CLASSICAL STATISTICAL ANALYSIS BASED ON A CERTAIN MULTIVARIATE COMPLEX GAUSSIAN DISTRIBUTION

By C. G. Khatri[1]

Gujarat University, Ahmedabad

1. Introduction and summary. N. R. Goodman [4] has discussed some aspects of the complex multivariate normal distribution, in particular, the analogue of the Wishart distribution and of multiple and partial correlations. We shall obtain maximum likelihood estimates for certain parameters and likelihood ratio tests of certain hypotheses arising in the study of such complex multivariate normal distributions. Although the general principles involved in the derivation of the distribution of the associated statistics are well known to mathematicians working on group representations (see, for example, Gelfand and Naimark [3], p. 24), it has seemed desirable to derive the needed results on this type in a way parallel to one method of obtaining the distributions of real multivariate analysis (Deemer and Olkin [2], Olkin [6] and Roy [7]). With the help of the results derived, we can handle the complex variates in the same manner as we do for real variates in the case of Gaussian distributions. Moreover, it can be noted that for every distributional result of classical multivariate Gaussian statistical analysis obtainable in closed (explicit) form, the counterpart analysis for complex Gaussian is also obtainable in closed (explicit) form with necessary changes.

It may be pointed out that the non-central distributions in this connection were derived independently by A. T. James [5] with the help of zonal polynomials of hermitian matrices, but we feel that sometimes the derivation of distributions with the help of Jacobian transformations may be useful.

2. Notations, integral and Jacobian transformations.

(*2.1*) *Notations.* Matrices will be denoted by bold face capital letters. The $p \times p$ identity matrix will be denoted by $\mathbf{I}_p$ or simply by $\mathbf{I}$, and any zero matrix by $\mathbf{0}$. The complex conjugate of a matrix $\mathbf{A}$ will be denoted by $\bar{\mathbf{A}}$, and the conjugate transpose by $\bar{\mathbf{A}}'$. A $p \times n$ matrix $\mathbf{U}$ will be said to be semi-unitary if $\mathbf{U}\bar{\mathbf{U}}' = \mathbf{I}_p$ for $p < n$ or $\bar{\mathbf{U}}'\mathbf{U} = \mathbf{I}_n$ for $n < p$. $\mathbf{A}^*$ will denote the differential of $\mathbf{A}$, that is, the matrix whose elements are the differentials of the elements of $\mathbf{A}$. For a matrix $\mathbf{A}$, det $\mathbf{A}$ or $|\mathbf{A}|$ will denote the determinant of $\mathbf{A}$ but, for a complex number a, $|a|$ will denote the absolute value of a. The notation $d\mathbf{A}$ will be used for the volume element commonly associated with $\mathbf{A}$. For example if $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$ with $\mathbf{A}_1$ and $\mathbf{A}_2$ real and $i = (-1)^{\frac{1}{2}}$, $d\mathbf{A} = d\mathbf{A}_1\, d\mathbf{A}_2$, and, if $\mathbf{A}$ is hermitian, $d\mathbf{A}_1 = \prod_k \prod_{j \geq k} da_{jk,1}$ and $d\mathbf{A}_2 = \prod_k \prod_{j>k} da_{jk,2}$. The Jacobian of the transformation $\mathbf{A} = \mathbf{F}\,(\mathbf{W})$ will be denoted by

$$J(\mathbf{A};\, \mathbf{W}) = \partial(\mathbf{A})/\partial(\mathbf{W}) = \partial(\mathbf{A}_1,\, \mathbf{A}_2)/\partial(\mathbf{W}_1,\, \mathbf{W}_2) = J(\mathbf{A}_1,\, \mathbf{A}_2;\, \mathbf{W}_1,\, \mathbf{W}_2).$$

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[1] Now at University of North Carolina on leave from Gujarat University.

$\int_{g(Z)=A} f(Z)\, dZ$ means a function h such that for any (open) set S in the space of $A = g(Z)$,

$$\int_S h(A)\, dA = \int_{[Z:g(Z)\epsilon S]} f(Z)\, dZ.$$

The density function of X: $r \times s$ having the complex multivariate normal distribution will be denoted by

$$CN(X: r \times s, \mu: r \times s, \Sigma: r \times r) = \Pi^{-rs}|\Sigma|^{-s} \exp[-\operatorname{tr} \Sigma^{-1}(X - \mu)(\overline{X - \mu})'],$$

where Σ is hermitian positive definite (hpd), and the density function of S: $r \times r$ distributed as complex Wishart will be denoted by $CW(S; r, n, \Sigma: r \times r) = \{\Gamma_r(n)\}^{-1}|\Sigma|^{-n}|S|^{n-r} \exp[-\operatorname{tr} \Sigma^{-1}S]$, where $\Gamma_r(n) = \Pi^{\frac{1}{2}r(r-1)}\{\prod_{j=1}^{r} \Gamma(n - j + 1)\}$ and $\Sigma: r \times r$ is hpd. If Σ is hpd, we can find a non-singular hermitian matrix P such that $\Sigma = P^2$, and for this case P will be denoted by $\Sigma^{\frac{1}{2}}$.

(*2.2*) (i) If $A: n \times p$ and $B: p \times m$, then $(AB)^* = A^*B + AB^*$.

(ii) If A is a hermitian matrix, then $(|A|)^* = \operatorname{tr}(\text{adj. } A)A^*$ and hence if $|A| > 0$, then $(\log |A|)^* = \operatorname{tr} A^{-1}A^*$.

(iii) $(A^{-1})^* = -A^{-1}A^*A^{-1}$ if A is nonsingular.

(iv) $U^*\bar{U}'$ or $\bar{U}'U^*$ is a skew hermitian matrix if U is a unitary matrix.

Proof. (i) follows from the definition of the differential of a matrix.

(ii) Since A is hermitian, $A = A_1 + iA_2$ gives A_1 a real symmetric matrix and A_2 a real skew-symmetric matrix, with $a_{jk} = a_{jk,1} + ia_{jk,2}$ and $\bar{a}_{jk} = a_{kj}$. Then

$$\begin{aligned}(\partial/\partial a_{jk,1})|A| &= (\text{cofactor of } a_{jj}) \text{ in } A & \text{if } j = k\\ &= (\text{cofactor of } a_{jk} + \text{cofactor of } a_{kj}) \text{ in } A & \text{if } j \neq k,\end{aligned}$$

and

$$(\partial/\partial a_{jk,2})|A| = i(\text{cofactor of } a_{jk} - \text{cofactor of } a_{kj}) \text{ in } A \quad \text{if } j \neq k.$$

Hence, it is easy to verify that $(|A|)^* = \operatorname{tr}(\text{adjoint } A)A^*$. The other part of (ii) is immediate.

(iii) and (iv) are obtained easily with the help of (i) and $A^{-1}A = I$, $\bar{U}'U = U\bar{U}' = I$.

(*2.3*) The Jacobian of the transformation $Z = GWH$ where $Z: p \times n$ and $W: p \times n$ are complex random matrices, and $G: p \times p$ and $H: n \times n$ are non-singular matrices is

$$J(Z; W) = |(\det G)|^{2n}|(\det H)|^{2p} = |G\bar{G}'|^n|H\bar{H}'|^p.$$

[For Z, W, G and H real matrices, $J(Z; W) = |G|^n|H|^p$.]

Proof. Let $G = G_1 + iG_2$, $H = H_1 + iH_2$, $Z = Z_1 + iZ_2 = GV$, $V = V_1 + iV_2 = WH$ and $W = W_1 + iW_2$. Then the transformation is

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} G_1 & -G_2 \\ G_2 & G_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \text{and} \quad (V_1\, V_2) = (W_1\, W_2) \begin{pmatrix} H_1 & H_2 \\ -H_2 & H_1 \end{pmatrix}.$$

Hence, $J(Z; W) = J(Z; V)J(V; W) = |G\bar{G}'|^n|H\bar{H}'|^p$, for the theorem (2.5) of

Goodman [4] gives

$$|(\det \mathbf{G})|^2 = |\mathbf{G}\bar{\mathbf{G}}'| = \begin{vmatrix} \mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{G}_2 & \mathbf{G}_1 \end{vmatrix} = \begin{vmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ -\mathbf{G}_2 & \mathbf{G}_1 \end{vmatrix}.$$

(*2.4*) Let $\mathbf{X}$: $p \times n$ be a complex random matrix, $\mathbf{M}$: $q \times n$ a given matrix of rank $q(\leqq n)$ and $\mathbf{A}$: $n \times n$ a hermitian positive definite matrix. Then for $n \geqq p + q$,

$$\int_{\substack{\mathbf{ZA}^{-1}\bar{\mathbf{Z}}'=\mathbf{S} \\ \mathbf{M}\bar{\mathbf{Z}}'=\bar{\mathbf{B}}'}} f(\mathbf{ZA}^{-1}\bar{\mathbf{Z}}', \mathbf{M}\bar{\mathbf{Z}}')\, d\mathbf{Z} = \Pi^{p(n-q)}\{\Gamma_p(n-q)\}^{-1}|\mathbf{A}|^p|\mathbf{MA}\bar{\mathbf{M}}'|^{-p}$$

$$\cdot f(\mathbf{S}, \bar{\mathbf{B}}')\, |\mathbf{S} - \mathbf{B}(\mathbf{MA}\bar{\mathbf{M}}')^{-1}\bar{\mathbf{B}}'|^{n-q-p}.$$

[If $\mathbf{Z}$, $\mathbf{M}$ and $\mathbf{A}$ are real matrices, then

$$\int_{\substack{\mathbf{ZA}^{-1}\mathbf{Z}'=\mathbf{S} \\ \mathbf{MZ}'=\mathbf{B}'}} f(\mathbf{ZA}^{-1}\mathbf{Z}', \mathbf{MZ}')\, d\mathbf{Z} = \Pi^{\frac{1}{2}p(n-q)-\frac{1}{4}p(p-1)}\left\{\prod_{j=1}^{p}\Gamma\left(\frac{n-q-j+1}{2}\right)\right\}^{-1}|\mathbf{A}|^{\frac{1}{2}p}$$

$$\cdot|\mathbf{MAM}'|^{-\frac{1}{2}p}f(\mathbf{S}, \mathbf{B}')\, |\mathbf{S} - \mathbf{B}(\mathbf{MAM}')^{-1}\mathbf{B}'|^{\frac{1}{2}(n-p-q-1)}.]$$

PROOF. Since $\mathbf{A}$ is hermitian positive definite (hpd), we can write it as $\mathbf{A} = \mathbf{T}^2$ where $\mathbf{T}$ is a hermitan matrix. Moreover $\mathbf{MT}$ is of rank q and so we can find a matrix $\mathbf{L}$: $(n - q) \times n$ such that

$$\mathbf{G} = \begin{pmatrix} \mathbf{MT} \\ \mathbf{L} \end{pmatrix} \quad \text{and} \quad \mathbf{G}\bar{\mathbf{G}}' = \begin{pmatrix} \mathbf{MA}\bar{\mathbf{M}}' & 0 \\ 0 & \mathbf{I}_{n-q} \end{pmatrix}.$$

