes solution for the weight function $g(\cdot)q_n(\cdot)$. From Theorem 1 of Section 2 the existence of a nonrandomized maximal solution $\delta_n(\cdot, \cdot)$ is guaranteed. The subsequent arguments of Section 6 will show that for large n, $\delta_n(\cdot, \cdot)$ is finite valued. Suppose $\eta(\cdot, \cdot)$ is an estimator which is as good as $\delta(\cdot, \cdot)$. Then

$$\text{(5.21)} \qquad \int_{-\infty}^{\infty} R(\delta_n, \theta)g(\theta)q_n(\theta)\, d\theta \leqq \int_{-\infty}^{\infty} R(\eta, \theta)g(\theta)q_n(\theta)\, d\theta$$
$$\leqq \int_{-\infty}^{\infty} R(\delta. \theta)g(\theta)g_n(\theta)\, d\theta.$$

Suppose it can be shown that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (R(\delta, \theta) - R(\delta_n, \theta))g(\theta)q_n(\theta)\, d\theta = 0. \tag{5.22}$$

Since

$$\text{if} \quad -\infty < \theta < \infty, \quad R(\delta, \theta) - R(\eta, \theta) \geqq 0, \tag{5.23}$$

and since $\lim_{n\to\infty} q_n(\theta) = 1$ for all θ, using (5.21), (5.22), (5.23) and Fatou's lemma

$$\int_{-\infty}^{\infty} (R(\delta, \theta) - R(\eta, \theta))g(\theta)\, d\theta = 0 \tag{5.24}$$

follows.

In Section 6 we prove the following results.

$$\text{For almost all } y(\mu), \quad \sup_x |\delta(x, y) - x - \delta_0(y)| = c(y) < \infty. \tag{5.25}$$

$$\text{For almost all } y(\mu), \quad \lim_{n\to\infty} \sup_x |\delta(x, y) - \delta_n(x, y)| = 0. \tag{5.26}$$

Define constants K_1 and K_2 as follows.

$$K_1 = \sup_\theta g(\theta). \tag{5.27}$$

$$K_2 = \sup\{|x| \mid \text{some } y, f(x, y) > 0\}. \tag{5.28}$$

The definitions of $\delta_0(\cdot)$ and $\delta(\cdot, \cdot)$ in (5.2) and (5.7) together with (2.6) imply that

$$\text{for almost all } y, \quad |\delta_0(y)| \leqq K_2\,; \quad \sup_x |\delta(x, y) - x| \leqq K_2\,; \quad \sup_x |\delta_n(x, y) - x| \leqq K_2, \quad n \geqq 1. \tag{5.29}$$

The conditions of (5.29) will be of interest to us in the case $W(\cdot)$ is unbounded. The hypotheses of Theorem 5 then imply that if the support of $\mu(\cdot)$ has at least two points then $K_2 < \infty$. Consequently in the case $W(\cdot)$ is unbounded the hypotheses of Theorem 5, (5.25), (5.26) and (5.29) imply there is a constant $K_3 \geqq K_2$ such that

$$\text{for almost all } y(\mu), \quad \sup_x |\delta(x, y) - x - \delta_0(y)| \leqq K_3\,; \quad \sup_x |\delta_n(x, y) - x - \delta_0(y)| \leqq K_3, \quad n \geqq 1. \tag{5.30}$$

Since from (5.30) follows that if $W(\cdot)$ is unbounded

$$W(\delta(x, y) - \theta) \leqq W(K_3 + x + \delta_0(y) - \theta) + W(-K_3 + x + \delta_0(y) - \theta), \tag{5.31}$$

from (5.1) it then follows that in all cases covered by Theorem 5,

$$\sup_\theta R(\delta, \theta) < \infty. \tag{5.32}$$

Then

$$(5.33)\quad \iiint |W(\delta(x,y)-\theta) - W(\delta_n(x,y)-\theta)| f(x-\theta, y) \cdot g(\theta) q_n(\theta)\, dx\, \mu(dy)\, d\theta < \infty.$$

By Fubini's theorem,

$$(5.34)\quad \begin{aligned} R_n &= \int (R(\delta,\theta) - R(\delta_n, \theta)) g(\theta) q_n(\theta)\, d\theta \\ &= \iiint (W(\delta(x,y)-\theta) - W(\delta_n(x,y)-\theta)) f(x-\theta, y) \\ &\qquad \cdot g(\theta) q_n(\theta)\, d\theta\, dx\, \mu(dy). \end{aligned}$$

By definition of $\delta(\cdot, \cdot)$, for almost all $y(\mu)$,

$$(5.35)\quad -\infty < \int (W(\delta(x,y)-\theta) - W(\delta_n(x,y)-\theta)) f(x-\theta)\, g(\theta)\, d\theta \leqq 0.$$

It follows that

$$(5.36)\quad \begin{aligned} R_n &\leqq \iiint (W(\delta(x,y)-\theta) - W(\delta_n(x,y)-\theta))\, f(x-\theta, y) \\ &\qquad \cdot (q_n(\theta) - q_n(x))\, g(\theta)\, d\theta\, dx\, \mu(dy) \\ &\leqq K_1 \iiint |W(\delta(x,y)-\theta) - W(\delta_n(x,y)-\theta)| f(x-\theta, y) \\ &\qquad \cdot |q_n(\theta) - q_n(x)|\, d\theta\, dx\, \mu(dy) \\ &= K_1 \iiint |W(\delta(x,y)-x+\theta) - W(\delta_n(x,y)-x+\theta)| f(\theta, y) \\ &\qquad \cdot |q_n(x-\theta) - q_n(x)|\, dx\, d\theta\, \mu(dy). \end{aligned}$$

We use the inequalities proven in Lemma 8.1 of Section 8.

$$(5.37)\quad \begin{aligned} |q_n(x-\theta) - q_n(x)| &= (1/n)^p q_n(x-\theta) q_n(x)\, \big|\, |x|^p - |x-\theta|^p \big| \\ &\leqq 2(1/n)^p q_n(x-\theta) q_n(x) (|\theta|^p + |\theta|\, |x|^{p-1}). \end{aligned}$$

Since $\sup_x (1+|x|^{p-1})/(1+|x|^p) \leqq 2$ and since $0 < p-1 \leqq 1$, so that

$$(5.38)\quad \begin{aligned} |x/n|^{p-1} &\leqq |(x-\theta)/n|^{p-1} + |\theta/n|^{p-1} \\ &\leqq (1 + |(x-\theta)/n|^{p-1})(1+|\theta|^{p-1}), \end{aligned}$$

it follows that

$$(5.39)\quad q_n(x-\theta)|x/n|^{p-1} \leqq 2(1+|\theta|^{p-1}).$$

Therefore

$$(5.40)\quad |q_n(x-\theta) - q_n(x)| \leqq 6(1/\mathrm{n}) q_n(x) |\theta| (1+|\theta|^{p-1}).$$

Then

$$(5.41)\quad \begin{aligned} R_n &\leqq 6K_1(1/n) \iiint |\, W(\delta(x, y) - x + \theta) - W(\delta_n(x, y) - x + \theta)| \\ &\qquad \cdot f(\theta, y)\, |\theta|\, (1 + |\theta|^{p-1})\, q_n(x)\, dx\, d\theta\, \mu(dy) \\ &= 6K_1 \iiint |W(\delta(nx, y) - nx + \theta) - W(\delta_n(nx, y) - nx + \theta)|\, f(\theta, y) \\ &\qquad \cdot |\theta|\, (1 + |\theta|^{p-1})\, q(x)\, dx\, d\theta\, \mu(dy). \end{aligned}$$

CASE I. $W(\cdot)$ *is bounded*. By virtue of (2.6), $W(\cdot)$ has at most a countable number of discontinuities. Therefore, for each (x, y)

$$(5.42)\quad \lim_{n\to\infty} |W(\delta(nx, y) - nx + \theta) - W(\delta_n(nx, y) - nx + \theta)| = 0 \quad \text{a.e., Lebesgue measure.}$$

Therefore by Fubini's theorem, (5.42) holds for almost all (θ, x, y). By the bounded convergence theorem,

$$(5.43)\quad \lim_{n\to\infty} R_n = 0$$

now follows.

CASE II. $W(\cdot)$ *is uniformly continuous*. Define for $\epsilon > 0$

$$(5.44)\quad \Delta(\epsilon) = \sup \{|W(x) - W(y)| \mid |x - y| \leqq \epsilon\}.$$

Then by (5.30)

$$(5.45)\quad \begin{aligned} &|W(\delta(nx, y) - nx + \theta) - W(\delta_n(nx, y) - nx + \theta)| \\ &\qquad \leqq \Delta(|\delta(nx, y) - \delta_n(nx, y)|) \leqq \Delta(2K_3). \end{aligned}$$

By (5.26) and the bounded convergence theorem $\lim_{n\to\infty} R_n = 0$ now follows.

CASE III. $W(\cdot)$ *is convex*. In this case, using (5.30) and (5.29),

$$(5.46)\quad \begin{aligned} &|W(\delta(nx, y) - nx + \theta) - W(\delta_n(nx, y) - nx + \theta)| \\ &\qquad \leqq |\delta(nx, y) - \delta_n(nx, y)|(|W'(K_3 + \delta_0(y) + \theta)| \\ &\qquad\qquad + |W'(-K_3 + \delta_0(y) + \theta)|) \\ &\qquad \leqq |\delta(nx, y) - \delta_n(nx, y)| \\ &\qquad\qquad \cdot (|W'(K_2 + K_3 + \theta)| + |W'(-K_2 - K_3 + \theta)|). \end{aligned}$$

Using (5.26), (5.40), (5.29) and the bounded convergence theorem, $\lim_{n\to\infty} R_n = 0$ follows.

CASE IV. $W(\cdot)$ *is a convex function on* $[0, \infty)$, $\sup_{x<0} W(\cdot) < \infty$. In this case, if $\theta \leqq 2K_3$ it is clear from (5.26) and (5.29), (5.30) that

$$(5.47)\quad \begin{aligned} 0 = \lim_{n\to\infty} \sup_y \int_{-\infty}^{\infty} &|W(\delta(nx, y) - nx + \theta) \\ &- W(\delta_n(nx, y) - nx + \theta)\,|\, q(x)dx. \end{aligned}$$

Therefore using the bounded convergence theorem

$$\begin{aligned}(5.48)\qquad 0 = \lim_{n\to\infty} \int \mu(dy) \int_{-\infty}^{2K_3} d\theta\, |\theta|\, (1 + |\theta|^{p-1})\, f(\theta, y) \int_{-\infty}^{\infty} dx \\ \cdot\, |W(\delta(nx, y) - nx + \theta) - W(\delta(nx, y) - nx + \theta)|\, q(x).\end{aligned}$$

If $\theta \geqq 2K_3$ then for almost all $y(\mu)$,

$$(5.49)\qquad \delta(nx, y) - nx + \theta \geqq 0 \quad \text{and} \quad \delta_n(nx, y) - nx + \theta \geqq 0.$$

Using the convexity of $W(\cdot)$ on $[0, \infty)$, and an argument similar to Case III, (5.46) may be established if $\theta \geqq 2K_3$. Using the bounded convergence theorem

$$\begin{aligned}(5.50)\qquad 0 = \lim_{n\to\infty} \int \mu(dy) \int_{2K_3}^{\infty} d\theta\, |\theta|\, (1 + |\theta|^{p-1})\, f(\theta, y) \int_{-\infty}^{\infty} dx \\ |W(\delta(nx, y) - nx + \theta) - W(\delta_n(nx, y) - nx + \theta)|\, q(x)\end{aligned}$$

provided

$$(5.51)\qquad \int_0^{\infty} |\theta|^p\, |W'(2K_3 + \theta)|\, f(\theta, y)\, d\theta\, \mu(dy) < \infty.$$

As (5.51) is covered by the assumptions made, the proof is complete.