Transforming $\mathbf{Z}$ by the relation $\mathbf{Z} = \mathbf{X}\bar{\mathbf{G}}'^{-1}\mathbf{T}$, we have, by (*2.2*), $J(\mathbf{Z}; \mathbf{X}) = |\mathbf{A}|^p|\mathbf{MA}\bar{\mathbf{M}}'|^{-p}$ and $\mathbf{S} = \mathbf{X}_1\bar{\mathbf{X}}_1' + \mathbf{B}(\mathbf{MA}\bar{\mathbf{M}}')^{-1}\bar{\mathbf{B}}'$, where $\mathbf{X} = (\mathbf{B}\ \mathbf{X}_1)$ and $\mathbf{B} = \mathbf{Z}\bar{\mathbf{M}}'$. Hence, we get

$$\int_{\substack{\mathbf{ZA}^{-1}\bar{\mathbf{Z}}'=\mathbf{S} \\ \mathbf{M}\bar{\mathbf{Z}}'=\bar{\mathbf{B}}'}} f(\mathbf{ZA}^{-1}\bar{\mathbf{Z}}', \mathbf{M}\bar{\mathbf{Z}}')\, d\mathbf{Z} = |\mathbf{A}|^p\, |\mathbf{MA}\bar{\mathbf{M}}'|^{-p} \int_{\mathbf{X}_1\bar{\mathbf{X}}_1'=\mathbf{R}} f_1(\mathbf{X}_1\bar{\mathbf{X}}_1')\, d\mathbf{X}_1,$$

where $\mathbf{X}_1$: $p \times (n - q)$, $\mathbf{R} = \mathbf{S} - \mathbf{B}(\mathbf{MA}\bar{\mathbf{M}}')^{-1}\bar{\mathbf{B}}'$ and $f_1(\mathbf{X}_1\bar{\mathbf{X}}_1') = f[\mathbf{X}_1\bar{\mathbf{X}}_1' + \mathbf{B}(\mathbf{MA}\bar{\mathbf{M}}')^{-1}\bar{\mathbf{B}}', \bar{\mathbf{B}}']$. Now applying the result corresponding to complex Wishart established by Goodman [4], namely, $\int_{\mathbf{X}_1\bar{\mathbf{X}}_1'=\mathbf{R}} d\mathbf{X}_1 = \Pi^{p(n-q)}\{\Gamma_p(n-q)\}^{-1}|\mathbf{R}|^{n-p-q}$, if $n - q \geqq p$, we get the result (*2.4*).

(*2.5*) The Jacobian of the transformation $\mathbf{T} = \mathbf{AG}$, where $\mathbf{T}$, $\mathbf{A}$ and $\mathbf{G}$ are lower triangular complex matrices of order p, is $J(\mathbf{T}; \mathbf{G}) = \prod_j |a_{jj}|^{2(p-j+1)}$, while $J(\mathbf{T}; \mathbf{A}) = \prod_j |g_{jj}|^{2j}$ if all the diagonal elements of $\mathbf{A}$, $\mathbf{G}$ and $\mathbf{T}$ are complex, or $J(\mathbf{T}; \mathbf{G}) = \prod_j a_{jj}^{2(p-j)+1}$, while $J(\mathbf{T}; \mathbf{A}) = \prod_j g_{jj}^{2j-1}$ if all the diagonal elements of $\mathbf{A}$, $\mathbf{G}$ and $\mathbf{T}$ are real. [If $\mathbf{A}$, $\mathbf{T}$ and $\mathbf{G}$ are real matrices, then $J(\mathbf{T}; \mathbf{G}) = \prod_j a_{jj}^{p-j+1}$, while $J(\mathbf{T}; \mathbf{A}) = \prod_j a_{jj}^j$.] Proof follows by noting the equations

$$t_{jk} = \prod_{q=k}^{j} a_{jq} g_{qk} \qquad \text{for } j \geqq k = 1, 2, \cdots, p.$$

(*2.6*) The Jacobian of the transformation $\mathbf{X} = \mathbf{TU}$, where $\mathbf{X}: p \times n$ is a complex random matrix, $\mathbf{T}$ is a lower triangular matrix and $\mathbf{U}: p \times n$ is at least a semi-unitary matrix ($n \geqq p$), is

$$J(\mathbf{X}; \mathbf{T}, \mathbf{U}) = 2^p \prod_j (t_{jj}^{2n-2j+1}) h_1(\mathbf{U})$$

if all the diagonal elements of $\mathbf{T}$ are positive and real and all the diagonal elements of $\mathbf{U}$ (i.e. u_{jj}, $j = 1, w, \cdots, p$) are complex, while $J(\mathbf{X}; \mathbf{T}, \mathbf{U}) = \prod_j (|t_{jj}|^{2(n-j)}) h_2(\mathbf{U})$ if all the diagonal elements of $\mathbf{T}$ are complex and all the diagonal elements of $\mathbf{U}$ are real. [When $\mathbf{X}$, $\mathbf{T}$ and $\mathbf{U}$ have real elements, then $J(\mathbf{X}; \mathbf{T}, \mathbf{U}) = 2^p \prod_j (t_{jj}^{n-j}) g(\mathbf{U})$.]

PROOF. We shall prove the result only for the latter case. Here, in $\mathbf{U}$, there are $2pn - p(p + 1)$ random variables. Let $\mathbf{V}$: $(n - p) \times n$ be a matrix of rank $(n - p)$ such that $\bar{\mathbf{Q}}' = (\bar{\mathbf{U}}' \ \bar{\mathbf{V}}')$ is a unitary matrix. Then $\mathbf{Q}^*\bar{\mathbf{Q}}'$ is skew hermitian giving n^2 different elements. Let $\mathbf{A} = \mathbf{U}^*\bar{\mathbf{Q}}'$. Then there are apparent $2pn - p^2$ different values in $\mathbf{A}$, but, in fact, there are only $2pn - p^2 - p$ random variables in $\mathbf{U}^*$. Hence, we shall assume that the imaginary parts in the diagonal elements of $\mathbf{A}$ (i.e. in $a_{jj} = a_{jj,1} + ia_{jj,2}$, the values of $a_{jj,2}$) can be determined in terms of other elements of $\mathbf{A}$. Now taking the differential of $\mathbf{X} = (\mathbf{T}, \mathbf{0})\mathbf{Q}$ and using (*2.2*), we get $\mathbf{T}^{-1}\mathbf{X}^*\bar{\mathbf{Q}}' = (\mathbf{T}^{-1}\mathbf{T}^* \ \mathbf{0}) + \mathbf{U}^*\bar{\mathbf{Q}}'$. Hence, if $\mathbf{W} = \mathbf{T}^{-1}\mathbf{X}^*\bar{\mathbf{Q}}'$, $\mathbf{G} = \mathbf{T}^{-1}\mathbf{T}^*$ and $\mathbf{A} = \mathbf{U}^*\bar{\mathbf{Q}}'$,

$$J(\mathbf{X}; \mathbf{T}, \mathbf{U}) = J(\mathbf{X}^*; \mathbf{T}^*, \mathbf{U}^*) = J(\mathbf{X}^*; \mathbf{W}) J(\mathbf{G}; \mathbf{T}^*) J(\mathbf{A}; \mathbf{U}^*)$$

and then using (*2.3*) for $J(\mathbf{X}^*; \mathbf{W})$, (*2.5*) for $J(\mathbf{G}; \mathbf{T}^*)$ and $h_2(\mathbf{U}) = J(\mathbf{A}; \mathbf{U}^*)$, we get the latter part of (*2.6*). Similarly the first part can be established.

(*2.7*) Let $\mathbf{U}: p \times n$ ($n \geqq p$) be at least semi-unitary. Then

$$\int_{\mathbf{U}\bar{\mathbf{U}}'=\mathbf{I}} h_1(\mathbf{U})\, d\mathbf{U} = \Pi^p \int_{\mathbf{U}\bar{\mathbf{U}}'=\mathbf{I}} h_2(\mathbf{U})\, d\mathbf{U} = \Pi^{pn}\{\Gamma_p(n)\}^{-1},$$

where $h_1(\mathbf{U})$ and $h_2(\mathbf{U})$ are defined in (*2.6*).

$$\left[\text{When } \mathbf{U} \text{ is real, } \int_{\mathbf{U}\mathbf{U}'=\mathbf{I}} g(\mathbf{U})\, d\mathbf{U} = \Pi^{\frac{1}{2}pn - \frac{1}{4}p(p+1)} \left\{\prod_{j=1}^{p} \Gamma\left(\frac{n-j+1}{2}\right)\right\}^{-1}.\right]$$

PROOF. Let $\mathbf{X}: p \times n$ be a random complex matrix, satisfying

$$\int \exp(-\operatorname{tr} \mathbf{X}\bar{\mathbf{X}}')\, d\mathbf{X} = \Pi^{pn}.$$

Applying the first part on the transformation given in (*2.6*), we get

$$2^p \int_{\mathbf{T}} \prod_j (t_{jj}^{2n-2j+1}) \exp[-\operatorname{tr} \mathbf{T}\bar{\mathbf{T}}']\, d\mathbf{T} \int_{\mathbf{U}\bar{\mathbf{U}}'=\mathbf{I}} h_1(\mathbf{U})\, d\mathbf{U} = \Pi^{pn}.$$

First, noting $\operatorname{tr} \mathbf{T}\bar{\mathbf{T}}' = \sum_{j=2}^{p} \sum_{k=1}^{j-1} (t_{jk,1}^2 + t_{jk,2}^2) + \sum_{j=1}^{p} t_{jj}^2$, and then, integrating over $\mathbf{T}$, we get the first part of (*2.7*). Similarly, the second part can be proved.

(*2.8*) The Jacobian of the transformation $\mathbf{S} = \mathbf{H}\mathbf{R}\bar{\mathbf{H}}'$, where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \bar{\mathbf{S}}'_{12} & \mathbf{S}_{22} \end{pmatrix} : p \times p \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \bar{\mathbf{R}}'_{12} & \mathbf{R}_{22} \end{pmatrix} : p \times p$$

are hermitian positive (semi-) definite matrices of rank $r(\leqq p)$ in random variables, $\mathbf{S}_{11} : r \times r$ nonsingular and $\mathbf{S}_{12} : r \times (p - r)$, and $\mathbf{R}_{11} : r \times r$ nonsingular and $\mathbf{R}_{12} : r \times (p - r)$, and $\bar{\mathbf{H}}' = (\bar{\mathbf{H}}_1' \, \bar{\mathbf{H}}_2')$ is a non-singular matrix with $\mathbf{H}_1 : r \times p$, is

$$J(\mathbf{S}; \mathbf{R}) = |\mathbf{H}_1\mathbf{R}\bar{\mathbf{H}}_1'|^{(p-r)}|\mathbf{H}\bar{\mathbf{H}}'|^{r}/|R_{11}|^{(p-r)}.$$

[When $\mathbf{S}$, $\mathbf{R}$ and $\mathbf{H}$ are real matrices and $\mathbf{S}$ and $\mathbf{R}$ are symmetric, then, $J(\mathbf{S}; \mathbf{R}) = |\mathbf{H}_1\mathbf{R}\mathbf{H}_1'|^{\frac{1}{2}(p-r)}|\mathbf{H}|^{r+1}/|\mathbf{R}_{11}|^{\frac{1}{2}(p-r)}$.]

Proof. Since $\mathbf{S}_{11} : r \times r$ and $\mathbf{R}_{11} : r \times r$ are hermitian positive definite, let us write $\mathbf{S}_{11} = \mathbf{T}_1\bar{\mathbf{T}}_1'$ and $\mathbf{R}_{11} = \mathbf{T}_3\bar{\mathbf{T}}_3'$ where $\mathbf{T}_1 : r \times r$ and $\mathbf{T}_3 : r \times r$ are lower triangular matrices with positive and real diagonal elements. Let $\mathbf{S}_{12} = \mathbf{T}_1\bar{\mathbf{T}}_2'$ and $\mathbf{R}_{12} = \mathbf{T}_3\bar{\mathbf{T}}_4'$. Then $\mathbf{S} = \mathbf{H}\mathbf{R}\bar{\mathbf{H}}'$ becomes

$$\begin{pmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{pmatrix} (\bar{\mathbf{T}}_1' \, \bar{\mathbf{T}}_2') = \mathbf{H} \begin{pmatrix} \mathbf{T}_3 \\ \mathbf{T}_4 \end{pmatrix} (\bar{\mathbf{T}}_3' \, \bar{\mathbf{T}}_4')\bar{\mathbf{H}}'.$$

Hence, we make the following successive transformations: $\mathbf{S}_{11} = \mathbf{T}_1\bar{\mathbf{T}}_1'$, $\mathbf{S}_{12} = \mathbf{T}_1\bar{\mathbf{T}}_2'$, $\mathbf{X}_1 = \mathbf{T}_1\mathbf{U}_1$, $\mathbf{X}_2 = \mathbf{T}_2\mathbf{U}_1$, $(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{H}(\mathbf{Y}_1\mathbf{Y}_2)$, $\mathbf{Y}_1 = \mathbf{T}_3\mathbf{U}_2$, $\mathbf{Y}_3 = \mathbf{T}_4\mathbf{U}_2$, $\mathbf{T}_3\bar{\mathbf{T}}_3' = \mathbf{R}_{11}$, $\mathbf{T}_3\bar{\mathbf{T}}_4' = \mathbf{R}_{12}$ and $\mathbf{U}_2 = \mathbf{U}\mathbf{U}_1$, where $\mathbf{U}_1 : r \times r$ and $\mathbf{U}_2 : r \times r$ are random unitary matrices, while $\mathbf{U}: r \times r$ is a unitary matrix. Hence

$$J(\mathbf{S}; \mathbf{R}) = J(\mathbf{S}_{11} ; \mathbf{T}_1)J(\mathbf{S}_{12} ; \mathbf{T}_2)J(\mathbf{T}_1, \mathbf{T}_2, \mathbf{U}_1 ; \mathbf{X}_1, \mathbf{X}_2)J(\mathbf{X}_1, \mathbf{X}_2 ; \mathbf{Y}_1, \mathbf{Y}_2)$$
$$\cdot J(\mathbf{Y}_1, \mathbf{Y}_2 ; \mathbf{T}_3, \mathbf{T}_4, \mathbf{U}_2)J(\mathbf{T}_3 ; \mathbf{R}_{11})J(\mathbf{T}_4 ; \mathbf{R}_{12})J(\mathbf{U}_2 ; \mathbf{U}_1).$$

Applying $J(\mathbf{S}_{11} ; \mathbf{T}_1) = 2^r\prod_{j=1}^{r} t_{jj,1}^{2r-2j+1}$ (see Goodman [4]), (*2.3*) and (*2.6*), we get

$$(2.8.1) \quad J(\mathbf{S}; \mathbf{R}) = |\mathbf{H}\bar{\mathbf{H}}'|^{r}|\mathbf{H}_1\mathbf{R}\bar{\mathbf{H}}_1'|^{(p-r)}\,|\mathbf{R}_{11}|^{-(p-r)}h_1(\mathbf{U}_2)\{h_1(\mathbf{U}_1)\}^{-1}J(\mathbf{U}_2 ; \mathbf{U}_1).$$

Now, we note that $\mathbf{U}_2 = \mathbf{U}\mathbf{U}_1$, i.e. $\mathbf{A}_2 = \mathbf{U}\mathbf{A}_1\bar{\mathbf{U}}'$ if $\mathbf{A}_2 = \mathbf{U}_2^*\bar{\mathbf{U}}_2'$ and $\mathbf{A}_1 = \mathbf{U}_1^*\bar{\mathbf{U}}_1'$. Since $\mathbf{A}_2$ and $\mathbf{A}_1$ are skew-hermitian random matrices, we get $i\mathbf{A}_2$ and $i\mathbf{A}_1$ the hermitian random matrices. With this notation,

$$(2.8.2) \quad \begin{aligned} J(\mathbf{U}_2 ; \mathbf{U}_1) &= J(\mathbf{U}_2^*; \mathbf{U}_1^*) = J(\mathbf{U}_2^*; \mathbf{A}_2)J(\mathbf{A}_2 ; \mathbf{A}_1)J(\mathbf{A}_1 ; \mathbf{U}_1^*) \\ &= \{h_1(\mathbf{U}_2)\}^{-1}\{h_1(\mathbf{U}_1)\} \end{aligned}$$

for $J(i\mathbf{A}_2 ; i\mathbf{A}_1) = 1$, (see Goodman [4]).

Using (2.8.2) in (2.8.1), we get (*2.8*).

(*2.9*) The Jacobian of the transformation $\mathbf{S} = \mathbf{U}\mathbf{D}_\lambda\bar{\mathbf{U}}'$, where $\mathbf{S}$ is a hermitian positive definite, $\mathbf{D}_\lambda$ is a diagonal matrix with real and distinct diagonal elements and $\mathbf{U}$ is a unitary matrix with real diagonal elements, is

$$J(\mathbf{S}; \mathbf{U}, \mathbf{D}_\lambda) = \prod_{k=1}^{p-1} \prod_{j=k+1}^{p} (\lambda_j - \lambda_k)^2 h_2(\mathbf{U}).$$

[If $\mathbf{S}$ and $\mathbf{U}$ are real matrices, then $J(\mathbf{S}; \mathbf{D}_\lambda, \mathbf{U}) = \prod_{k=1}^{p-1} \prod_{j=k+1}^{p} |\lambda_j - \lambda_k| g(\mathbf{U})$.]

PROOF. Taking differential of $\mathbf{S}$, we get $\bar{\mathbf{U}}'\mathbf{S}^*\mathbf{U} = \bar{\mathbf{U}}'\mathbf{U}^*\mathbf{D}_\lambda + \mathbf{D}_\lambda^* + \mathbf{D}_\lambda \bar{\mathbf{U}}'^*\mathbf{U}$. Let $\bar{\mathbf{A}}' = \bar{\mathbf{U}}'\mathbf{U}^* = -\mathbf{A}$ for $\mathbf{A}$ is a skew-hermitian matrix, and $\mathbf{W} = \bar{\mathbf{U}}'\mathbf{S}^*\mathbf{U}$. Then

$$\begin{aligned} &\mathbf{W} = \mathbf{D}_\lambda \mathbf{A} - \mathbf{A}\mathbf{D}_\lambda + \mathbf{D}_\lambda^* \quad \text{or} \quad w_{jj,1} = \lambda_j^*, \\ &w_{jk,t} = (\lambda_j - \lambda_k) a_{jk,t} \text{ for } t = 1, 2 \quad \text{and} \quad p \geqq j > k = 1, 2, \cdots, p-1. \end{aligned} \tag{2.9.1}$$

Now,

$$\begin{aligned} J(\mathbf{S}; \mathbf{U}, \mathbf{D}_\lambda) &= J(\mathbf{S}^*; \mathbf{U}^*, \mathbf{D}_\lambda) = J(\mathbf{S}^*; \mathbf{W}) J(\mathbf{W}; \mathbf{A}, \mathbf{D}_\lambda^*) J(\mathbf{A}; \mathbf{U}^*) \\ &= J(\mathbf{W}; \mathbf{A}, \mathbf{D}_\lambda^*) h_2(\mathbf{U}) = [\prod_{k=1}^{p-1} \prod_{j=k+1}^{p} (\lambda_j - \lambda_k)^2] h_2(\mathbf{U}). \end{aligned}$$

3. Maximum likelihood estimates of certain complex matrices.

(3.1) Point estimation. Let $\mathbf{S}: p \times n$ be a complex random matrix whose density function is

$$L = CN(\mathbf{Z}; \boldsymbol{\mu}\mathbf{M}, \boldsymbol{\Sigma}), \tag{3.1.1}$$

where $\boldsymbol{\Sigma}: p \times p$ is hpd, $\boldsymbol{\mu}: p \times q$ is a complex matrix and $\mathbf{M}: q \times n$ is either a given complex matrix of rank $q(\leqq n)$, or has a distribution, not depending on the parameters $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$. To derive the maximum likelihood estimates of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$, with the help of *(2.2)*, we note that

$$\begin{aligned} (\log L)^* = -n \operatorname{tr} (\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}^*) &+ \operatorname{tr} [\boldsymbol{\Sigma}^{-1}(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})\overline{(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}^*] \\ &+ 2 \text{ R.P. } \{\operatorname{tr} [\boldsymbol{\Sigma}^{-1}(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})\bar{\mathbf{M}}'\bar{\boldsymbol{\mu}}'^*]\}, \end{aligned} \tag{3.1.2}$$

where R.P. $(\cdot)$ = real part of $(\cdot)$.

Moreover, we note that if $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1 + i\boldsymbol{\Sigma}_2$ and $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + i\boldsymbol{\mu}_2$, then

$$\begin{aligned} \partial \log L / \partial \sigma_{jk,t} &= \text{coefficient of } \sigma_{jk,t}^* \text{ in } (\log L)^*; \\ &\qquad j, k = 1, 2, \cdots, p; t = 1, 2 \\ &= 0 \text{ for } j = k \text{ and } t = 2, \end{aligned} \tag{3.1.3a}$$

and

$$\frac{\partial \log L}{\partial \mu_{jk,t}} = \text{coefficient of } \mu_{jk,t}^* \text{ in } (\log L)^*; \begin{cases} j = 1, 2, \cdots, p \\ k = 1, 2, \cdots, q \text{ and} \\ t = 1, 2. \end{cases} \tag{3.1.3b}$$

Hence if

$$\frac{\partial \log L}{\partial \boldsymbol{\Sigma}_t} = \left(\frac{\partial \log L}{\partial \sigma_{jk,t}} \epsilon_{jk} \right) \quad \text{and} \quad \frac{\partial \log L}{\partial \boldsymbol{\mu}_t} = \left(\frac{\partial \log L}{\partial \mu_{jk,t}} \right); \quad (t = 1, 2), \tag{3.1.4}$$

where $\epsilon_{jk} = 1$ if $j = k$ and $= \frac{1}{2}$ if $j \neq k$, then from (3.1.2), we get

$$(3.1.5a) \qquad \frac{\partial \log L}{\partial \mathbf{\Sigma}_1} - i\,\frac{\partial \log L}{\partial \mathbf{\Sigma}_2} = \mathbf{\Sigma}^{-1}(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})(\overline{\mathbf{Z} - \boldsymbol{\mu}\mathbf{M}})'\mathbf{\Sigma}^{-1} - n\mathbf{\Sigma}^{-1}$$

and

$$(3.1.5b) \qquad \frac{\partial \log L}{\partial \boldsymbol{\mu}_1} + i\,\frac{\partial \log L}{\partial \boldsymbol{\mu}_2} = \mathbf{\Sigma}^{-1}(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})\overline{\mathbf{M}}'.$$