6. Convergence lemmas. In order to complete the discussion of Section 5 we verify (5.25) and (5.26). The proofs of these results are given in a series of lemmas below. Lemmas 6.1 to 6.4 lead to the proof of Lemma 6.5 from which (5.25) follows. Lemma 6.6 contains the result (5.26).

The definitions of $\delta_0(\cdot)$ and $\delta(\cdot, \cdot)$ in (5.2) and (5.7) imply the following relations.

(6.1) For almost all $y(\mu)$, the function $\int_{-\infty}^{\infty} W(c + x) f(x - \delta_0(y), y)\, dx$ has its minimum value at a single point $c = 0$.

For almost all $y(\mu)$, the function

$$(6.2)\qquad \int_{-\infty}^{\infty} W(c - x + \theta) f(-\delta_0(y) + \theta, y) g(\delta_0(y) + x - \theta)\, d\theta$$

has its minimum value at a single point $c = \delta(x, y) - \delta_0(x)$.

Since for each y, the weight function $g_y^*(\cdot)$ defined by

$$(6.3)\qquad g_y^*(\theta) = g(\delta_0(y) + \theta)$$

satisfies (5.5) and (5.6), it is sufficient to consider $f(\cdot)$ as a function of a single real variable, and the estimator $\delta(\cdot, \cdot)$ as a function of a single real variable.

LEMMA 6.1. *Let $g(\cdot)$ be a bounded Borel measurable function of a real variable. Suppose $\{x_n, n \geqq 1\}$ is a real number sequence such that* $\lim_{n\to\infty} x_n = x$. *There is an integer sequence $\{a_n, n \geqq 1\}$ and a set N of Lebesgue measure zero such that if $\theta \not\in N$,*

$$\lim_{n\to\infty} g(x_{a_n} - \theta) = g(x - \theta). \tag{6.4}$$

PROOF. Define a sequence $\{g_n(\cdot), n \geqq 1\}$ of functions as follows.

$$\text{if} \quad |x| \leqq n, \quad g_n(x) = g(x); \qquad \text{if} \quad |x| > n, \quad g_n(x) = 0. \tag{6.5}$$

As is well known,

$$\text{if} \quad m \geqq 1, \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} |g_m(x_n - \theta) - g_m(x - \theta)|\, d\theta = 0. \tag{6.6}$$

We now use a diagonalization process. Let $\{a_{1,n}, n \geqq 1\}$ be an integer sequence, $N(1)$ a set of Lebesgue measure zero, such that

$$\text{if} \quad \theta \not\in N(1), \qquad \lim_{n\to\infty} g_1(x_{a_{1,n}} - \theta) = g_1(x - \theta). \tag{6.7}$$

If $m \geqq 1$, let $\{a_{m+1,n}, n \geqq 1\}$ be a subsequence of $\{a_{m,n}, n \geqq 1\}$ and $N(m+1)$ be a set of Lebesgue measure zero such that

$$\text{if} \quad \theta \not\in N(m+1), \qquad \lim_{n\to\infty} g_{m+1}(x_{a_{m+1,n}} - \theta) = g_{m+1}(x - \theta). \tag{6.8}$$

Define

$$N = \bigcup_{m=1}^{\infty} N(m), \quad \text{and if} \quad n \geqq 1, \qquad a_n = a_{n,n}. \tag{6.9}$$

It is easily verified that the set N and the sequence $\{a_n, n \geqq 1\}$ satisfy the conclusion of the lemma.

LEMMA 6.2. *Suppose*

$$\text{for all real numbers } c, \quad \int_{-\infty}^{\infty} W(c + x) f(x)\, dx < \infty. \tag{6.10}$$

If $g(\cdot)$ is a bounded non-negative Borel measurable function and if $F(\cdot)$ is defined by

$$F(x) = \inf_c \int_{-\infty}^{\infty} W(c - \theta) f(x - \theta) g(\theta)\, d\theta, \tag{6.11}$$

then $F(\cdot)$ is a continuous function.

PROOF. We will show first that

$$F(x) \leqq \liminf_{y\to x} F(y). \tag{6.12}$$

If this does not hold then there is an $\epsilon > 0$ and a sequence $\{x_n, n \geqq 1\}$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and if} \quad n \geqq 1, \quad F(x_n) + \epsilon \leqq F(x). \tag{6.13}$$

In view of Lemma 6.1 we may assume that if $\theta \not\in N$, $\lim_{n\to\infty} g(x_n - \theta) = g(x - \theta)$. Using (6.11) and (6.13) there is a sequence $\{c_n, n \geqq 1\}$ such that

$$\epsilon/2 + \int_{-\infty}^{\infty} W(c_n - x_n + \theta) f(\theta)\, g(x_n - \theta)\, d\theta \leqq F(x). \tag{6.14}$$

Without loss of generality we may suppose $\lim_{n\to\infty} c_n = c_0$, provided values $\pm\infty$ are allowed. By Fatou's lemma it follows that

$$\epsilon/2 + \int_{-\infty}^{\infty} W(c_0 - x + \theta) f(\theta)\, g(x - \theta)\, d\theta \leqq F(x). \tag{6.15}$$

Since $\int_{-\infty}^{\infty} W(c - x + \theta)f(\theta)g(x - \theta)\, d\theta$ is a continuous function of $c \in [-\infty, \infty]$, (6.15) contradicts the definition of $F(x)$.

We next show $\limsup_{y \to x} F(y) \leqq F(x)$. To show this we need to show that for each real number c, the function $H(\cdot)$ defined by

$$H(y) = \int_{-\infty}^{\infty} W(c - y + \theta) f(\theta)\, g(y - \theta)\, d\theta \tag{6.16}$$

is a continuous function. Suppose $\epsilon > 0$ and $\{y_n, n \geqq 1\}$ is a sequence satisfying

$$\lim_{n\to\infty} y_n = y, \qquad \liminf_{n\to\infty} |H(y_n) - H(y)| \geqq \epsilon. \tag{6.17}$$

Using Lemma 6.1 we may assume without loss of generality that if $\theta \not\in N$, $\lim_{n\to\infty} g(y_n - \theta) = g(y - \theta)$. Also we may assume $\sup_n |y_n - y| \leqq 1$. Then, if $K_1 = \sup_\theta g(\theta)$, it follows that

$$\begin{aligned} \text{if} \quad n \geqq 1, \qquad & W(c - y_n + \theta)g(y_n - \theta) \\ & \leqq K_1(W(c - y - 1 + \theta) + W(c - y + 1 + \theta)). \end{aligned} \tag{6.18}$$

As the right side of (6.18) is integrable, by the bounded convergence theorem

$$\lim_{n\to\infty} H(y_n) = H(y) \tag{6.19}$$

follows. This contradicts (6.17). Therefore no such sequence $\{y_n, n \geqq 1\}$ can exist and continuity of $H(\cdot)$ is proven. Let $\epsilon > 0$ be given. By definition of $F(\cdot)$ there is a number c such that

$$F(x) + \epsilon/2 > \int_{-\infty}^{\infty} W(c - x + \theta) f(\theta)\, g(x - \theta)\, d\theta. \tag{6.20}$$

Using the continuity of $H(\cdot)$, there is a number ϵ_1 such that

$$\begin{aligned} \text{if} \quad |x - y| < \epsilon_1 \quad \text{then} \qquad F(y) &\leqq \int_{-\infty}^{\infty} W(c - y + \theta) f(\theta)\, g(y - \theta)\, d\theta \\ &\leqq \epsilon/2 + \int_{-\infty}^{\infty} W(c - x + \theta) f(\theta)\, g(x - \theta)\, d\theta < \epsilon + F(x). \end{aligned} \tag{6.21}$$

Therefore

$$\text{if} \quad \epsilon > 0, \qquad \limsup_{y\to x} F(y) \leqq \epsilon + F(x). \tag{6.22}$$

(6.12) and (6.22) complete the proof of Lemma 6.2.

Lemma 6.3. *Suppose* (6.10) *holds and the weight function* $g(\cdot)$ *is bounded. Suppose the function* $\int_{-\infty}^{\infty} W(c - \theta)f(x - \theta)g(\theta)\, d\theta$ *has its minimum value at a single point* $c = \delta(x)$, *finite in value. Then* $\delta(\cdot)$ *is a continuous function.*

Proof. If $\delta(\cdot)$ is not continuous at x then we may find a sequence $\{x_n, n \geqq 1\}$ such that $\lim_{n\to\infty} x_n = x$, and if $\theta \not\in N$, $\lim_{n\to\infty} g(x_n - \theta) = g(x - \theta)$ (see Lemma 6.1), and $\lim_{n\to\infty} \delta(x_n) = c \neq \delta(x)$. $c = \pm\infty$ is a possible value for c. Define

$F(\cdot)$ by

$$\text{(6.23)}\quad \text{if}\quad -\infty < y < \infty, \qquad F(y) = \int_{-\infty}^{\infty} W(\delta(y) - \theta)\, f(y - \theta)\, g(\theta)\, d\theta.$$

By Lemma 6.2 $F(\cdot)$ is a continuous function. Therefore

$$\text{(6.24)}\qquad \lim_{n\to\infty} F(x_n) = F(x).$$

By Fatou's lemma,

$$\text{(6.25)}\qquad \begin{aligned} &\int_{-\infty}^{\infty} W(c - x + \theta)\, f(\theta)\, g(x - \theta)\, d\theta \\ &\quad \leqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} W(\delta(x_n) - x_n + \theta)\, f(\theta)\, g(x_n - \theta)\, d\theta \\ &\quad = \liminf_{n\to\infty} F(x_n) = F(x). \end{aligned}$$

It follows from the hypotheses of the lemma that $c = \delta(x)$. This contradiction shows $\delta(\cdot)$ must be continuous.