From (3.1.5), we get the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ as

$$(3.1.6a) \qquad \boldsymbol{\beta} = \mathbf{Z}\overline{\mathbf{M}}'(\mathbf{M}\overline{\mathbf{M}}')^{-1}$$

and

$$(3.1.6b) \qquad \boldsymbol{\psi} = n^{-1}[\mathbf{Z}\overline{\mathbf{Z}}' - \boldsymbol{\beta}(\mathbf{M}\overline{\mathbf{M}}')\overline{\boldsymbol{\beta}}'] = n^{-1}\mathbf{Z}[\mathbf{I} - \overline{\mathbf{M}}'(\mathbf{M}\overline{\mathbf{M}}')^{-1}\mathbf{M}]\overline{\mathbf{Z}}'.$$

It is easy to verify that (3.1.1) is equivalent to

$$(3.1.7) \qquad L = \Pi^{-pn}|\mathbf{\Sigma}|^{-n} \exp\,[-n \operatorname{tr} \mathbf{\Sigma}^{-1}\boldsymbol{\psi} - \operatorname{tr} \mathbf{\Sigma}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu})(\mathbf{M}\overline{\mathbf{M}}')(\overline{\boldsymbol{\beta} - \boldsymbol{\mu}})']$$

and so applying (*2.4*) for $\mathbf{Z}\overline{\mathbf{Z}}'$ and $\boldsymbol{\beta} = \mathbf{Z}\overline{\mathbf{M}}'(\mathbf{M}\overline{\mathbf{M}}')^{-1}$, we can easily verify that $\boldsymbol{\beta}$ and $n\boldsymbol{\psi}$ are independently distributed, the distribution of $n\boldsymbol{\psi}$ is $CW(n\boldsymbol{\psi};\, p,\, n - q,\, \mathbf{\Sigma})$ and $\boldsymbol{\beta}$ has the density function

$$(3.1.8) \qquad \Pi^{-pq}|\mathbf{\Sigma}|^{-q}|\mathbf{M}\overline{\mathbf{M}}'|^{p} \exp\,[-\operatorname{tr} \mathbf{\Sigma}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu})(\mathbf{M}\overline{\mathbf{M}}')(\overline{\boldsymbol{\beta} - \boldsymbol{\mu}})'].$$

(*3.2*) *Likelihood ratio statistic.* To test the null hypothesis $H_0(\boldsymbol{\mu} = \mathbf{0})$ against $H(\boldsymbol{\mu} \neq \mathbf{0})$, we apply likelihood ratio method. Here we have first to determine the maximum value of L under H_0 (denoted as $\max_{H_0} L$) and under H (i.e. $\max_H L$).

It is easy to verify by the technique given in (*3.1*) that

$$(3.2.1) \qquad (\max_{H_0} L) = \Pi^{-pn}|n^{-1}(\mathbf{Z}\overline{\mathbf{Z}}')|^{-n} \exp\,(-np)$$

and

$$(3.2.2) \qquad (\max_H L) = \Pi^{-pn}|\boldsymbol{\psi}|^{-n} \exp\,(-np).$$

Hence,

$$(3.2.3) \qquad \Lambda = \{(\max_{H_0} L)/(\max_H L)\}^{1/n} = |\boldsymbol{\psi}|/|\boldsymbol{\psi} + n^{-1}\boldsymbol{\beta}(\mathbf{M}\overline{\mathbf{M}}')\overline{\boldsymbol{\beta}}'|.$$

The likelihood ratio criterion gives us the critical region

$$(3.2.4) \qquad \Lambda < \lambda_1,$$

where $\Pr(\Lambda \leqq \lambda_1 \mid H_0) = \alpha$. The distribution of Λ under H_0 and that for the large values of n are given in Sections (*5.2*) and (*5.3*). Since (3.2.4) is parallel to the likelihood ratio criterion for real Gaussian variables, we give below two other parallel critical regions:

$$(3.2.5) \qquad \xi = n^{-1} \operatorname{tr} \{\boldsymbol{\beta}(\mathbf{M}\overline{\mathbf{M}}')\overline{\boldsymbol{\beta}}'\boldsymbol{\psi}^{-1}\} > \lambda_2$$

and

(3.2.6) $$\eta = \text{max ch root of } \{n^{-1}\boldsymbol{\beta}(\mathbf{M}\bar{\mathbf{M}}')\bar{\boldsymbol{\beta}}'\boldsymbol{\psi}^{-1}\} > \lambda_3,$$

where $\Pr(\xi > \lambda_2 \mid H_0) = \Pr(\eta > \lambda_3 \mid H_0) = \alpha$. The joint distribution of the characteristic roots of $n^{-1}\boldsymbol{\beta}(\mathbf{M}\bar{\mathbf{M}}')\bar{\boldsymbol{\beta}}'\boldsymbol{\psi}^{-1}$ are given in Section 7. It may be noted that when $q = 1$, $\xi = \eta = (1 - \Lambda)/\Lambda$ is distributed as non-central beta of second kind $(p, n - 1)$ with $(\mathbf{M}\bar{\mathbf{M}}')n^{-1}\, \bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ as the non-central parameter (refer to Section 5). Section 6 deals with the distribution $\boldsymbol{\beta}(\mathbf{M}\bar{\mathbf{M}}')\bar{\boldsymbol{\beta}}'$.

(*3.3*) Let us write $\boldsymbol{\mu} = (\boldsymbol{\mu}_1\ \boldsymbol{\mu}_2)$, $\bar{\mathbf{M}}' = (\bar{\mathbf{M}}_1'\ \bar{\mathbf{M}}_2')$ and

$$\mathbf{A} = \mathbf{M}\bar{\mathbf{M}}' = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \bar{\mathbf{A}}_{12}' & \mathbf{A}_{22} \end{pmatrix},$$

where $\boldsymbol{\mu}_1 : p \times q_1$, $\boldsymbol{\mu}_2 : p \times q_2$, $q_2 = q - q_1$, $\mathbf{M}_1 : p \times q_1$, $\mathbf{M}_2 : p \times q_2$, $\mathbf{A}_{11} : q_1 \times q_1$, $\mathbf{A}_{12} : q_1 \times q_2$ and $\mathbf{A}_{22} : q_2 \times q_2$. Here, we consider the null hypothesis $H_0(\boldsymbol{\mu}_1 = \mathbf{0})$ against the alternative $H(\boldsymbol{\mu}_1 \neq \mathbf{0})$. As before, we obtain

(3.3.1) $$(\max_{H_0} L) = \Pi^{-pn}|n^{-1}\boldsymbol{\psi}_w|^{-n} \exp(-np),$$

where $\boldsymbol{\psi}_w = n^{-1}[\mathbf{Z}\bar{\mathbf{Z}}' - \boldsymbol{\beta}_{2,0}\mathbf{A}_{22}\bar{\boldsymbol{\beta}}_{2,0}']$, $\boldsymbol{\beta}_{2,0} = \mathbf{Z}\bar{\mathbf{M}}_2'\mathbf{A}_{22}^{-1}$. Now if

$$\boldsymbol{\beta}_{1,0} = (\mathbf{Z}\bar{\mathbf{M}}_1' - \boldsymbol{\beta}_{2,0}\bar{\mathbf{A}}_{12}')\mathbf{A}_{11.2}^{-1},$$

then $E(\boldsymbol{\beta}_{1,0}) = \boldsymbol{\mu}_1$ and

(3.3.2) $$\boldsymbol{\psi}_w = \boldsymbol{\psi} + n^{-1}\boldsymbol{\beta}_{1,0}\mathbf{A}_{11.2}\bar{\boldsymbol{\beta}}_{1.0}',$$

where $\boldsymbol{\psi}$ is defined in (3.1.6b) and $\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\bar{\mathbf{A}}_{12}'$.

Hence the critical region for the likelihood ratio criterion is

(3.3.3) $$\Lambda_1 = |\boldsymbol{\psi}|/|\boldsymbol{\psi} + n^{-1}\boldsymbol{\beta}_{1,0}\mathbf{A}_{11.2}\bar{\boldsymbol{\beta}}_{1.0}'| < \lambda_1,$$

where $\Pr(\Lambda_1 < \lambda_1 \mid H_0) = \alpha$. This is parallel to real variates given by Anderson ([1], p. 211). The other two criteria are

(3.3.4) $$\xi_1 = \operatorname{tr}[\boldsymbol{\beta}_{1,0}\mathbf{A}_{11.2}\bar{\boldsymbol{\beta}}_{1,0}'(n\boldsymbol{\psi})^{-1}] > \lambda_2$$

and

(3.3.5) $$\eta_1 = \text{max ch root of } [\boldsymbol{\beta}_{1,0}\mathbf{A}_{11.2}\bar{\boldsymbol{\beta}}_{1,0}'(n\boldsymbol{\psi})^{-1}] > \lambda_3,$$

where $\Pr(\xi_1 > \lambda_2 \mid H_0) = \Pr(\eta_1 > \lambda_3 \mid H_0) = \alpha$.

Under the null hypothesis, the distributions of Λ_1, ξ_1 and η_1 are the same as those of Λ, ξ and η of Section (*3.2*) by changing q to q_1.

4. Non-central complex Wishart distribution.

(*4.1*) Let $\mathbf{Z}: p \times n$, a complex random matrix, have a density function,

(4.1.1) $$CN(\mathbf{Z}; \boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\Sigma}: p \times p$ is hpd and $\boldsymbol{\mu}: p \times n$ is of rank $t \leqq p \leqq n$. Since the rank of

$\Sigma^{-1}\mu$ is t, we can find $\mathbf{U}: p \times t$, a semi-unitary matrix and $\mathbf{V}: n \times n$, a unitary matrix such that

$$\Sigma^{-1}\mu = \mathbf{U}(\mathbf{D}_\lambda\ \mathbf{0})\mathbf{V} \quad \text{and so} \quad \Sigma^{-1}\mu\bar{\mu}'\Sigma^{-1} = \mathbf{U}\mathbf{D}_\lambda^2\bar{\mathbf{U}}', \tag{4.1.2}$$

where $\mathbf{D}_\lambda : t \times t$ is a diagonal matrix with real diagonal elements.

Under the above conditions we are interested in deriving the distribution of $\mathbf{Z}\bar{\mathbf{Z}}' = \mathbf{S}: p \times p$. Let $\mathbf{Z} = \mathbf{YV}$. Then the Jacobian of the transformation is one and $\mathbf{Y}\bar{\mathbf{Y}}' = \mathbf{Z}\bar{\mathbf{Z}}' = \mathbf{S}$. Now, we apply (*2.4*) for getting the joint distribution of $\mathbf{S} = \mathbf{Y}\bar{\mathbf{Y}}'$ and $\bar{\mathbf{X}}_1' = (\mathbf{I}_t\ \mathbf{0})\bar{\mathbf{Y}}'$ and then integrating over $\mathbf{X}_1$, we get the density function of $\mathbf{S}$ as

$$\begin{aligned}\Pi^{-pt}|\Sigma|^{-n}\{\Gamma_p(n-t)\}^{-1}\int |\mathbf{S} - \mathbf{X}_1\bar{\mathbf{X}}_1'|^{n-t-p}& \\ \cdot \exp\,[-\operatorname{tr}\Sigma^{-1}(\mathbf{S} + \mu\bar{\mu}') + 2\text{ R.P. tr }\mathbf{U}\mathbf{D}_\lambda\bar{\mathbf{X}}_1']\,d\mathbf{X}_1&,\end{aligned} \tag{4.1.3}$$

where $\mathbf{S}$ is hpd and integration over $\mathbf{X}_1$ is such that $\mathbf{S} - \mathbf{X}_1\bar{\mathbf{X}}_1'$ is hpd. Note that we have used the condition $n \geqq p + t$, but when $\mathbf{X}_1 : p \times t$ will be completely integrated, then we shall only require the condition $n \geqq p$. Since $\mathbf{S}^{\frac{1}{2}}\mathbf{U}\mathbf{D}_\lambda$ is of rank t, we can find two unitary matrices $\mathbf{U}_1 : p \times p$ and $\mathbf{V}_1 : t \times t$ such that

$$\mathbf{S}^{\frac{1}{2}}\mathbf{U}\mathbf{D}_\lambda = \mathbf{U}_1\begin{pmatrix}\mathbf{D}_w \\ \mathbf{0}\end{pmatrix}\mathbf{V}_1, \tag{4.1.4}$$

where $\mathbf{D}_w : t \times t$ is a diagonal matrix with real diagonal elements w_j ($j = 1, 2, \cdots, t$) and w_j^2 are the nonzero ch roots of $(\Sigma^{-1}\mu\bar{\mu}'\Sigma^{-1}\mathbf{S})$. Let in (4.1.3), $\bar{\mathbf{X}}' = \mathbf{V}_1\bar{\mathbf{X}}_1'\mathbf{S}^{-\frac{1}{2}}\mathbf{U}_1$. Then (4.1.3) can be rewritten as

$$\begin{aligned}\Pi^{-pt}|\Sigma|^{-n}\{\Gamma_p(n-t)\}^{-1}|\mathbf{S}|^{n-p}\exp\,(-\operatorname{tr}\Sigma^{-1}\mathbf{S} - \operatorname{tr}\Sigma^{-1}\mu\bar{\mu}')& \\ \cdot\int |\mathbf{I}_p - \mathbf{X}\bar{\mathbf{X}}'|^{n-t-p}\exp\,[2\text{ R.P. tr }(\mathbf{D}_w\bar{\mathbf{X}}')]\,d\mathbf{X}&,\end{aligned} \tag{4.1.5}$$

where $\mathbf{S}$ is hpd, $\operatorname{tr}(\mathbf{D}_w\bar{\mathbf{X}}')$ means $\sum_{j=1}^t (\mathbf{D}_w\bar{\mathbf{X}}')_{jj}$, and $\mathbf{X}: p \times t$ is integrated with the condition that $\mathbf{I}_p - \mathbf{X}\bar{\mathbf{X}}'$ is hpd. (4.1.5) can be simplified, but we are not interested at this stage.

Particular cases

When $t = 0$, we have the distribution of $\mathbf{S} = \mathbf{Z}\bar{\mathbf{Z}}'$ as

$$CW(\mathbf{S}; p, n, \Sigma)\,d\mathbf{S}, \tag{4.1.6}$$

where $\mathbf{S}$ is hpd.

When $t = 1$, we have the distribution of $\mathbf{S} = \mathbf{Z}\bar{\mathbf{Z}}'$ as follows:

$$CW(\mathbf{S};\ p,\ n,\ \Sigma)\ \exp\ (-\alpha)\textstyle\sum_{j=0}^{\infty}\alpha_1^j\Gamma(n)/\{j!\,\Gamma(n+j)\}\ d\mathbf{S}, \tag{4.1.7}$$

where $\alpha_1 = \operatorname{tr}(\bar{\mu}'\Sigma^{-1}\mathbf{S}\Sigma^{-1}\mu)$, $\alpha = \operatorname{tr}(\bar{\mu}'\Sigma^{-1}\mu)$ and $\mathbf{S}$ is hpd.

(*4.2*) We give below the moments of $|\mathbf{S}|$ only for $t = 0$ and $t = 1$. When $t = 0$,

$$E|\mathbf{S}|^r = |\Sigma|^r\Gamma_p(n+r)/\Gamma_p(n) \quad \text{for} \quad r = 0, 1, 2, \cdots. \tag{4.2.1}$$

When $t = 1$,

$$E|\mathbf{S}|^r = |\boldsymbol{\Sigma}|^r\Gamma_p(n+r)\{\Gamma_p(n)\}^{-1} \cdot\exp(-\alpha)\sum_{j=0}^{\infty}\alpha^j\Gamma(n+r+j)\Gamma(n)/\{j!\,\Gamma(n+r)\Gamma(p+j)\}, \tag{4.2.2}$$

where $\alpha = \mathrm{tr}\,(\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$ and $r = 0, 1, 2, \cdots$.

5. Non-central complex multivariate Beta distribution.

(5.1) Let the joint density function of complex random matrices $\mathbf{X}: p \times q$ and $\mathbf{S}: p \times p$ be

$$CN(\mathbf{X};\boldsymbol{\mu},\boldsymbol{\Sigma})CW(\mathbf{S};p,n,\boldsymbol{\Sigma}). \tag{5.1.1}$$

In this section, we shall obtain the distribution $\mathbf{R} = \bar{\mathbf{X}}'\mathbf{S}^{-1}\mathbf{X}$ when (i) $q \leqq p$ and (ii) $q > p$.

(i) Let us consider first $q \leqq p$. Using the transformation $\mathbf{Y} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{X}$ and $\mathbf{S}_1 = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{S}\boldsymbol{\Sigma}^{-\frac{1}{2}}$, we get the joint density function of $\mathbf{Y}$ and $\mathbf{S}_1$ as

$$CN(\mathbf{Y};\boldsymbol{\nu},\mathbf{I})CW(\mathbf{S}_1;p,n,\mathbf{I}), \tag{5.1.2}$$

where $\boldsymbol{\nu} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}$ and $\mathbf{R} = \bar{\mathbf{X}}'\mathbf{S}^{-1}\mathbf{X} = \bar{\mathbf{Y}}'\mathbf{S}_1^{-1}\mathbf{Y}$. Since $q \leqq p$, $\bar{\mathbf{Y}}'\mathbf{Y}$ is hpd and $\mathbf{Y}(\bar{\mathbf{Y}}'\mathbf{Y})^{-\frac{1}{2}} = \boldsymbol{\Delta}_1 : p \times q$ is semi-unitary. Hence, we can have $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_1\,\boldsymbol{\Delta}_2): p \times p$ a unitary matrix. Then

$$\mathbf{R} = \bar{\mathbf{Y}}'\boldsymbol{\Delta}[\bar{\boldsymbol{\Delta}}'\mathbf{S}_1\boldsymbol{\Delta}]^{-1}\bar{\boldsymbol{\Delta}}'\mathbf{Y} = (\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}\mathbf{G}_{1.2}^{-1}(\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}, \tag{5.1.3}$$

where

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \bar{\mathbf{G}}_{12}' & \mathbf{G}_{22} \end{pmatrix} = \bar{\boldsymbol{\Delta}}'\mathbf{S}\boldsymbol{\Delta}$$

and $\mathbf{G}_{1.2} = (\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\bar{\mathbf{G}}_{12}'): q \times q$. Now

$J(\mathbf{S};\mathbf{R},\mathbf{G}_{12},\mathbf{G}_{22}) = J(\mathbf{S};\mathbf{G})J(\mathbf{G}_{11};\mathbf{G}_{1.2})J(\mathbf{G}_{1.2};\mathbf{R}) = J(\mathbf{G}_{1.2};\mathbf{R}) = |\bar{\mathbf{Y}}'\mathbf{Y}|^q|\mathbf{R}|^{-2q}$,

with the help of $\mathbf{G}_{1.2}^* = -(\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}\mathbf{R}^{-1}\mathbf{R}^*\mathbf{R}^{-1}(\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}$ and *(2.8)*. Integrating over $\mathbf{G}_{12}$ and $\mathbf{G}_{22}$, we get the joint density function of $\mathbf{Y}$ and $\mathbf{R}$ as

$$\Pi^{-pq}\{\Gamma_q(n+q-p)\}^{-1}|\bar{\mathbf{Y}}'\mathbf{Y}|^{n+q-p}|\mathbf{R}|^{-(n+2q-p)} \cdot\exp[-\mathrm{tr}\,\{\overline{(\mathbf{Y}-\boldsymbol{\nu})}'(\mathbf{Y}-\boldsymbol{\nu}) - \bar{\mathbf{Y}}'\mathbf{Y}\mathbf{R}^{-1}\}]. \tag{5.1.4}$$

Using the transformation $\mathbf{Z} = \mathbf{Y}(\mathbf{I} + \mathbf{R}^{-1}): p \times q$, the Jacobian is $J(\mathbf{Y};\mathbf{Z}) = |\mathbf{I} + \mathbf{R}^{-1}|^{-p}$ and integrating over $\mathbf{Z}$, we get the density function of $\mathbf{R}$ as

$$\{\Gamma_q(n+q-p)\}^{-1}|\mathbf{R}|^{p-q}|\mathbf{I}+\mathbf{R}|^{-(n+q)} \cdot\exp[-\mathrm{tr}\,\{\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\mathbf{I}+\mathbf{R})^{-1}\}]E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}, \tag{5.1.5}$$

where $\mathbf{R}$ is hpd and

$$E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p} = \int |\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}CN(\mathbf{Z}:p\times q;\boldsymbol{\nu}(\mathbf{I}+\mathbf{R}^{-1})^{-\frac{1}{2}},\mathbf{I})\,d\mathbf{Z}. \tag{5.1.6}$$

With the help of (5.1.5), we can obtain the density function of

$$\mathbf{V} = \bar{\mathbf{X}}'(\mathbf{S} + \mathbf{X}\bar{\mathbf{X}}')^{-1}\mathbf{X} = \mathbf{I} - (\mathbf{I} + \mathbf{R})^{-1},$$

for $J(\mathbf{R}; \mathbf{V}) = |\mathbf{I} - \mathbf{V}|^{-2q}$, as

$$\{\Gamma_q(n + q - p)\}^{-1}|\mathbf{V}|^{p-q}|\mathbf{I} - \mathbf{V}|^{n-p} \cdot \exp[-\operatorname{tr}\{\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\mathbf{I} - \mathbf{V})\}]E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}, \tag{5.1.7}$$

where $\mathbf{V}$ and $\mathbf{I} - \mathbf{V}$ are hpd and $E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}$ is the same as (5.1.6) by substituting $\boldsymbol{\nu}(\mathbf{I} + \mathbf{R}^{-1})^{-\frac{1}{2}}$ by $\boldsymbol{\nu}\mathbf{V}^{\frac{1}{2}}$.