Lemma 6.4. *Suppose $g(\cdot)$ is a bounded non-negative Borel measurable function such that*

$$\text{(6.26)}\qquad \alpha = \lim_{\theta\to-\infty} g(\theta), \quad \alpha > 0; \qquad \beta = \lim_{\theta\to\infty} g(\theta), \quad \beta > 0.$$

Suppose (6.10) *holds. Suppose*

$$\text{(6.27)}\qquad \begin{aligned} &\textit{the function} \quad \int_{-\infty}^{\infty} W(c - \theta)\, f(x - \theta)\, g(\theta)\, d\theta \\ &\quad \textit{has a unique minimum at} \quad c = \delta(x), \quad \textit{a finite value.} \end{aligned}$$

Suppose

$$\text{(6.28)}\qquad \textit{the function} \quad \int_{-\infty}^{\infty} W(c - \theta)\, f(x - \theta)\, d\theta$$

has a unique minimum at $c = 0$. *Then* $\lim_{|x|\to\infty} (\delta(x) - x) = 0$.

Proof. By definition of $\delta(\cdot)$,

$$\text{(6.29)}\quad \int_{-\infty}^{\infty} W(\delta(x) - x + \theta)\, f(\theta)\, g(x - \theta)\, d\theta \leqq \int_{-\infty}^{\infty} W(\theta)\, f(\theta)\, g(x - \theta)\, d\theta.$$

We consider first the case $\lim_{x\to\infty} (\delta(x) - x) = 0$. By the bounded convergence theorem

$$\text{(6.30)}\qquad \lim_{x\to\infty} \int_{-\infty}^{\infty} W(\theta)\, f(\theta)\, g(x - \theta)\, d\theta = \beta \int_{-\infty}^{\infty} W(\theta)\, f(\theta)\, d\theta.$$

Suppose there is a sequence $\{x_n, n \geqq 1\}$ such that

$$\text{(6.31)}\qquad \lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} (\delta(x_n) - x_n) = c.$$

Values $\pm\infty$ are allowed for c.

By Fatou's lemma and (6.30)

$$\beta \int_{-\infty}^{\infty} W(c+\theta) f(\theta)\, d\theta \leqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} W(\delta(x_n) - x_n + \theta) \cdot f(\theta)\, g(x_n - \theta)\, d\theta \leqq \beta \int_{-\infty}^{\infty} W(\theta) f(\theta)\, d\theta. \tag{6.32}$$

Therefore since (6.10) is assumed, $c = 0$ follows. Consequently it is proven that $\lim_{x\to\infty} (\delta(x) - x) = 0$. By a similar argument $\lim_{x\to-\infty} (\delta(x) - x) = 0$. As a summary of preceding results,

LEMMA 6.5. *Suppose $g(\cdot)$ is a bounded non-negative Borel measurable function and that* (6.10), (6.26), (6.27) *and* (6.28) *hold. Then $\delta(\cdot)$ is a continuous function and* $\lim_{|x|\to\infty} (\delta(x) - x) = 0$.

If the conclusion of Lemma 6.5 is stated in terms of the problem of Section 5 and the changes of variable indicated in (6.1), (6.2) and (6.3), the result stated in (5.25) follows.

In the following the functions $q(\cdot)$, $q_n(\cdot)$, $n \geqq 1$ are as in (5.20). We will suppose $\delta_n(\cdot)$, $n \geqq 1$, is a Bayes solution for the weight function $g(\cdot)q_n(\cdot)$.

LEMMA 6.6. *Suppose the hypotheses of Lemma* 6.5 *are satisfied. In addition, suppose*

$$\int_{-\infty}^{\infty} |x|^p W(x) f(x)\, dx < \infty. \tag{6.33}$$

Then

$$\lim_{n\to\infty} \sup_x |\delta(x) - \delta_n(x)| = 0. \tag{6.34}$$

PROOF. We will assume the denial of (6.34) and obtain a contradiction. In the contrary case there is an $\epsilon > 0$ and a sequence $\{x_n, n \geqq 1\}$ such that

$$\text{if} \quad n \geqq 1, \quad |\delta(x_n) - \delta_n(x_n)| \geqq \epsilon. \tag{6.35}$$

We may without loss of generality assume at once that

$$\lim_{n\to\infty} x_n/n = d \quad \text{and} \quad \lim_{n\to\infty} (\delta(x_n) - \delta_n(x_n)) = c. \tag{6.36}$$

$\pm\infty$ are possible values of d and c. We consider below three cases.

CASE I. $d = \pm\infty$. The argument if $d = -\infty$ is exactly parallel to the argument if $d = +\infty$. We give the proof only for $d = +\infty$. By Lemma 8.1 Section 8,

$$(1 + |x/n|^p)/(1 + |(x-\theta)/n|^p) \leqq 2(1 + |\theta|^p). \tag{6.37}$$

Also

$$1 = \lim_{n\to\infty} (1 + |x_n/n|^p)/(1 + |(x_n - \theta)/n|^p). \tag{6.38}$$

Consequently if (6.33) holds, by the bounded convergence theorem,

$$\lim_{n\to\infty} (1 + |x_n/n|^p) \int_{-\infty}^{\infty} W(\theta) f(\theta)\, g(x_n - \theta)\, q_n(x_n - \theta)\, d\theta = \beta \int_{-\infty}^{\infty} W(\theta) f(\theta)\, d\theta. \tag{6.39}$$

Define functions $H_n(\cdot)$, $n \geqq 1$ by

$$H_n(x) = \int_{-\infty}^{\infty} W(\delta_n(x) - x + \theta)f(\theta)g(x - \theta)q_n(x - \theta)\, d\theta. \tag{6.40}$$

By Lemma 6.4, $\lim_{x\to\infty} (\delta(x) - x) = 0$. Therefore from (6.36) follows

$$\lim_{n\to\infty} x_n - \delta_n(x_n) = c.$$

Apply Fatou's lemma to obtain

$$\lim_{n\to\infty} (1 + |x_n/n|^p)H_n(x_n) \geqq \beta \int_{-\infty}^{\infty} W(-c + \theta)f(\theta)\, d\theta. \tag{6.41}$$

From the definition of $\delta(\cdot)$,

$$H_n(x) \leqq \int_{-\infty}^{\infty} W(\theta)f(\theta)g(x - \theta)q_n(x - \theta)\, d\theta. \tag{6.42}$$

Therefore from (6.39), (6.41) and (6.42) it follows that

$$\beta \int_{-\infty}^{\infty} W(-c + \theta)f(\theta)\, d\theta \leqq \beta \int_{-\infty}^{\infty} W(\theta)f(\theta)\, d\theta. \tag{6.43}$$

By (6.10), $c = 0$ follows.

Case II. $-\infty < d < \infty$, $\limsup_n |x_n| = \infty$. In Case II we may suppose without loss of generality that $\lim_{n\to\infty} |x_n| = \infty$. We consider the case that $\limsup_{n\to\infty} x_n = \infty$. Then we may suppose without loss of generality (by otherwise taking subsequences) that $\lim_{n\to\infty} x_n = \infty$. The case $\liminf_{n\to\infty} x_n = -\infty$ may be treated by a parallel argument. As in Case I, $\lim_{n\to\infty} (\delta(x_n) - x_n) = 0$. Since we assume throughout that (6.36) holds, $\lim_{n\to\infty} (x_n - \delta_n(x_n)) = c$. Also,

$$\lim_{n\to\infty} q_n(x_n - \theta) = q(d). \tag{6.44}$$

Using the bounded convergence theorem and (6.42),

$$\limsup_{n\to\infty} H_n(x_n) \leqq \beta q(d) \int_{-\infty}^{\infty} W(\theta)f(\theta)\, d\theta. \tag{6.45}$$

Applying Fatou's lemma,

$$\liminf_{n\to\infty} H_n(x_n) \geqq \beta q(d) \int_{-\infty}^{\infty} W(-c + \theta)f(\theta)\, d\theta. \tag{6.46}$$

(6.45) and (6.46) together imply $c = 0$.

Case III. *The sequence* $\{x_n, n \geqq 1\}$ *is bounded.* We may then suppose (by considering subsequences) that

$$x = \lim_{n\to\infty} x_n; \qquad \text{if } \theta \not\in N, \ \lim_{n\to\infty} g(x_n - \theta) = g(x - \theta). \tag{6.47}$$

(see Lemma 6.1)

By definition of $\delta_n(\cdot)$,

$$\begin{aligned}(6.48)\quad &\int_{-\infty}^{\infty} W(\delta_n(x_n) - x_n + \theta)f(\theta)g(x_n - \theta)q_n(x_n - \theta)\, d\theta \\ &\qquad \leqq \int_{-\infty}^{\infty} W(\delta(x_n) - x_n + \theta)f(\theta)g(x_n - \theta)q_n(x_n - \theta)\, d\theta.\end{aligned}$$

By Lemma 6.5, $\lim_{n\to\infty} (\delta(x_n) - x_n) = \delta(x) - x$, and $\sup_x |\delta(x) - x| \leqq K_4$. Then

$$\begin{aligned}(6.49)\quad &W(\delta(x_n) - x_n + \theta)g(x_n - \theta)q_n(x_n - \theta) \\ &\qquad \leqq K_1(W(K_4 + \theta) + W(-K_4 + \theta)).\end{aligned}$$

By hypothesis the right side of (6.49) is $f(\cdot)$ integrable. By the bounded convergence theorem,

$$(6.50)\quad \limsup_{n\to\infty} H_n(x_n) \leqq \int_{-\infty}^{\infty} W(\delta(x) - x + \theta)f(\theta)g(x - \theta)\, d\theta.$$

By assumption

$$\begin{aligned}(6.51)\quad \lim_{n\to\infty} (\delta_n(x_n) - x_n) &= \lim_{n\to\infty} (\delta_n(x_n) - \delta(x_n)) \\ &\qquad + \lim_{n\to\infty}(\delta(x_n) - x_n) = -c + (\delta(x) - x).\end{aligned}$$

Using Fatou's lemma,

$$(6.52)\quad \liminf_{n\to\infty} H_n(x_n) \geqq \int_{-\infty}^{\infty} W(-c + \delta(x) - x + \theta)f(\theta)g(x - \theta)\, d\theta.$$

From (6.50), (6.52) and the hypothesis of the lemma, $c = 0$ follows. That completes the proof of Lemma 6.6.

(5.26) of Section 5 is a consequence of Lemma 6.6 and the changes of variable introduced at the start of this section.

7. A minimax estimator of $\theta \in [0, \infty)$. Throughout this section we will use the notation introduced in Section 4. We will assume in this section that the loss function $W(\cdot)$ is a strictly convex function satisfying (2.6). We suppose $W(\cdot)$ satisfies

$$(7.1)\quad \text{for all real numbers } c, \qquad \iint W(c + x)f(x, y)\, dx\, \mu(dy) < \infty.$$

Throughout $W'(\cdot)$ is the right continuous right derivative of $W(\cdot)$.