In particular, with the help of (4.2.2), we shall only write explicitly the density function of $\mathbf{R}$ in the linear case as

$$\begin{aligned}&[\Gamma_q(n + p)/\{\Gamma_q(p)\Gamma_q(n + q - p)\}]|\mathbf{R}|^{p-q}|\mathbf{I} + \mathbf{R}|^{-(n+q)}\\ &\cdot\exp[-\operatorname{tr}(\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})]\textstyle\sum_{j=0}^{\infty}[\operatorname{tr}\{\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\mathbf{I} + \mathbf{R}^{-1})^{-1}\}]^j\\ &\cdot\Gamma(n + q + j)\Gamma(p)/\{j!\,\Gamma(n + q)\Gamma(p + j)\}.\end{aligned} \tag{5.1.8}$$

(ii) Let $q > p$. Here the rank of $\mathbf{R} = \bar{\mathbf{X}}'\mathbf{S}^{-1}\mathbf{X} = \bar{\mathbf{Y}}'\mathbf{S}_1^{-1}\mathbf{Y}$ is p. Let

$$\mathbf{R} = \begin{pmatrix}\mathbf{R}_{11} & \mathbf{R}_{12}\\ \mathbf{R}_{12}' & \mathbf{R}_{22}\end{pmatrix}$$

have $\mathbf{R}_{11} : p \times p$ and $\mathbf{R}_{12} : p \times (q - p)$ as random elements, and $\mathbf{R}_{22} = \bar{\mathbf{R}}_{12}'\mathbf{R}_{11}^{-1}\mathbf{R}_{12}$. Let $\mathbf{Y} = (\mathbf{Y}_1\ \mathbf{Y}_2)$ where $\mathbf{Y}_1 : p \times p$ is non-singular and $\mathbf{Y}_2 : p \times (q - p)$. We use the transformation $\mathbf{R}_{11} = \bar{\mathbf{Y}}_1'\mathbf{S}_1^{-1}\mathbf{Y}_1$ and $\mathbf{R}_{12} = \bar{\mathbf{Y}}_1'\mathbf{S}_1^{-1}\mathbf{Y}_2$ when $\mathbf{Y}_1$ is fixed. The Jacobian of the transformation is $J(\mathbf{S}_1, \mathbf{Y}_2; \mathbf{R}_{11}, \mathbf{R}_{12}) = J(\mathbf{Y}_2; \mathbf{R}_{12})J(\mathbf{S}_1; \mathbf{R}_{11}) = |\bar{\mathbf{Y}}_1'\mathbf{Y}_1|^q|\mathbf{R}_{11}|^{-2q}$. Moreover, we note that $(\mathbf{I} + \mathbf{R}_{11}^{-1} + \mathbf{R}_{11}^{-1}\mathbf{R}_{12}\bar{\mathbf{R}}_{12}'\mathbf{R}_{11}^{-1})^{-1} = \mathbf{R}_{11} - (\mathbf{R}_{11}\,\mathbf{R}_{12})(\mathbf{I} + \mathbf{R})^{-1}\overline{(\mathbf{R}_{11}\,\mathbf{R}_{12})}'$. Then it can be shown that the density function of $\mathbf{R}$ for $q > p$ is

$$\Pi^{-p(q-p)}\{\Gamma_p(n)\}^{-1}|\mathbf{R}_{11}|^{p-q}|\mathbf{I} + \mathbf{R}|^{-(n+q)} \cdot\exp[-\operatorname{tr}\{\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\mathbf{I} + \mathbf{R})^{-1}]E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}, \tag{5.1.9}$$

where $\mathbf{R}_{11}$ is hpd and

$$E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p} = \int|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}CN(\mathbf{Z}: p \times p; \boldsymbol{\delta}, \mathbf{I})\,d\mathbf{Z}, \tag{5.1.10}$$

with

$$\boldsymbol{\delta} = \boldsymbol{\nu}\begin{pmatrix}\mathbf{I}\\ \bar{\mathbf{R}}_{12}'\mathbf{R}_{11}^{-1}\end{pmatrix}\left[\mathbf{R}_{11} - (\mathbf{R}_{11}\,\mathbf{R}_{12})(\mathbf{I} + \mathbf{R})^{-1}\begin{pmatrix}\mathbf{R}_{11}\\ \bar{\mathbf{R}}_{12}'\end{pmatrix}\right]^{\frac{1}{2}}.$$

We note that

$$\boldsymbol{\delta}\bar{\boldsymbol{\delta}}' = \boldsymbol{\nu}\mathbf{R}(\mathbf{I} + \mathbf{R})^{-1}\bar{\boldsymbol{\nu}}' = \boldsymbol{\nu}(\mathbf{I} + \mathbf{R})^{-1}\mathbf{R}\bar{\boldsymbol{\nu}}' \tag{5.1.11}$$

or the nonzero ch roots of $\boldsymbol{\delta}\bar{\boldsymbol{\delta}}'$ are the nonzero ch roots of $[\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{R}(\mathbf{I} + \mathbf{R})^{-1}]$.

For the distribution of $\mathbf{V}$, the Jacobian of the transformation $\mathbf{V} = \mathbf{I} - (\mathbf{I} + \mathbf{R})^{-1} = \bar{\mathbf{X}}'(\mathbf{S} + \mathbf{X}\bar{\mathbf{X}}')^{-1}\mathbf{X} = \bar{\mathbf{Y}}'(\mathbf{S}_1 + \mathbf{Y}\bar{\mathbf{Y}}')^{-1}\mathbf{Y}$ is complicated. Hence, we transform $\mathbf{S}_1 + \mathbf{Y}\bar{\mathbf{Y}}' = \mathbf{S}_2$ and then apply the same technique used for the distribution of $\mathbf{R}$ for $q > p$. Finally, we shall get the density function of $\mathbf{V}$ as follows:

$$\Pi^{-p(q-p)}\{\Gamma_p(n)\}^{-1}|\mathbf{V}_{11}|^{p-q}|\mathbf{I} - \mathbf{V}|^{n-p} \cdot \exp\left[-\operatorname{tr}\{\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\mathbf{I} - \mathbf{V})\}\right]E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}, \tag{5.1.12}$$

where $\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \bar{\mathbf{V}}_{12}' & \mathbf{V}_{22} \end{pmatrix}$, $\mathbf{V}_{11} : p \times p$ is hpd, $\mathbf{V}_{22} = \bar{\mathbf{V}}_{12}'\mathbf{V}_{11}^{-1}\mathbf{V}_{12}$, $\mathbf{I} - \mathbf{V}$ is hpd and $E|\bar{\mathbf{Z}}'\mathbf{Z}|^{n+q-p}$ is the same as (5.1.10) after substituting $\boldsymbol{\delta}$ for $\boldsymbol{\nu}\begin{pmatrix} \mathbf{I} \\ \bar{\mathbf{V}}_{12}' & \mathbf{V}_{11}^{-1} \end{pmatrix}\mathbf{V}_{11}^{\frac{1}{2}}$, $\mathbf{V}_{12} : p \times (p - q)$.

Comparing the distribution of (5.1.12) and (5.1.9), we can write down the Jacobian of the transformation $\mathbf{V} = \mathbf{I} - (\mathbf{I} + \mathbf{R})^{-1}$ as

$$J(\mathbf{R};\mathbf{V}) = |\mathbf{R}_{11}|^{q-p}|\mathbf{I} - \mathbf{V}|^{-p-q}|\mathbf{V}_{11}|^{p-q} = |\mathbf{I} - \mathbf{V}|^{-2q} \tag{5.1.13}$$

for $|\mathbf{R}_{11}| = |\mathbf{V}_{11}||\mathbf{I} - \mathbf{V}|^{-1}$. (Note that this is the same expression when $\mathbf{R}$ is nonsingular.)

[When $\mathbf{R}$ and $\mathbf{V}$ are real matrices, $J(\mathbf{R};\mathbf{V}) = |\mathbf{I} - \mathbf{V}|^{-q-1}$.]

When the rank of $\boldsymbol{\mu}$ is one, with the help of (4.2.2) one can obtain easily the density functions of $\mathbf{R}$ and $\mathbf{V}$ from (5.1.9) and (5.1.12) respectively.

(5.2) To establish the distribution of $\Lambda = |\mathbf{S}|/|\mathbf{S} + \mathbf{X}\bar{\mathbf{X}}'| = |\mathbf{I} - \mathbf{V}| = |\mathbf{I} + \mathbf{R}|^{-1}$, where $\mathbf{S}, \mathbf{V}, \mathbf{R}$ and $\mathbf{X}$ are defined in *(5.1)*.

For a linear case, (i.e. rank of $\boldsymbol{\mu}$ is one), it is easy to verify from (5.1.8) [and from (5.1.9)] that

$$\begin{aligned} E(\Lambda^r) &= E(|\mathbf{I} + \mathbf{V}|^{-r}) \\ &= [\{\Gamma_p(n + q)\Gamma_p(n + r)\}/\{\Gamma_p(n)\Gamma_p(n + q + r)\}] \\ &\quad \cdot \exp(-\alpha) \textstyle\sum_{j=0}^{\infty} \alpha^j\Gamma(n + q + j)\Gamma(n + q + r)/ \\ &\qquad \{j!\,\Gamma(n + q)\Gamma(n + q + r + j)\}, \end{aligned} \tag{5.2.1}$$

where $\alpha = \operatorname{tr}(\bar{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$ and $n \geqq$ maximum (p, q).

Let $x_1, x_2, \cdots, x_p$ be independent real Beta variates with the density functions given by

$$[\Gamma(n + q - j + 1)/\{\Gamma(q)\Gamma(n - j + 1)\}]x_j^{n-j}(1 - x_j)^{q-1} \quad \text{for } j = 2, 3, \cdots, p \tag{5.2.2}$$

and

$$\exp(-\alpha)\textstyle\sum_{k=0}^{\infty} \alpha^k[\Gamma(n + q + k)/\{k!\,\Gamma(q)\Gamma(n + k)\}]x_1^{n+k-1}(1 - x_1)^{q-1}. \tag{5.2.3}$$

Then it is easy to verify that

$$E(\Lambda^r) = \textstyle\prod_{j=1}^p E(x_j^r) = E(\prod_{j=1}^p x_j)^r. \tag{5.2.4}$$

Hence the distribution of Λ is the same as the product of p independent real Beta variates x_j ($j = 1, 2, \cdots, p$) given by (5.2.2) and (5.2.3).

When $\boldsymbol{\mu} = \mathbf{0}$, the above results remain valid.

(*5.3*) *An asymptotic distribution of* Λ. Let us consider a general expression

$$E(\Lambda^{nh}) = \prod_{j=1}^{p} \left\{ \frac{\Gamma(n + p_1 + nh - j + 1)\Gamma(n + p_2 - j + 1)}{\Gamma(n + p_2 + nh - j + 1)\Gamma(n + p_1 - j + 1)} \right\}, \tag{5.3.1}$$

where $p_2 \geqq p_1$. Then using the result given by Anderson ([1], (23), p. 207) for large values of n, it can be shown that

$$\begin{aligned} &\Pr(-m \log \Lambda \leqq \xi) \\ &\quad = \Pr(\chi_f^2 \leqq \xi) + r_2 m^{-2}[\Pr(\chi_{f+4}^2 \leqq \xi) - \Pr(\chi_f^2 \leqq \xi)] + O(m^{-3}), \end{aligned} \tag{5.3.2}$$

where $m = 2n + p_1 + p_2 - p$, $f = 2(p_2 - p_1)p$ and

$$r_2 = p(p_2 - p_1)[p^2 + (p_2 - p_1)^2 - 2]/3.$$

Hence for the first approximation, we get

(5.3.3) $-m$ Log Λ will be distributed asymptotically as χ^2 with f degrees of freedom where $m = 2n + p_2 + p_1 - p$ and $f = 2p(p_2 - p_1)$. We note that in (*5.2*), $p_2 = q$, $p_1 = 0$ and $p = p$.

6. Distribution of s.s. and s.p. matrix due to regression coefficients.

(*6.1*) Let the distribution of $\mathbf{Z}: p \times n$ (a complex random matrix) be given by (3.1.1) and the distribution of $\boldsymbol{\beta} = \mathbf{Z}\bar{\mathbf{M}}'(\mathbf{M}\bar{\mathbf{M}}')^{-1}$, the maximum likelihood estimate of $\boldsymbol{\mu}$ called a regression matrix of $\mathbf{Z}$ on $\mathbf{M}$, is given by (3.1.8). The s.s. and s.p. matrix due to $\boldsymbol{\mu}$ is $\mathbf{B} = \boldsymbol{\beta}(\mathbf{M}\bar{\mathbf{M}}')\bar{\boldsymbol{\beta}}'$. In this section we shall obtain the distribution of $\mathbf{B}$ when $\mathbf{M}: q \times n$ is fixed. Since $\mathbf{M}\bar{\mathbf{M}}'$ is hpd, we can write it as $\mathbf{M}\bar{\mathbf{M}}' = [(\mathbf{M}\bar{\mathbf{M}}')^{\frac{1}{2}}]^2$, where $(\mathbf{M}\bar{\mathbf{M}}')^{\frac{1}{2}}$ is hermitian. Then the distribution of $\boldsymbol{\beta}(\mathbf{M}\bar{\mathbf{M}}')^{\frac{1}{2}} = \mathbf{X}: p \times q$ is

$$\Pi^{-pq}|\boldsymbol{\Sigma}|^{-q} \exp[-\operatorname{tr} \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\nu})(\overline{\mathbf{X} - \boldsymbol{\nu}})'] \, d\mathbf{X}, \tag{6.1.1}$$

where $\boldsymbol{\nu} = \boldsymbol{\mu}(\mathbf{M}\bar{\mathbf{M}}')^{\frac{1}{2}}$, and $\mathbf{B} = \mathbf{X}\bar{\mathbf{X}}'$. When $q \geqq p$, the distribution of $\mathbf{B}$ is non-central Wishart which is considered in Section 4. When $q < p$, then the distribution of $\mathbf{B}$ will be called a complex pseudo-Wishart.

(*6.2*) Now let us suppose that $\mathbf{M}: q \times n$ given in (3.1.1) is distributed as

$$\Pi^{-qn}|\boldsymbol{\Sigma}_1|^{-n} \exp(-\operatorname{tr} \boldsymbol{\Sigma}_1^{-1}\mathbf{M}\bar{\mathbf{M}}') \, d\mathbf{M}. \tag{6.2.1}$$

Using the distribution of $\mathbf{M}$ in (3.1.8), we use (*2.4*) and obtain the joint distribution of $(\mathbf{M}\bar{\mathbf{M}}')$ and $\boldsymbol{\beta}$. Then integrating over $(\mathbf{M}\bar{\mathbf{M}}')$, we obtain the distribution of $\boldsymbol{\beta}$ as

$$\Pi^{-pq}|\boldsymbol{\Sigma}_1|^{p}|\boldsymbol{\Sigma}|^{-q}\{\Gamma_q(n+p)\}\{\Gamma_q(n)\}^{-1}|\mathbf{I} + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu})\boldsymbol{\Sigma}_1(\overline{\boldsymbol{\beta} - \boldsymbol{\mu}})'|^{-n-p} \, d\boldsymbol{\beta}. \tag{6.2.2}$$

If the rank of $\mathfrak{u}$ is t and w_j^2's are the nonzero ch roots of $(\Sigma^{-1}\mathfrak{u}\mathbf{M}\bar{\mathbf{M}}'\bar{\mathfrak{u}}'\Sigma^{-1}\mathbf{B})$, then the joint density function $\mathbf{M}$ and $\mathbf{B}$: $p \times p$ is

$$(6.2.3) \quad \begin{aligned} &\Pi^{-pt-qn}|\Sigma_1|^{-n}|\Sigma|^{-q}\{\Gamma_p(q-t)\}^{-1}|\mathbf{B}|^{q-p}\exp(-\mathrm{tr}\,\Sigma^{-1}\mathbf{B}) \\ &\cdot\exp[-\mathrm{tr}\,(\Sigma_1^{-1}+\bar{\mathfrak{u}}'\Sigma^{-1}\mathfrak{u})\mathbf{M}\bar{\mathbf{M}}']\int_{\mathbf{X}:p\times t}|\mathbf{I}_p-\mathbf{X}\bar{\mathbf{X}}'|^{q-t-p} \\ &\cdot\exp(2\,\mathrm{R.P.}\,\mathrm{tr}\,\mathbf{D}_w\bar{\mathbf{X}}')\,d\mathbf{X}. \end{aligned}$$

When $t = 0$ the density function of $\mathbf{B}$ is $CW(\mathbf{B}; p, q, \Sigma)$.

For $t = 1$, we integrate over $\mathbf{M}$ and get the density function of $\mathbf{B}$ as

$$(6.2.4) \quad \begin{aligned} &\{\Gamma_p(q)\}^{-1}|\Sigma|^{-q}|\mathbf{I}+\Sigma^{-1}\mathfrak{u}\Sigma_1\bar{\mathfrak{u}}'|^{-n}|\mathbf{B}|^{q-p}\exp(-\mathrm{tr}\,\Sigma^{-1}\mathbf{B}) \\ &\cdot\sum_{j=0}^{\infty}[\mathrm{tr}\,\{(\mathbf{I}+\Sigma^{-1}\mathfrak{u}\Sigma_1\bar{\mathfrak{u}}')^{-1}\Sigma^{-1}\mathfrak{u}\Sigma_1\bar{\mathfrak{u}}'\Sigma^{-1}\mathbf{B}\}]^j\tbinom{n+j-1}{j}\Gamma(q)\{\Gamma(q+j)\}^{-1}. \end{aligned}$$

7. Distribution of the ch roots of certain hermitian matrices.

(7.1) Let the joint distribution of complex random matrices $\mathbf{X}$: $p \times q$ and $\mathbf{S}$: $p \times p$ be given by (5.1.1). We note that the ch roots of $[\mathbf{S}^{-1}(\mathbf{X}\bar{\mathbf{X}}')]$ are equal to corresponding ch roots of $[\mathbf{S}_1^{-1}(\mathbf{Y}\bar{\mathbf{Y}}')]$ where $\mathbf{Y}$ and $\mathbf{S}_1$ have the joint density function given by (5.1.2). Moreover, when $q \leqq p$, we have shown in *(5.1)* that the distribution of $\mathbf{R} = \bar{\mathbf{X}}'\mathbf{S}^{-1}\mathbf{X}$ is the same as that of $\mathbf{R} = (\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}\mathbf{G}_{1.2}^{-1}(\bar{\mathbf{Y}}'\mathbf{Y})^{\frac{1}{2}}$, where $\mathbf{Y}$: $p \times q$ and $\mathbf{G}_{1.2}$: $q \times q$ are independent having respective density functions $CN(\mathbf{Y}; \mathfrak{v}, \mathbf{I})$ and $CW(\mathbf{G}_{1.2}; q, n+q-p, \mathbf{I})$. The nonzero ch roots of $(\mathbf{S}^{-1}\mathbf{X}\bar{\mathbf{X}}')$ are the same as the ch roots of $\mathbf{R}$ when $q \leqq p$. Now when $q \geqq p$, the ch roots of $[\mathbf{S}^{-1}\mathbf{X}\bar{\mathbf{X}}']$ are the same as those of $\mathbf{T} = (\mathbf{Y}\bar{\mathbf{Y}}')^{\frac{1}{2}}\mathbf{S}_1^{-1}(\mathbf{Y}\bar{\mathbf{Y}}')^{\frac{1}{2}}$ where $\mathbf{Y}$: $p \times q$ and $\mathbf{S}_1$: $p \times p$ are independent having respective density functions $CN(\mathbf{Y}; \mathfrak{v}, \mathbf{I})$ and $CW(\mathbf{S}_1; p, n, \mathbf{I})$. Now the form of the statistics $\mathbf{R}$ and $\mathbf{T}$ are similar. Hence, if we know the distribution of the ch roots of $\mathbf{T}$ when $q \geqq p$, we can easily write down the distribution of the ch roots of $\mathbf{R}$ when $q \leqq p$, by making the substitution

$$(7.1.1) \qquad (p, q, n) \to (q, p, n+q-p).$$

We may compare the result (7.1.1) with that mentioned by Roy ([7], p. 46) for real variates.

Hence, we shall only consider the case when $q \geqq p$. For simplicity of the results, we shall consider $\mathfrak{v} = \mathbf{0}$. In this case the density function of $\mathbf{T}$: $p \times p$ can be written as

$$(7.1.2) \qquad [\Gamma_p(n+q)/\{\Gamma_p(n)\Gamma_p(q)\}]|\mathbf{T}|^{q-p}|\mathbf{I}+\mathbf{T}|^{-(n+q)}.$$

Let us transform $\mathbf{T}$ by the relation $\mathbf{T} = \mathbf{U}\mathbf{D}_\varphi\bar{\mathbf{U}}'$ where $\mathbf{U}$: $p \times p$ is a unitary matrix with real diagonal elements and $\mathbf{D}_\varphi$ is a diagonal matrix with $\varphi_j > 0$, real and distinct which are the ch roots of $[\mathbf{S}^{-1}(\mathbf{X}\bar{\mathbf{X}}')]$ (or $\mathbf{T}$). By *(2.9)*, the Jacobian of the transformaton $J(\mathbf{R}; \mathbf{U}, \mathbf{D}_\varphi) = h_2(\mathbf{U})[\prod_{k=1}^{p-1}\prod_{j=k+1}^{p}(\varphi_j - \varphi_k)^2]$. Using this in (7.1.1) and integrating over $\mathbf{U}$ with the help of *(2.7)*, we get the joint density function of $0 \leqq \varphi_1 \leqq \varphi_2 \leqq \cdots \leqq \varphi_p < \infty$ as

$$(7.1.3) \qquad c\prod_{j=1}^{p}[\varphi_j^{q-p}(1+\varphi_j)^{-n-q}][\prod_{k=1}^{p-1}\prod_{j=k+1}^{p}(\varphi_j-\varphi_k)^2],$$

where

$$c = \prod_{j=1}^{p} [\Gamma(n + q - j + 1)/ \{\Gamma(n - j + 1)\Gamma(p - j + 1)\Gamma(q - j + 1)\}]. \tag{7.1.4}$$

For $q \geqq p$, the joint density function of the ch roots of $[\mathbf{X}\bar{\mathbf{X}}'(\mathbf{S} + \mathbf{X}\bar{\mathbf{X}}')^{-1}]$, which are $f_j = \varphi_j/(1 + \varphi_j)$ for $j = 1, 2, \cdots, p$, is given by

$$c\prod_{i=1}^{p} [f_j^{q-p}(1 - f_j)^{n-p}][\prod_{k=1}^{p-1} \prod_{j=k+1}^{p} (f_j - f_k)^2], \tag{7.1.5}$$

where $0 \leqq f_1 \leqq \cdots \leqq f_p \leqq 1$, and c is defined in (7.1.4).

We may compare the above results with those given by Roy ([7], p. 35) or Anderson ([1], p. 318) for real Gaussian variates.