Under the assumptions we have made there is a uniquely defined function $\delta(\cdot, \cdot)$ satisfying for almost all $y(\mu)$, for all x

$$\begin{aligned}(7.2)\quad &\int_0^{\infty} W(\delta(x, y) - \theta)f(x - \theta, y)\, d\theta = \inf_c \int_0^{\infty} W(c - \theta)f(x - \theta, y)\, d\theta; \\ &0 = \int_0^{\infty} W'(\delta(x, y) - \theta)f(x - \theta, y)\, d\theta.\end{aligned}$$

Existence follows from the results of Section 2. Uniqueness follows from the fact $W'(\cdot)$ is strictly increasing.

Throughout $\delta_0(\cdot)$ will be the (measurable) solution of

$$0 = \int_{-\infty}^{\infty} W'(\delta_0(y) + x)f(x, y)\, dx. \tag{7.3}$$

Our proof that $\delta(\cdot, \cdot)$ is minimax requires the integrability condition

$$\iint_{-\infty}^{\infty} |x + \delta_0(y)|\, |W'(x + \delta_0(y))|\, f(x, y)\, dx\, \mu(dy) < \infty. \tag{7.4}$$

Throughout R will be the constant risk of the estimator $x + \delta_0(y)$.

The main result of this section is the following theorem.

THEOREM 6. *Let $R(\delta, \cdot)$ be the risk function of $\delta(\cdot, \cdot)$. If* (7.1) *holds then*

$$\textit{if} \quad \theta \geqq 0, \quad R(\delta, \theta) \leqq R; \tag{7.5}$$

$$\lim_{\theta \to \infty} R(\delta, \theta) = R. \tag{7.6}$$

If (7.4) *holds then $\delta(\cdot, \cdot)$ is a minimax estimator of $\theta \in [0, \infty)$.*

The methods of the preceding sections do not seem strong enough to prove the $\delta(\cdot, \cdot)$ of this section admissible. In Section 8 we give an admissibility theorem for square error which implies in the case of square error, if for some $\epsilon > 0$,

$$\iint |x|^{3+\epsilon} f(x, y)\, dx\, \mu(dy) < \infty \tag{7.7}$$

then the estimator $\delta(\cdot, \cdot)$ is admissible. Consequently the results of Section 7 and Section 8 include the result of Katz [8] when $f(x) = (1/(2\pi)^{\frac{1}{2}}) \exp(-(1/2)x^2)$.

Although $\delta(\cdot, \cdot)$ may be shown to be admissible and minimax it may have very surprising properties. In the case of square error and $f(x) = (n - 1)/x^n$, $-\infty < x \leqq -1$, $f(x) = 0$, $x > -1$, n even, we find

$$\begin{aligned} &\text{if} \quad x \leqq -1, \quad \delta(x) = -x/(n-2), \\ &\qquad\qquad \text{if} \quad x > -1, \quad \delta(x) = x + (n-1)/(n-2). \end{aligned} \tag{7.8}$$

The estimator $\delta(\cdot, \cdot)$ may therefore have the property that the smaller the observed value the larger the estimated value!

If $\delta_0^*(\ )$ is a measurable function defined on E_{n-1} the statistician may elect to use as his density function

$$f^*(x, y) = f(x - \delta_0^*(y), y). \tag{7.9}$$

Provided (7.1) and (7.4) hold for $f^*(\cdot, \cdot)$ a minimax estimator $\delta^*(\cdot, \cdot)$ of $\theta \in [0, \infty)$ is defined by

$$0 = \int_0^{\infty} W'(\delta^*(x, y) - \theta) f^*(x - \theta, y)\, d\theta. \tag{7.10}$$

If the loss is square error and for some $\epsilon > 0$

$$\iint |x + \delta_0^*(x)|^{3+\epsilon} f(x, y)\, dx\, \mu(dy) < \infty \tag{7.11}$$

then it follows from Section 8 that $\delta^*(\cdot, \cdot)$ is admissible as an estimator of $\theta \varepsilon [0, \infty)$.

In order to prove Theorem 6, we first develop some properties of $\delta(\cdot, \cdot)$.

LEMMA 7.1. *For all* (x, y) *such that*

$$\int_0^\infty f(x - \theta, y)\, d\theta > 0, \qquad \delta(x, y) \geqq 0. \tag{7.12}$$

PROOF. If $\delta(x, y) < 0$ and $\theta \geqq 0$ then $W'(\delta(x, y) - \theta) < 0$. Therefore

$$0 > \int_0^\infty W'(\delta(x, y) - \theta) f(x - \theta, y)\, d\theta. \tag{7.13}$$

This contradicts (7.3).

LEMMA 7.2. *Define a function* $a(\cdot)$ *by*

$$a(y) = \inf \left\{ x \,\middle|\, \int_0^\infty f(x - \theta, y)\, d\theta > 0 \right\}. \tag{7.14}$$

For each y, $\delta(x, y) - x$ *is a nonincreasing function of* x, $x > a(y)$. *If* $x > a(y)$ *then* $\delta(x, y) \geqq x + \delta_0(y)$.

$$\lim_{x \to \infty} (\delta(x, y) - x) = \delta_0(y).$$

PROOF. By change of variables in (7.2),

$$0 = \int_{-\infty}^x W'(\delta(x, y) - x + \theta) f(\theta, y)\, d\theta. \tag{7.15}$$

If $x > a(y)$ and $\delta(x, y) - x < \delta_0(y)$ it would follow that

$$0 = \int_{-\infty}^x W'(\delta(x, y) - x + \theta) f(\theta, y)\, d\theta < \int_{-\infty}^x W'(\delta_0(y) + \theta) f(\theta, y)\, d\theta. \tag{7.16}$$

Since

$$0 = \int_{-\infty}^\infty W'(\delta_0(y) + \theta) f(\theta, y)\, d\theta \tag{7.17}$$

it follows that

$$\int_x^\infty W'(\delta_0(y) + \theta) f(\theta, y)\, d\theta \geqq 0. \tag{7.18}$$

But (7.16) and (7.18) contradict (7.17). Therefore $\delta(x, y) \geqq x + \delta_0(y)$ follows.

Suppose for some y and x_1, x_2 that $a(y) \leqq x_1 < x_2$ and $\delta(x_2, y) - x_2 >$

$\delta(x_1, y) - x_1$. Since $W'(\cdot)$ is a strictly increasing function,

$$(7.19)\qquad \begin{aligned} 0 &= \int_{-\infty}^{x_1} W'(\delta(x_1, y) - x_1 + \theta)f(\theta, y)\, d\theta \\ &< \int_{-\infty}^{x_1} W'(\delta(x_2, y) - x_2 + \theta)f(\theta, y)\, d\theta. \end{aligned}$$

This implies

$$(7.20)\qquad \delta(x_2, y) - x_2 + x_1 > 0$$

since otherwise the integrand of the right side of (7.19) would be nonpositive. Therefore

$$(7.21)\qquad \int_{x_1}^{x_2} W'(\delta(x_2, y) - x_2 + \theta)f(\theta, y)\, d\theta \geqq 0.$$

(7.19) and (7.21) together contradict (7.15). Therefore $\delta(x_2, y) - x_2 \leqq \delta(x_1, y) - x_1$.

It follows that $\lim_{x\to\infty} (\delta(x, y) - x) = c(y)$ exists. By the monotone convergence theorem, for any real number x_0,

$$(7.22)\qquad \begin{aligned} 0 &\geqq \lim_{x\to\infty} \int_{-\infty}^{x_0} W'(\delta(x, y) - x + \theta)f(\theta, y)\, d\theta \\ &= \int_{-\infty}^{x_0} W'(c(y) + \theta)f(\theta, y)\, d\theta. \end{aligned}$$

Since (7.22) holds for all x_0,

$$(7.23)\qquad 0 \geqq W'(c(y) + \theta)f(\theta, y)\, d\theta.$$

This implies

$$(7.24)\qquad \delta_0(y) \geqq c(y) = \lim_{x\to\infty} (\delta(x, y) - x).$$

By the first part of Lemma 7.2, $c(y) = \delta_0(y)$ follows.

Lemma 7.3. *If $R(\delta, \cdot)$ is the risk function of δ,*

$$(7.25)\qquad \lim_{\theta\to\infty} R(\delta, \theta) = R = \iint W(\delta_0(y) + x)f(x, y)\, dx\mu(dy).$$

Proof. Let $W_-(\cdot)$ and $W_+(\cdot)$ be monotone functions such that for all x, $W(x) = W_-(x) + W_+(x)$. Then

$$(7.26)\qquad \begin{aligned} R(\delta, \theta) &= \iint W(\delta(x, y) - \theta)f(x - \theta, y)\, dx\mu(dy) \\ &= \iint W(\delta(x + \theta, y) - (x + \theta) + x)f(x, y)\, dx\mu(dy). \end{aligned}$$

Since $W_-(\cdot)$ and $W_+(\cdot)$ are monotone functions, by the monotone convergence theorem and Lemma 7.2,

$$(7.27)\quad \begin{aligned} \lim_{\theta\to\infty} \iint W_-(\delta(x+\theta, y) - (x+\theta) + x)f(x, y)\, dx\mu(dy) \\ = \iint W_-(\delta_0(y) + x)f(x, y)\, dx\mu(dy); \\ \lim_{\theta\to\infty} \iint W_+(\delta(x+\theta, y) - (x+\theta) + x)f(x, y)\, dx\mu(dy) \\ = \iint W_+(\delta_0(y) + x)f(x, y)\, dx\mu(dy). \end{aligned}$$

Since $W(x) = W_-(x) + W_+(x)$ for all x, Lemma 7.3 follows.

LEMMA 7.4. *Suppose for each y there are real numbers $b(y)$ and $c(y)$ satisfying*

$$(7.28)\quad b(y) < c(y) \quad \text{and} \quad 1 = \int_{b(y)}^{c(y)} f(x, y)\, dx.$$

If $R(\delta, \cdot)$ is the risk function of $\delta(\cdot, \cdot)$ then

$$(7.29)\quad \text{if} \quad \theta \geqq 0, \quad R(\delta, \theta) \leqq R; \qquad R(\delta, 0) = R.$$

PROOF. Let R be as in (7.25). From (7.28),

$$(7.30)\quad \begin{aligned} \text{if} \quad x > c(y), \qquad \int_{-\infty}^{x} W'(\delta(x, y) - x + \theta)f(\theta, y)\, d\theta \\ = \int_{-\infty}^{\infty} W'(\delta(x, y) - x + \theta)f(\theta, y)\, d\theta. \end{aligned}$$

Therefore,

$$(7.31)\quad \text{if} \quad x > c(y), \qquad \delta(x, y) = x + \delta_0(y).$$

By change of variables,

$$(7.32)\quad R(\delta, \theta) = \iint_{b(y)}^{c(y)} W(\delta(x+\theta, y) - \theta)f(x, y)\, dx\mu(dy).$$

We consider the functions

$$(7.33)\quad R(\delta, \theta, y) = \int_{b(y)}^{c(y)} W(\delta(x+\theta, y) - \theta)f(x, y)\, dx$$

and

$$(7.34)\quad R(y) = \int_{b(y)}^{c(y)} W(\delta_0(y) + x)f(x, y)\, dx.$$

From (7.31) it follows that

$$(7.35)\quad \text{if} \quad \theta > c(y) - b(y) \quad \text{then} \quad R(\delta, \theta, y) = R(y).$$

The function

$$\int_0^\infty (R(y) - R(\delta, \theta + d, y))\, d\theta = H(d) \tag{7.36}$$

is the signed area under the curve $R(y) - R(\delta, \theta + d, y)$ from d to ∞. If we can show that for $d \geqq 0$ this is a decreasing function of d, it will follow that if $\theta, d \geqq 0$, $R(y) \geqq R(\delta, \theta + d, y)$. For by Lemma 7.3, $\lim_{d\to\infty} R(\delta, \theta + d, y) = R(y)$. Suppose $d_1 < d_2$. From (7.36),

$$\begin{aligned} H(d_1) - H(d_2) &= \int_0^\infty (R(\delta, \theta + d_2, y) - R(\delta, \theta + d_1, y))\, d\theta \\ &= \int_0^\infty d\theta \int_{-\infty}^\infty (W(\delta(x + d_2, y) - d_2 - \theta) \\ &\qquad - W(\delta(x + d_1, y) - d_1 - \theta))f(x - \theta, y)\, dx. \end{aligned} \tag{7.37}$$

It follows from (7.28) and (7.31) that the double integral in (7.37) is absolutely convergent. The order of integration may be interchanged. We use below the fact $W'(\cdot)$ is a strictly increasing function and the inequality

$$\text{for all } z_1, z_2, \quad W(z_1) - W(z_2) \geqq (z_1 - z_2)W'(z_2). \tag{7.38}$$

By Lemma 7.2,

$$\begin{aligned} &\text{if } x + d_1 > a(y) \text{ then} \\ &\qquad (\delta(x + d_2, y) - d_2) - (\delta(x + d_1, y) - d_1) \leqq 0; \\ &\text{if } x > a(y) \text{ then } \delta(x + d_1, y) - d_1 \leqq \delta(x). \end{aligned} \tag{7.39}$$

We may suppose without loss of generality that $a(y) = b(y)$, as follows from (7.14) and (7.28). The conditions $x - \theta > b(y)$ and $\theta > 0$ then imply $x > \theta + b(y) > b(y) = a(y)$. Since $0 \leqq d_1 < d_2$, (7.39) is valid for all x in the range of integration in (7.37). Therefore

$$\begin{aligned} &H(d_1) - H(d_2) \\ &\geqq \int_{-\infty}^\infty dx \int_0^\infty ((\delta(x + d_2, y) - d_2) - (\delta(x + d_1, y) - d_1)) \\ &\qquad \cdot W'(\delta(x + d_1, y) - d_1 - \theta)f(x - \theta, y)\, d\theta \\ &\geqq \int_{-\infty}^\infty dx\, ((\delta(x + d_2, y) - d_2 - (\delta(x + d_1, y) - d_1)) \\ &\qquad \cdot \int_0^\infty W'(\delta(x) - \theta)f(x - \theta, y)\, d\theta = 0. \end{aligned} \tag{7.40}$$

Therefore it follows that for all $0 \leqq d_1 < d_2$, $H(d_1) - H(d_2) \geqq 0$.

Using the fact $\delta(x, y) - x$ is a nonincreasing function of x and using the decomposition $W(\cdot) = W_-(\cdot) + W_+(\cdot)$ it is easily shown that for each y,

$R(\delta, \theta, y)$ is a continuous function of θ. The result, if $0 \leqq d_1 < d_2$ then $H(d_1) - H(d_2) \geqq 0$ is equivalent to

$$\int_{d_1}^{d_2} (R(y) - R(\delta, \theta, y))\, d\theta \geqq 0. \tag{7.41}$$

From the continuity of $R(\delta, \cdot, y)$ it now follows that

$$\text{if} \quad \theta \geqq 0, \qquad R(y) \geqq R(\delta, \theta, y). \tag{7.42}$$

Integration of both sides of the inequality in (7.42) with respect to the measure $\mu(\cdot)$ gives inequality (7.29).

Repetition of the above argument for $\theta < 0$ will show

$$\int_0^d (R(y) - R(\delta, \theta, y))\, d\theta \geqq 0$$

for all real numbers d. It follows from the continuity of $R(\delta, \cdot, y)$ that $R(y) = R(\delta, 0, y)$. Integration of this equality with respect to $\mu(\cdot)$ gives the final statement of the lemma.

We now generalize Lemma 7.4 to apply to all density functions $f(\cdot, \cdot)$ satisfying the hypotheses of this section. Suppose $f(\cdot, \cdot)$ is given. For each integer $n \geqq 1$ define a function $f_n(\cdot, \cdot)$ by

$$\begin{aligned} &\text{for all } y, \quad \text{if} \quad |x| \leqq n, \qquad f_n(x, y) = f(x, y), \\ &\qquad\qquad\qquad \text{if} \quad |x| > n, \qquad f_n(x, y) = 0. \end{aligned} \tag{7.43}$$

For each integer $n \geqq 1$ define functions $\delta_n^*(\cdot)$, $\delta_n(\cdot, \cdot)$, $R(\delta_n, \theta)$, and the constant R_n by

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} W'(\delta_n^*(y) + x) f_n(x, y)\, dx; \\ 0 &= \int_0^{\infty} W'(\delta_n(x, y) - \theta) f_n(x - \theta, y)\, d\theta; \\ R(\delta_n, \theta) &= \iint_{-\infty}^{\infty} W(\delta_n(x, y) - \theta) f_n(x - \theta, y)\, dx\mu(dy); \\ \text{and} \quad R_n &= \iint_{-\infty}^{\infty} W(\delta_n^*(y) + x) f_n(x, y)\, dx. \end{aligned} \tag{7.44}$$

We will show below that $x + \delta_n^*(y)$ is a minimax estimator of $\theta \,\varepsilon\, (-\infty, \infty)$ for the family of $L_1(-\infty, \infty)$ functions $\{f_n(\cdot - \theta, y), -\infty < \theta < \infty\}$; see Lemma 7.6. It follows that

$$R_n \leqq \iint W(x + \delta_0(y)) f_n(x, y)\, dx\mu(dy) \leqq R. \tag{7.45}$$

By Lemma 7.4,

$$\text{if} \quad \theta \geqq 0, \quad R_n(\delta_n, \theta) \leqq R_n. \tag{7.46}$$

We show immediately following (7.48) that

$$\lim_{n\to\infty} \delta_n(x, y) = \delta(x, y). \tag{7.47}$$

By Fatou's lemma, if $\theta \geqq 0$,

$$\begin{aligned} R(\delta, \theta) &\leqq \liminf_{n\to\infty} \iint W(\delta_n(x, y) - \theta) f_n(x - \theta, y)\, dx\mu(dy) \\ &= \liminf_{n\to\infty} R_n(\delta_n, \theta) = \liminf_{n\to\infty} R_n \leqq R. \end{aligned} \tag{7.48}$$

To verify (7.47) suppose $\{a_n, n \geqq 1\}$ is an integer sequence, $\lim_{n\to\infty} a_n = \infty$, and for given (x, y), $\lim_{n\to\infty} \delta_{a_n}(x, y) = c$. By definition of $\delta_n(\cdot, \cdot)$, $n \geqq 1$,

$$\begin{aligned} \int_0^\infty W(\delta_n(x, y) - \theta) f_n(x - \theta, y)\, d\theta &\leqq \int_0^\infty W(\delta(x, y) - \theta) \\ \cdot f_n(x - \theta, y)\, d\theta &\leqq \int_0^\infty W(\delta(x, y) - \theta) f(x - \theta, y)\, d\theta. \end{aligned} \tag{7.49}$$

By Fatou's lemma and (7.49),

$$\begin{aligned} \int_0^\infty W(c - \theta) f(x - \theta, y)\, d\theta &\leqq \liminf_{n\to\infty} \int_0^\infty W(\delta_n(x, y) - \theta) \\ \cdot f_n(x - \theta, y)\, d\theta &\leqq \int_0^\infty W(\delta(x, y) - \theta) f(x - \theta, y)\, d\theta. \end{aligned} \tag{7.50}$$

By definition of $\delta(\cdot, \cdot)$, $c = \delta(x, y)$ follows.

It remains to show that the validity of (7.48) implies $\delta(\cdot, \cdot)$ is a minimax estimator of $\theta \in [0, \infty)$. If not there is an $\epsilon > 0$ and an estimate $\delta_1(\cdot, \cdot)$ satisfying if $\theta \geqq 0$ then $R(\delta_1, \theta) + \epsilon \leqq R$.

Then relative to the family

$$\{f(x - \delta_0(y) - \theta, y), -\infty < \theta < \infty\} \tag{7.51}$$

the risk function of the estimator $\delta_1(x - \delta_0(y), y)$ satisfies

$$\begin{aligned} &\text{if} \quad \theta \geqq 0 \quad \text{then} \\ &\iint W(\delta_1(x - \delta_0(y), y) - \theta) f(x - \delta_0(y) - \theta, y)\, dx\mu(dy) \\ &\qquad = \iint W(\delta_1(x, y) - \theta) f(x - \theta, y)\, dx\mu(dy) \\ &\qquad \leqq -\epsilon + \iint W(x + \delta_0(y) - \theta) f(x - \theta, y)\, dx\mu(dy) \\ &\qquad = -\epsilon + \iint W(x - \theta) f(x - \delta_0(y) - \theta, y)\, dx\mu(dy). \end{aligned} \tag{7.52}$$

Also

$$\int W'(x-\theta)f(x-\delta_0(y)-\theta,y)\,d\theta = \int W'(\theta+\delta_0(y))f(\theta,y)\,d\theta = 0. \tag{7.53}$$

We may therefore suppose for the remainder of the argument that $\delta_0(y) \equiv 0$.

Let $\delta_N(\cdot,\ \cdot)$ be the Bayes estimator for the uniform distribution of θ on $[-N, N]$. The estimator $\delta_1(x + a, y) - a$ has as risk function $R(\delta_1, \theta + a)$ satisfying $R(\delta_1, \theta + a) \leqq R - \epsilon$ if $\theta \geqq -a$. Consequently if $a = N$, it follows that

$$(1/2N)\int_{-N}^{N} R(\delta_N, \theta)\,d\theta \leqq R - \epsilon. \tag{7.54}$$

We will show that if (7.4) holds then

$$\lim_{N\to\infty} (1/2N)\int_{-N}^{N} R(\delta_N, \theta)\,d\theta = R. \tag{7.55}$$

This contradiction shows the estimator $\delta_1(\cdot,\ \cdot)$ cannot exist.