If the distribution of a complex random matrix $\mathbf{X}: p \times q$ is given by

$$CN(\mathbf{X}; \mathbf{0}, \mathbf{I})\, d\mathbf{X}, \tag{7.1.6}$$

then the joint density function of $0 \leqq \psi_1 \leqq \psi_2 \leqq \cdots \leqq \psi_p \leqq 1$, the ch roots of $\mathbf{X}\bar{\mathbf{X}}'$ (when $q \geqq p$), is given by

$$[\prod_{j=1}^{p} \Gamma(p - j + 1)\Gamma(q - j + 1)]^{-1}\prod_{i=1}^{p} [\psi_j^{q-p} \exp(-\psi_j)] \cdot[\prod_{k=1}^{p-1} \prod_{j=k+1}^{p} (\psi_j - \psi_k)^2]. \tag{7.1.7}$$

It may be interesting to note that if we put $\lambda_j = n\varphi_j$ in (7.1.3) [or $\lambda_j = nf_j$ in (7.1.5)] and then take limit as $n \rightarrow \infty$, it is easy to see that the asymptotic distribution of λ_j $(j = 1, 2, \cdots, q)$ is the same as that given in (7.1.7) by substituting λ_j for ψ_j.

(*7.2*) Let $\mathbf{S}: (p + q) \times (p + q)$, a random hpd matrix, be distributed as

$$\{\Gamma_{p+q}(n)\}^{-1}|\mathbf{\Sigma}|^{-n}|\mathbf{S}|^{n-p-q} \exp(-\operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S})\, d\mathbf{S}. \tag{7.2.1}$$

Let us partition the matrices $\mathbf{S}$ and $\mathbf{\Sigma}$ as follows:

$$\mathbf{S} = \begin{array}{cc} \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \bar{\mathbf{S}}_{12}' & \mathbf{S}_{22} \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array}, \quad \mathbf{\Sigma} = \begin{array}{cc} \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \bar{\mathbf{\Sigma}}_{12}' & \mathbf{\Sigma}_{22} \end{pmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p & \quad q \end{matrix} & \end{array} \tag{7.2.2}$$

and let $\mathbf{\Sigma}_{11.2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\bar{\mathbf{\Sigma}}_{12}'$. It can be verified that

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{22}^{-1} \end{pmatrix} + \begin{pmatrix} \mathbf{I}_p \\ \mathbf{\Sigma}_{22}^{-1}\bar{\mathbf{\Sigma}}_{12}' \end{pmatrix} \mathbf{\Sigma}_{11.2}^{-1}(\mathbf{I}_p \quad - \mathbf{\Sigma}_{12}\, \mathbf{\Sigma}_{22}^{-1}). \tag{7.2.3}$$

Then the joint density function of $\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\bar{\mathbf{S}}_{12}'$, $\mathbf{X} = \mathbf{S}_{12}\mathbf{S}_{22}^{-1}$ and $\mathbf{S}_{22}$ is written as

$$\{\Gamma_{p+q}(n)\}^{-1}|\mathbf{\Sigma}|^{-n}|\mathbf{S}_{11.2}|^{n-p-q}|\mathbf{S}_{22}|^{n+p-q} \exp(-\operatorname{tr} \mathbf{\Sigma}_{11.2}^{-1}\mathbf{S}_{11.2}) \cdot\exp[-\operatorname{tr} \mathbf{\Sigma}_{11.2}^{-1}(\mathbf{X} - \boldsymbol{\mu})\mathbf{S}_{22}(\overline{\mathbf{X} - \boldsymbol{\mu}})' - \operatorname{tr} \mathbf{\Sigma}_{22}^{-1}\mathbf{S}_{22}], \tag{7.2.4}$$

where $\boldsymbol{\mu} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$. From (7.2.4), it is easy to see that $\mathbf{S}_{11.2}$ and a joint function of $\mathbf{X}$ and $\mathbf{S}_{22}$ are independently distributed. Using section (*6.2*), we can obtain the distribution of $\mathbf{B} = \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\bar{\mathbf{S}}_{12}' = \mathbf{X}\mathbf{S}_{22}\bar{\mathbf{X}}'$ when $p \leqq q \leqq n$. When the rank of $\boldsymbol{\mu}$ is one, we can write down the explicit density function of $\mathbf{B}$ with the help of (6.2.4) as

$$\begin{aligned} (7.2.5) \qquad & \{\Gamma_p(q)\}^{-1}|\mathbf{I} - \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\bar{\boldsymbol{\Sigma}}_{12}'|^n|\boldsymbol{\Sigma}_{11.2}|^{-q}|\mathbf{B}|^{q-p}\exp(-\operatorname{tr}\boldsymbol{\Sigma}_{11.2}^{-1}\mathbf{B}) \\ & \cdot \textstyle\sum_{j=0}^{\infty}[\operatorname{tr}(\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\bar{\boldsymbol{\Sigma}}_{12}'\boldsymbol{\Sigma}_{11.2}^{-1}\mathbf{B})]^j\binom{n+j-1}{j}\Gamma(q)\{\Gamma(q+j)\}^{-1}. \end{aligned}$$

In the special case, let $p = 1$. Then $S_{11.2}$ and B are scaler quantities. Using the distribution of $S_{11.2}$ from (7.2.4) and the distribution of B from (7.2.5), we get the distribution of $r = B/(S_{11.2} + B)$ [which is defined by Goodman [4] as *multiple coherence*] as

$$\begin{aligned} (7.2.6) \qquad & (1 - \zeta)^n r^{q-1}(1 - r)^{n-q-1} \\ & \cdot \textstyle\sum_{j=0}^{\infty}(\zeta r)^j\binom{n+j-1}{j}\Gamma(n+j)/\Gamma(q+j)\Gamma(n-\nu), \end{aligned}$$

where $\zeta = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\bar{\boldsymbol{\Sigma}}_{12}'/\Sigma_{11}$.

For the distribution of the ch roots of $(\mathbf{S}_{11.2}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\bar{\mathbf{S}}_{12}') = \mathbf{P}$, it may be noted as (7.1.1) that if we know the distribution of the ch roots of $\mathbf{P}$ for $p \leqq q \leqq n$, we can write down the distribution of the nonzero ch roots of $\mathbf{P}$ for $p \geqq q$ by substituting

$$(7.2.7) \qquad (p, q, n - q) \to (q, p, n - p).$$

For the case $p \leqq q \leqq n$ and $\boldsymbol{\Sigma}_{12} = \mathbf{0}$, using (7.2.4) it can be shown as in section (*7.1*), the joint density function of $0 \leqq \phi_1 \leqq \phi_2 \leqq \cdots \leqq \phi_p < \infty$ (ϕ_j's the ch roots of $\mathbf{P}$) is the same as (7.1.3) by replacing n by $n - q$.

Now let $\mathbf{S}$ and $\boldsymbol{\Sigma}$ be partitioned in three rows and three columns as

$$(7.2.8) \qquad \mathbf{S} = \begin{array}{cc} \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13} \\ \bar{\mathbf{S}}_{12}' & \mathbf{S}_{22} & \mathbf{S}_{23} \\ \bar{\mathbf{S}}_{13}' & \bar{\mathbf{S}}_{23}' & \mathbf{S}_{33} \end{pmatrix} & \begin{matrix} a \\ b \\ c \end{matrix} \\ \begin{matrix} \;a\; & \;\;\;b\;\;\; & \;c\; \end{matrix} & \end{array} \text{ and } \boldsymbol{\Sigma} = \begin{array}{cc} \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\ \bar{\boldsymbol{\Sigma}}_{12}' & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \bar{\boldsymbol{\Sigma}}_{13}' & \bar{\boldsymbol{\Sigma}}_{23}' & \boldsymbol{\Sigma}_{33} \end{pmatrix} & \begin{matrix} a \\ b \\ c \end{matrix} \\ \begin{matrix} \;a\; & \;\;\;b\;\;\; & \;c\; \end{matrix} & \end{array},$$

where $a + b + c = p + q$. Let $\boldsymbol{\Sigma}_{jk.3} = \boldsymbol{\Sigma}_{jk} - \boldsymbol{\Sigma}_{j3}\boldsymbol{\Sigma}_{33}^{-1}\bar{\boldsymbol{\Sigma}}_{k3}'$ and $\mathbf{S}_{jk.3} = \mathbf{S}_{jk} - \mathbf{S}_{j3}\mathbf{S}_{33}^{-1}\bar{\mathbf{S}}_{k3}'$ $(j, k = 1, 2)$. Let us transform $\mathbf{S}_{11}$, $\mathbf{S}_{12}$ and $\mathbf{S}_{22}$ respectively to $\mathbf{S}_{11.3}$, $\mathbf{S}_{12.3}$ and $\mathbf{S}_{22.3}$. Then for integration over $\mathbf{S}_{13}$, $\mathbf{S}_{23}$ and $\mathbf{S}_{33}$, we use the inverse of $\boldsymbol{\Sigma}$ similar to one mentioned in (7.2.3), and thus we arrive at the joint density function of $\mathbf{S}_{11.3}$, $\mathbf{S}_{12.3}$ and $\mathbf{S}_{22.3}$ as follows:

$$(7.2.9) \qquad \{\Gamma_{a+b}(n - c)\}^{-1}|\boldsymbol{\Sigma}_{.3}|^{-(n-c)}|\mathbf{S}_{.3}|^{n-c-a-b}\exp(-\operatorname{tr}\boldsymbol{\Sigma}_{.3}^{-1}\mathbf{S}_{.3}),$$

where

$$\boldsymbol{\Sigma}_{.3} = \begin{pmatrix} \boldsymbol{\Sigma}_{11.3} & \boldsymbol{\Sigma}_{12.3} \\ \bar{\boldsymbol{\Sigma}}_{12.3}' & \boldsymbol{\Sigma}_{22.3} \end{pmatrix} \text{ and } \mathbf{S}_{.3} = \begin{pmatrix} \mathbf{S}_{11.3} & \mathbf{S}_{12.3} \\ \bar{\mathbf{S}}_{12.3}' & \mathbf{S}_{22.3} \end{pmatrix}.$$

Thus, we find that the distribution of the ch roots of $\mathbf{S}_{11.3}^{-1}\mathbf{S}_{12.3}\,\mathbf{S}_{22.3}^{-1}\bar{\mathbf{S}}_{12.3}'$ can be obtained from that of $\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\bar{\mathbf{S}}_{12}'$ by substituting $(p, q, n) \to (a, b, n - c)$. When $a = 1$, we write down below the density function of $r_{.3} = \mathbf{S}_{12.3}\mathbf{S}_{22.3}^{-1}\bar{\mathbf{S}}_{12.3}'/S_{11.3}$ as

$$(7.2.10) \quad \begin{aligned} &(1 - \zeta_{.3})^{n-c} r_{.3}^{b-1}(1 - r_{.3})^{n-c-b-1} \\ &\quad \cdot \textstyle\sum_{j=0}^{\infty} (\zeta_{.3} r_{.3})^j \binom{n-c+j-1}{j} \Gamma(n - c + j)/\Gamma(b + j)\Gamma(n - c - b), \end{aligned}$$

where $\zeta_{.3} = \mathbf{\Sigma}_{12.3}\mathbf{\Sigma}_{22.3}^{-1}\bar{\mathbf{\Sigma}}_{12.3}'/\Sigma_{11.3}$. We shall call $r_{.3}$ as sample *partial multiple coherence*. When $b = 1$, $r_{.3}$ is defined by Goodman [4] as *partial coherence*.

All the above results are comparable to those for real Gaussian variates.

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