From the defining relation

$$0 = \int_{-N}^{N} W'(\delta_N(x,y)-\theta)f(x-\theta,y)\,d\theta, \tag{7.56}$$

by change of variable,

$$0 = \int_{x-N}^{x+N} W'(\delta_N(x,y)-x+\theta)f(\theta,y)\,d\theta. \tag{7.57}$$

This implies the inequalities

$$\delta_N(x,y) - x + (x+N) \geqq 0, \qquad \delta_N(x,y) - x + (x-N) \leqq 0. \tag{7.58}$$

That is,

$$|\delta_N(x,y)| \leqq N. \tag{7.59}$$

Define x^* by $x^* + N = x$. Then from (7.57) follows

$$0 = \lim_{N\to\infty} \int_{x^*}^{x^*+2N} W'(\delta_N(x^*+N,y) - x^* - N + \theta)f(\theta,y)\,d\theta. \tag{7.60}$$

From (7.60) it follows that

$$d_+(y) = \lim_{N\to\infty} (\delta_N(x+N,y) - x - N) \tag{7.61}$$

exists. By use of a substitution $x^{**} = x + N$ it may be shown that

$$d_-(y) = \lim_{N\to\infty} (\delta_N(x-N,y) - x + N) \tag{7.62}$$

exists. Using the convexity of $W(\cdot)$,

$$
\begin{aligned}
&(1/2N) \int_{-N}^{N} \iint (W(x-\theta) - W(\delta_N(x,y) - \theta)) f(x-\theta, y)\, dx\mu(dy)\, d\theta \\
&\leqq (1/2N) \iiint_{-N}^{N} (x - \delta_N(x,y)) W'(x-\theta) f(x-\theta, y)\, d\theta\, dx\mu(dy) \\
&= -(1/2N) \iiint_{N}^{\infty} (x - \delta_N(x,y)) W'(x-\theta) f(x-\theta, y)\, d\theta\, dx\mu(dy) \\
&\quad - (1/2N) \iiint_{-\infty}^{-N} (x - \delta_N(x,y)) W'(x-\theta) f(x-\theta, y)\, d\theta\, dx\mu(dy) \\
&= -(1/2N) \iint_{-\infty}^{\infty} \int_{-\infty}^{x-N} (x - \delta_N(x,y)) W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy) \qquad (7.63)\\
&\quad - (1/2N) \iint_{-\infty}^{\infty} \int_{x+N}^{\infty} (x - \delta_N(x,y)) W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy) \\
&= -(1/2N) \iint_{-\infty}^{\infty} \int_{-\infty}^{x} (x + N - \delta_N(x+N, y)) W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy) \\
&\quad + (1/2N) \iint_{-\infty}^{\infty} \int_{-\infty}^{x} (x - N - \delta_N(x-N, y)) W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy) \\
&= (1/2N) \iint_{-\infty}^{\infty} \int_{-\infty}^{x} ((x - N - \delta_N(x-N, y)) \\
&\qquad\qquad - (x + N - \delta_N(x+N, y))) W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy).
\end{aligned}
$$

$$(7.64) \qquad |(x - N - \delta_N(x-N, y)) - (x + N - \delta_N(x+N, y))|/(2N) \leqq 2$$

for all x, y. As shown above in (7.61), (7.62)

$$
\begin{aligned}
-d_+(y) &= \lim_{N\to\infty} x + N - \delta_N(x+N, y) \quad \text{and} \\
-d_-(y) &= \lim_{N\to\infty} (x - N) - \delta_N(x-N, y)
\end{aligned}
$$

exist and are finite. Consequently the quantity (7.64) bounded by 2 tends to zero for each x, y. It will follow by the bounded convergence theorem that if

$$(7.65) \qquad \iint_{-\infty}^{\infty} \left| \int_{-\infty}^{x} W'(\theta) f(\theta, y)\, d\theta \right| dx\mu(dy) < \infty$$

then $\lim_{N\to\infty} 1/2N \int_{-N}^{N} (R - R(\delta_N, \theta))\, d\theta = 0$. Since $\int_{-\infty}^{\infty} W'(\theta) f(\theta, y)\, d\theta = 0$ and since $\int_{-\infty}^{x} W'(\theta) f(\theta, y)\, d\theta \leqq 0$, $-\infty < x < \infty$, it suffices to show

$$(7.66) \qquad \iint_{-\infty}^{\infty} \int_{-\infty}^{x} W'(\theta) f(\theta, y)\, d\theta\, dx\mu(dy) > -\infty.$$

If $\iint_{-\infty}^{\infty} \theta |W'(\theta)| f(\theta, y)\, d\theta\mu(dy) < \infty$ then integration by parts gives the value of (7.66) to be $-\iint_{-\infty}^{\infty} \theta W'(\theta) f(\theta, y)\, d\theta\mu(dy)$. That completes the proof of the following lemma and the theorem of this section.

LEMMA 7.6. *Suppose (7.4) holds. Let $\delta_N(\cdot, \cdot)$ be the Bayes estimate corresponding to the uniform distribution on $[-N, N]$. Then*

$$\lim_{N\to\infty} (1/2N) \textstyle\int_{-N}^{N} R(\delta_N, \theta)\, d\theta = R.$$

It follows that $x + \delta^*(y)$ is a minimax estimate of $\theta \,\varepsilon\, (-\infty, \infty)$.

8. An admissibility result for square error. The results of Section 6 and Section 7 do not give information about the admissibility of a generalized Bayes estimator $\delta(\cdot)$ relative to a weight function $g(\cdot)$ if $g(\cdot)$ is unbounded or if for example $g(\theta) = 0, \theta < 0$. In the case of square error it is possible to give an explicit formula for $\delta(\cdot)$. Using methods similar to those of Stein [11] it is possible to obtain a better admissibility theorem. We will continue to use the notation of earlier sections so as to include the case of n observations.

THEOREM 7. *Suppose $g(\cdot)$ is a non-negative Borel measurable function satisfying*

$$\begin{aligned} &\text{there exist constants } c_1 > 0,\ c_2 \geqq 0 \text{ and } \alpha, \\ &0 < \alpha < 1 \quad \text{such that if} \quad -\infty < \theta < \infty, \quad g(\theta) \leqq c_1 + c_2 |\theta|^\alpha. \end{aligned} \tag{8.1}$$

Suppose there is a β satisfying $2 \geqq \beta > 1 + \alpha$ and

$$\iint |x|^{2+\alpha+\beta} f(x, y)\, dx\mu(dy) < \infty. \tag{8.2}$$

Let

$$\Omega = \{\theta \mid \text{for all } \lambda > 0, \int_{\theta-\lambda}^{\theta+\lambda} g(x)\, dx > 0\}. \tag{8.3}$$

If $\delta(\cdot, \cdot)$ is the generalized Bayes estimator for $g(\cdot)$ then $\delta(\cdot, \cdot)$ is an admissible estimator of $\theta \,\varepsilon\, \Omega$.

To prove Theorem 7 we will use a sequence of *a priori* density functions $\{q_\sigma(\cdot), \sigma > 0\}$ defined by

$$q(\theta) = 1/(1 + |\theta|^\beta),\ q_\sigma(\theta) = (1/\sigma) q(\theta/\sigma). \tag{8.4}$$

We assume $1 < \beta < 2$ and do not include normalization. In view of the discussion in Section 5, since the loss function is strictly convex, it is sufficient to prove almost admissibility.

In the calculation below we will need the following inequalities.

LEMMA 8.1. *If $1 < \beta < 2$ then*

$$\left| |z + \lambda|^\beta - |z|^\beta \right| \leqq 2 |\lambda| (|\lambda|^{\beta-1} + |z|^{\beta-1}); \tag{8.5}$$

$$1 + |\lambda|^\beta \leqq 2(1 + |z - \lambda|^\beta)(1 + |z|^\beta). \tag{8.6}$$

PROOF. For all z,

$$|z|^\beta = \beta \int_0^z \operatorname{sign}(w)\, |w|^{\beta-1}\, dw. \tag{8.7}$$

Therefore

$$(8.8)\qquad \begin{aligned} \left| |z+\lambda|^{\beta} - |z|^{\beta} \right| &= \beta \left| \int_z^{z+\lambda} \operatorname{sign}(w)\, |w|^{\beta-1}\, dw \right| \\ &\leq \beta\, |\lambda| \max\left(|z|^{\beta-1}, |z+\lambda|^{\beta-1}\right) \leqq \beta\, |\lambda| \left(|z|^{\beta-1} + |\lambda|^{\beta-1}\right). \end{aligned}$$

Since $\beta < 2$, (8.5) follows.

To prove (8.6)

$$(8.9)\qquad \begin{aligned} 1 + |\lambda|^{\beta} = 1 + |\lambda - z + z|^{\beta} &\leqq 2^{\beta-1}(1 + |\lambda - z|^{\beta} + |z|^{\beta}) \\ &\leqq 2(1 + |\lambda - z|^{\beta})(1 + |z|^{\beta}). \end{aligned}$$

Let $\delta_\sigma(\cdot, \cdot)$ be the Bayes solution corresponding to the $L_1(-\infty, \infty)$ function $g(\cdot)q_\sigma(\cdot)$. It suffices to prove

$$(8.10)\qquad \lim_{\sigma\to\infty} \sigma \int_{-\infty}^{\infty} (R(\delta, \theta) - R(\delta_\sigma, \theta))g(\theta)q_\sigma(\theta)\, d\theta = 0.$$

A direct calculation will show

$$(8.11)\qquad \begin{aligned} A_\sigma &= \int_{-\infty}^{\infty} (R(\delta, \theta) - R(\delta_\sigma, \theta))g(\theta)q_\sigma(\theta)\, d\theta \\ &= \iiint (\delta(x, y) - \delta_\sigma(x, y))^2 f(x - \theta, y)g(\theta)q_\sigma(\theta)\, d\theta\, dx\mu(dy). \end{aligned}$$

Also

$$(8.12)\qquad \delta(x, y) - \delta_\sigma(x, y) = \frac{\int (\delta(x, y) - \eta)f(x - \eta, y)g(\eta)q_\sigma(\eta)\, d\eta}{\int f(x - \eta, y)g(\eta)q_\sigma(\eta)\, d\eta}.$$

By definition of $\delta(\cdot, \cdot)$

$$(8.13)\qquad 0 = \int (\delta(x, y) - \eta)f(x - \eta, y)g(\eta)\, d\eta.$$

From (8.11), (8.12) and (8.13) follows

$$(8.14)\qquad \begin{aligned} &A_\sigma \\ &= \iint \frac{\left(\int (\delta(x, y) - \eta)f(x - \eta, y)g(\eta)(q_\sigma(\eta) - q_\sigma(x))\, d\eta\right)^2}{\int f(x - \eta, y)g(\eta)q_\sigma(\eta)\, d\eta}\, dx\mu(dy) \\ &\leqq \iiint (\delta(x, y) - \eta)^2 \left[1 - \frac{q_\sigma(x)}{q_\sigma(\eta)}\right]^2 g(\eta) \\ &\qquad \cdot q_\sigma(\eta)f(x - \eta, y)\, d\eta\, dx\mu(dy). \end{aligned}$$

Let

$$B_\sigma = q_\sigma^2(x)q_\sigma(\eta)(q_\sigma^{-1}(x) - q_\sigma^{-1}(\eta))^2. \tag{8.15}$$

By Lemma 8.1;

$$\begin{aligned} B_\sigma &\leqq 8\sigma^2 q_\sigma^2(x)q_\sigma(\eta)\left|\frac{\eta - x}{\sigma}\right|^2\left(\left|\frac{\eta - x}{\sigma}\right|^{\beta-1} + \left|\frac{x}{\sigma}\right|^{\beta-1}\right)^2 \\ &\leqq 16\sigma^2 q_\sigma^2(x)q_\sigma(\eta)\left|\frac{\eta - x}{\sigma}\right|^2\left(\left|\frac{\eta - x}{\sigma}\right|^{2\beta-2} + \left|\frac{x}{\sigma}\right|^{2\beta-2}\right). \end{aligned} \tag{8.16}$$

Again by Lemma 8.1,

$$q_\sigma(\eta)\left|\frac{\eta - x}{\sigma}\right|^\beta \leqq (2/\sigma)(1 + |x/\sigma|^\beta) = 2\sigma^{-2}q_\sigma^{-1}(x); \tag{8.17}$$

$$\begin{aligned} q_\sigma(\eta)\left|\frac{\eta - x}{\sigma}\right|^2 &= \sigma^{1-(2/\beta)}(q_\sigma(\eta))^{2-(2/\beta)} \\ &\cdot\left(\frac{|(\eta - x)/\sigma|^\beta}{1 + |\eta/\sigma|^\beta}\right)^{(2/\beta)-1}|(\eta - x)/\sigma|^\beta \leqq 4\sigma^{-(2/\beta)}(q_\sigma(x))^{(\beta-2)/\beta}\cdot|(\eta - x)/\sigma|^\beta. \end{aligned} \tag{8.18}$$

Also

$$|x/\sigma|^{2\beta-2} \leqq (1 + |x/\sigma|)^{(2\beta-2)/\beta} = \sigma^{(2-2\beta)/\beta}(q_\sigma(x))^{(2-2\beta)/\beta}. \tag{8.19}$$

From (8.16) to (8.19) follows

$$B_\sigma \leqq 96 q_\sigma(x)|(\eta - x)/\sigma|^\beta, \tag{8.20}$$

and

$$\begin{aligned} A_\sigma \leqq 96\iiint (\delta(x, y) - \eta)^2\,|(x - \eta)/\sigma|^\beta f(x - \eta, y) \\ \cdot g(\eta)q_\sigma(x)\,d\eta\,dx\mu(dy). \end{aligned} \tag{8.21}$$

Define

$$r_1 = (\beta + 2)/2 \quad\text{and}\quad r_2 = (\beta + 2)/\beta. \tag{8.22}$$

Then

$$1 = (1/r_1) + (1/r_2). \tag{8.23}$$

Applying the Hölder inequality to (8.21) gives

$$\begin{aligned} &\int (\delta(x, y) - \eta)^2\,|(x - \eta)/\sigma|^\beta f(x - \eta, y)g(\eta)\,d\eta \\ &\leqq \left(\int (\delta(x, y) - \eta)^{\beta+2} f(x - \eta, y)g(\eta)\,d\eta\right)^{1/r_1} \\ &\qquad\cdot\left(\int |(x - \eta)/\sigma|^{\beta+2} f(x - \eta, y)g(\eta)\,d\eta\right)^{1/r_2}. \end{aligned} \tag{8.24}$$

We now obtain an upper bound on the first term of the product in (8.24).

$$\int (\delta(x, y) - \eta)^{\beta+2} f(x - \eta, y)g(\eta)\, d\eta$$

$$\begin{aligned} &= \int \left(\frac{\int(\theta - \eta)f(x - \theta, y)g(\theta)\, d\theta}{\int f(x - \theta, y)g(\theta)\, d\theta}\right)^{\beta+2} f(x - \eta, y)g(\eta)\, d\eta \\ &\leqq \int \frac{\int |\theta - \eta|^{\beta+2} f(x - \theta, y)g(\theta)\, d\theta}{\int f(x - \theta, y)g(\theta)\, d\theta} f(x - \eta, y)g(\eta)\, d\eta \\ &\leqq 8 \int \frac{\int |\theta - x|^{\beta+2} f(x - \theta, y)g(\theta)\, d\theta}{\int f(x - \theta, y)g(\theta)\, d\theta} f(x - \eta, y)g(\eta)\, d\eta \\ &+ 8 \int \frac{\int |x - \eta|^{\beta+2} f(x - \theta, y)g(\theta)\, d\theta}{\int f(x - \theta, y)g(\theta)\, d\theta} f(x - \eta, y)g(\eta)\, d\eta \\ &\qquad = 16 \int |x - \eta|^{\beta+2} f(x - \eta, y)g(\eta)\, d\eta. \end{aligned} \tag{8.25}$$

From (8.21), (8.24) and (8.25) we obtain

$$A_\sigma \leqq 1536\sigma^{-\beta} \iiint |x - \eta|^{\beta+2} f(x - \eta, y)g(\eta)q_\sigma(x)\, d\eta\, dx\mu(dy). \tag{8.26}$$

Using (8.1) there is a constant K_1 such that

$$g(\eta) \leqq K_1 + K_1|x - \eta|^\alpha + K_1|x|^\alpha. \tag{8.27}$$

Substitution of (8.27) into (8.26) gives for a suitable constant K_2

$$\begin{aligned} A_\sigma &\leqq K_2\sigma^{-\beta} \iiint |x - \eta|^{\beta+2} f(x - \eta, y)q_\sigma(x)\, d\eta\, dx\mu(dy) \\ &+ K_2\sigma^{-\beta} \iiint |x - \eta|^{\alpha+\beta+2} f(x - \eta, y)q_\sigma(x)\, d\eta\, dx\mu(dy) \\ &\qquad + K_2\sigma^{-\beta} \iiint |x - \eta|^{\beta+2} f(x - \eta, y)\, |x|^\alpha\, q_\sigma(x)\, d\eta\, dx\mu(dy). \end{aligned} \tag{8.28}$$

Therefore if

$$\iint |x|^{\alpha+\beta+2} f(x, y)\, dx\mu(dy) < \infty, \tag{8.29}$$

then since $\int_{-\infty}^{\infty} q_\sigma(x)\, dx$ does not depend on σ, we may find a constant K_3 so large that

$$A_\sigma \leqq K_3\sigma^{-\beta} + K_3\sigma^{-\beta-\alpha} \int_{-\infty}^{\infty} |x|^\alpha\, q(x)\, dx. \tag{8.30}$$

For the integral in (8.30) to be finite,

$$\beta - \alpha > 1. \tag{8.31}$$

To complete the proof we verify (8.10).

$$0 \leqq \sigma A_\sigma \leqq K_3 \sigma^{-(\beta-1)} + K_3 \sigma^{-(\beta-\alpha-1)} \int_{-\infty}^{\infty} |x|^\alpha q(x)\, dx. \tag{8.32}$$

Since the right side of (8.32) tends to zero as $\sigma \to \infty$, (8.10) follows.

9. A nonparametric problem. Throughout this section we will suppose the loss function W is strictly convex. Ω will be a class of density functions defined on the real line such that $f(\cdot) \varepsilon \Omega$ and if $-\infty < \theta < \infty, f(\cdot - \theta) \varepsilon \Omega$. We suppose further that there is a $\tau > 0$ such that

$$(1/(2\pi\tau)^{\frac{1}{2}}) \exp(-(1/2\tau)x^2) \varepsilon \Omega. \tag{9.1}$$

We assume that

$$\text{if} \quad f \varepsilon \Omega \quad \text{then} \int_{-\infty}^{\infty} W(x) f(x)\, dx < \infty. \tag{9.2}$$

Since $W(0) = 0$, W is non-negative and strictly convex, condition (9.2) implies

$$\text{if} \quad f \varepsilon \Omega, \quad \int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty. \tag{9.3}$$

We define a function $\theta : \Omega \to$ reals by

$$\text{if} \quad f \varepsilon \Omega, \quad \theta(f) = \int_{-\infty}^{\infty} x f(x)\, dx. \tag{9.4}$$

The nonparametric problem we consider is the admissibility of the average of n observations as an estimator of $\theta(f)$. It is the purpose of this section to prove the following theorem.

THEOREM 8. *Suppose $W(\cdot)$ is symmetric, that is,*

$$\textit{for all } x, \qquad W(-x) = W(x). \tag{9.5}$$

Given the hypotheses above, among all estimators based on n independent observations from the same population the estimator δ defined by $\delta(x_1, \cdots, x_n) = (x_1 + \cdots + x_n)/n$ is an admissible estimator of $\theta(\cdot)$.

If $M \geqq 1$ is an integer the sequential procedure, take M observations, estimate $\delta(x_1, \cdots, x_M) = (1/M) \sum_{i=1}^{M} x_i$, is admissible in the class of all sequential procedures η satisfying, expected sample size of $\eta \leqq M$ for all $f \varepsilon \Omega$.

The proof, following, of this theorem, uses the results of Blyth [2]. It will be seen from the argument below that the proof of admissibility in the sequential case consists for the most part in improving on the admissibility argument of Blyth, op. cit., in the normal case.

In the case of n observations suppose that $\delta^*(\cdot)$ is an estimator as good as $\delta(x_1, \cdots, x_n) = (1/n) \sum_{i=1}^{n} x_i$. Then for all density functions $f \varepsilon \Omega$.

$$R(\delta^*, f) \leqq R(\delta, f). \tag{9.6}$$

By hypothesis there is a $\tau > 0$ such that if $-\infty < \theta < \infty$,

$$f(x, \theta) = (1/(2\pi\tau)^{\frac{1}{2}}) \exp((-1/2\tau)(x - \theta)^2) \varepsilon \Omega. \tag{9.7}$$

It follows immediately from the results of Blyth [2] that for all normal density functions $f(\cdot, \theta)$ of this set,

$$R(\delta^*, f(\cdot, \theta)) = R(\delta, f(\cdot, \theta)). \tag{9.8}$$

Define

$$A = \left\{(x_1, \cdots, x_n) \mid \delta^*(x_1, \cdots, x_n) \neq (1/n) \sum_{i=1}^{n} x_i\right\}. \tag{9.9}$$

If the set A has positive n-dimensional Lebesgue measure and if $-\infty < \theta < \infty$, then

$$0 < \int \cdots \int_A (1/(2\pi\tau)^{\frac{1}{2}})^n \exp (1/2\tau) \sum_{i=1}^{n} (x_i - \theta)^2 \prod_{i=1}^{n} dx_i . \tag{9.10}$$

From the strict convexity of $W(\cdot)$ and from (9.10) it follows at once for the estimator $(1/2)(\delta^* + \delta)$ that

$$R((1/2)(\delta^* + \delta), f(\cdot, \theta)) < R(\delta, f(\cdot, \theta)), \quad -\infty < \theta < \infty. \tag{9.11}$$

This contradiction shows A must be a set of Lebesgue measure zero. It follows that any estimator $\delta^*(\cdot)$ as good as $\delta(\cdot)$ relative to Ω can differ from $\delta(\cdot)$ only on a set of Lebesgue measure zero. Therefore

$$R(\eta, f) = R(\delta, f), \quad \text{all } f \varepsilon \Omega. \tag{9.12}$$

That completes the proof of the first part of the theorem.

To prove the second part of Theorem 8, suppose $M \geqq 1$ is an integer. Throughout the following δ will be the sequential procedure defined as follows. Take M observations (independent identically distributed.) Estimate $\delta_M(x_1, \cdots, x_M) = 1/M \sum_{i=1}^{M} x_i$. Suppose η is a sequential procedure. We write $N(\eta)$ for the stopping variable of this procedure and if $n \geqq 0$, $\eta_n(\cdot)$ is the estimator used given $N(\eta) = n$. We will suppose of η that

$$\begin{aligned} &E_f N(\eta) \leqq M, \quad \text{all } f \varepsilon \Omega, \text{ and} \\ &\qquad E_f W(\eta_N - \theta(f)) \leqq E_f W(\delta_M - \theta(f)), \quad \text{all } f \varepsilon \Omega. \end{aligned} \tag{9.13}$$

We will show in the sequel that the procedures η and δ can differ only on sets of measure zero.

Given $c > 0$ define for any sequential procedure η^*,

$$R_c(\eta^*, f) = cE_f N(\eta^*) + E_f W(\eta_N^* - \theta(f)). \tag{9.14}$$

It is shown by Blyth, op. cit., that there exist constants $c > 0$ for which relative to the risk function defined by (9.14) the procedure δ is minimax. We will suppose in the sequel one such value of c has been chosen and will not subsequently use c as a subscript. From (9.13) it follows that

$$R(\eta, f) \leqq R(\delta, f), \quad \text{all } f \varepsilon \Omega. \tag{9.15}$$

For density functions of the form (9.7) we will write

$$R(\eta, \theta) \leqq R(\delta, \theta). \tag{9.16}$$

For the remainder of the argument we suppose the observations are independent identically distributed normal random variables having $f(\cdot - \theta)$ of (9.7) as density function.

Since

$$\sup_\theta R(\delta, \theta) < \infty, \tag{9.17}$$

it follows η takes at least one observation with probability one. If $n \geqq 1$ let $\beta_n(x_1, \cdots, x_n)$ be the probability that $N(\eta) = n$ conditional on the sequence $\{x_i, i \geqq 1\}$ of values having been observed.

We suppose θ a random variable with *a priori* distribution

$$(1/(2\pi)^{\frac{1}{2}}\sigma) \exp(-(1/2\sigma^2)\theta^2), \qquad \sigma > 0. \tag{9.18}$$

Consequently for each n, the random variables $X_1, \cdots, X_n, \theta$ have a joint normal distribution. The conditional density of θ given $X_1 = x_1, \cdots, X_n = x_n$ is

$$\begin{aligned} &((n\sigma^2 + 1)/2\pi\sigma^2)^{\frac{1}{2}} \\ &\quad \cdot \exp - ((n\sigma^2 + 1)/2\sigma^2)(\theta - (\sigma^2/(n\sigma^2 + 1)) \sum_{i=1}^n x_i)^2. \end{aligned} \tag{9.19}$$

The joint density of $X_1, \cdots, X_n$ is a joint normal density function

$$p_{n,\sigma}(\cdot). \tag{9.20}$$

We define constants

$$\gamma_{n,\sigma} = \int \cdots \int \beta_n(x_1, \cdots, x_n) p_{n,\sigma}(x_1, \cdots, x_n) \prod_{i=1}^n dx_i, \qquad n \geqq 1, \sigma > 0. \tag{9.21}$$

It is easily verified that

$$\sum_{n=1}^\infty \gamma_{n,\sigma} = 1, \qquad \sigma > 0. \tag{9.22}$$

Following Blyth, op. cit., define a function $g(\cdot)$ by

$$\text{if} \quad m > 0, \qquad g(m) = (1/(2\pi)^{\frac{1}{2}}) \int W(\theta/m^{\frac{1}{2}}) \exp(-(1/2)\theta^2)\, d\theta. \tag{9.23}$$

If $n \geqq 1$, the Bayes terminal decision rule $\delta_{n,\sigma}(\cdot)$ does not depend on the stopping rule. It was noted by Blyth, op. cit., that for the class of loss functions considered,

$$\text{if} \quad n \geqq 1, \qquad \delta_{n,\sigma}(x_1, \cdots, x_n) = (\sigma^2/(n\sigma^2 + 1)) \sum_{i=1}^n x_i. \tag{9.24}$$

It follows that

$$\begin{aligned}
&\int \cdots \int \beta_n(x_1, \cdots, x_n) W(\eta_n(x_1, \cdots, x_n) - \theta) \\
(9.25) \qquad &\cdot (1/(2\pi\tau)^{\frac{1}{2}})^n \{\exp(-(1/2\tau) \sum_{i=1}^{n} (x_i - \theta)^2\}\{(1/(2\pi)^{\frac{1}{2}}\sigma) \\
&\cdot \exp - (1/2\sigma^2)\theta^2\} \, d\theta \prod_{i=1}^{n} dx_i \geqq \gamma_{n,\sigma} g(n/\tau + 1/\sigma^2).
\end{aligned}$$

The procedure δ has constant risk

$$(9.26) \qquad R = R(\delta, \theta), \qquad -\infty < \theta < \infty.$$

Therefore from (9.25) it follows that

$$\begin{aligned}
(9.27) \qquad R &\geqq \int_{-\infty}^{\infty} R(\eta, \theta)(1/(2\pi)^{\frac{1}{2}}\sigma) \exp(-1/2\sigma^2)\theta^2 \, d\theta \\
&\geqq \sum_{n=1}^{\infty} \gamma_{n,\sigma}(cn + g(n/\tau + 1/\sigma^2)).
\end{aligned}$$

From the definition (9.23) of $g(\cdot)$ and from (9.1) it follows that $g(\cdot)$ is a strictly decreasing function and

$$(9.28) \qquad g(m) < \infty \quad \text{if} \quad m \geqq 1/\tau, \qquad \lim_{m \to 0} g(m) = \infty.$$

It is shown by Blyth, op. cit., that $g(\cdot)$ is a strictly convex function. The constant R has the value

$$(9.29) \qquad R = cM + g(M/\tau).$$

We may without loss of generality suppose the constant c is chosen so that

$$(9.30) \qquad cM + g(M/\tau) < c(M + 1) + g((M + 1)/\tau)$$

and yet have R the minimum value of the numbers $\{cn + g(n/\tau), n \geqq 1\}$. The strictly convex function $cm + g(m)$ then has its minimum value somewhere between $m = M/\tau$ and $m = (M + 1)/\tau$. By the continuity of $g(\cdot)$ and the remarks just made there is $\epsilon > 0$ and a constant $K_1 > 0$ such that

$$\begin{aligned}
(9.31) \qquad &\text{if} \quad \sigma > K_1 \quad \text{and} \quad n \neq M, \\
&(cn + g(n/\tau + 1/\sigma^2)) - (cM + g(M/\tau + 1/\sigma^2)) > \epsilon.
\end{aligned}$$

It is shown by Blyth, op. cit., that there is a constant $K_2 > K_1$ such that if $\sigma > K_2$ then the Bayes risk of the Bayes decision procedure is

$$(9.32) \qquad cM + g(M/\tau + 1/\sigma^2).$$

Since the derivative $g'(\cdot)$ of $g(\cdot)$ is negative and strictly increasing,

$$\begin{aligned}
(9.33) \qquad R &- (cM + g(M/\tau + 1/\sigma^2)) \\
&= g(M/\tau) - g(M/\tau + 1/\sigma^2) \leqq (1/\sigma^2)(-g'(M/\tau)).
\end{aligned}$$

It follows from (9.27) and (9.32), (9.33) that

$$0 \leqq \sum_{n=1}^{\infty} \gamma_{n,\sigma}(cn + g(n/\tau + 1/\sigma^2)) - (cM + g(M/\tau + 1/\sigma^2)) \leqq (1/\sigma^2)(-g'(M/\tau)). \tag{9.34}$$

By (9.22),

$$\gamma_{M,\sigma} - 1 = -\sum_{n \neq M} \gamma_{n,\sigma}. \tag{9.35}$$

It follows that

$$0 \leqq \sum_{n \neq M} \gamma_{n,\sigma}((cn + g(n/\tau + 1/\sigma^2)) - (cM + g(M/\tau + 1/\sigma^2)) \leqq (1/\sigma^2)(-g'(M/\tau)). \tag{9.36}$$

By (9.31) if $\sigma > K_2$, each term of the sum in (9.36) is non-negative. Using (9.31) and (9.36) it follows that

$$\text{if} \quad n \geqq 1, \quad n \neq M \quad \text{then} \quad \lim_{\sigma\to\infty} \sigma\gamma_{n,\sigma} = 0. \tag{9.37}$$

By Fatou's lemma, K_3 appropriately chosen,

$$\liminf_{\sigma\to\infty} \sigma\gamma_{n,\sigma} \geqq K_3 \int \cdots \int \liminf_{\sigma\to\infty} \beta_n(x_1, \cdots, x_n) \cdot \{\exp(-1/2) \sum (x_i - \theta)^2\}\{\exp(-1/2\sigma^2)\theta^2\}\, d\theta \prod_{i=1}^{n} dx_i. \tag{9.38}$$

It follows that

(9.39) if $n \geqq 1$, $n \neq M$, and if $\liminf_{\sigma\to\infty} \sigma\gamma_{n,\sigma} = 0$ then $\beta_n(\cdot) = 0$ except on a set on n-dimensional Lebesgue measure zero.

Therefore (9.36) and (9.39) together imply the sequential procedure η takes M observations with probability one. That is,

(9.40) $\beta_M(\cdot) = 1$ except on a set of M dimensional Lebesgue measure zero.

The proof of Theorem 8 in the sequential case now follows from the proof in the fixed sample size case.

The result we have obtained answers a question left open by Blyth, op. cit. Blyth proves admissibility of δ within the class of sequential procedures having continuous risk function. It follows from our argument that if η is as good as δ then η cannot take more that $M + 1$ observations even if the restriction (9.30) is removed. Therefore $R(\eta, \cdot)$ must be continuous. ($W(\cdot)$ convex is not needed for the proof of (9.40). The hypotheses of Blyth are sufficient here.)

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