ON THE CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be a sequence of independent random variables (r.v.'s) with zero mean and finite standard deviation σ_i, $1 \leqq i \leqq n$. According to the central limit theorem, the normed sum

$$Y_n = (1/s_n) \sum_{i=1}^{n} X_i,$$

where $s_n = \sum_{i=1}^{n} \sigma_i^2$, is under certain additional conditions approximatively normally distributed. We will here examine the convergence of the moments and the absolute moments of Y_n towards the corresponding moments of the normal distribution. The results in this general case are stated in Theorem 3 and Theorem 4, but, in order to avoid repetition and unnecessary complication, explicit proofs will only be given in the case of equally distributed random variables. (Theorem 1 and Theorem 2).

2. Two lemmas concerning Fourier-Stieltjes transforms.

Lemma 1. *Let $H(x)$ be a function of bounded variation on $(-\infty, \infty)$, with the Fourier-Stieltjes transform*

$$h(t) = \int_{-\infty}^{\infty} e^{ixt}\, dH(x),$$

the moments

$$\gamma_j = \int_{-\infty}^{\infty} x^j\, dH(x), \qquad j = 0, 1, 2, \cdots,$$

and the absolute moments

$$\delta_r = \int_{-\infty}^{\infty} |x|^r |dH(x)|, \qquad r > 0.$$

(a) *If $\delta_r < \infty$, $r > 0$, then*

$$h(t) = \sum_{j=0}^{s} (\gamma_j (it)^j / j!) + c_r \delta_r \theta(t) |t|^r, \qquad -\infty < t < \infty,$$

where s is the greatest integer less than r, c_r is a finite constant, only depending on r and $|\theta(t)| \leqq 1$ for all t.

(b) *If $\delta_k < \infty$, for an integer $k > 0$, then*

$$h(t) = \sum_{j=0}^{k} (\gamma_j (it)^j / j!) + o(t^k).$$

For a proof, see Loève [6], p. 199.

Lemma 2. *Let $F(x)$ be a distribution function* (d.f.) *on $(-\infty, \infty)$ satisfying*

$$\int_{-\infty}^{\infty} x\, dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\, dF(x) = 1 \quad \text{and} \quad \beta_p = \int_{-\infty}^{\infty} |x|^p\, dF(x) < \infty,$$

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where $2 < p \leqq 3$, *and let* $f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$ *be the corresponding characteristic function* (ch.f.). *Then there exists two absolute constants* c' *and* c'', *so that for every* $n > 0$ *and every interval* I *of length* $c'/\beta_p^{1/(p-2)}$, *we have*

$$\int_I |f(t)|^n\, dt < c''/\sqrt{n}.$$

This Lemma was proved by Esseen ([4], pp. 94 and 100) for $p = 3$ and can be proved in a similar way for $2 < p < 3$.

3. The case of identically distributed variables. In this case we assume without loss of generality that the standard deviation of the variables is one, so that

$$Y_n = (1/\sqrt{n}) \textstyle\sum_{i=1}^{n} X_i . \tag{1}$$

Now Y_n has the ch.f. $f_n(t) = f^n(t/\sqrt{n})$, where $f(t)$ is the ch.f. of each X_i, and $\lim_{n\to\infty} f_n(t) = e^{-t^2/2}$.

In order to estimate the rate of convergence, we expand the function $e^{t^2/2} f^n(t/\sqrt{n})$ in powers of t, and arrange the terms in increasing powers of $1/\sqrt{n}$. If, for example $\beta_k = E|X_i|^k < \infty$, we obtain

$$e^{t^2/2} f^n(t/\sqrt{n}) = 1 + \textstyle\sum_{j=1}^{k-2} n^{-j/2} P_j(it) + o(1/n^{(k-2)/2}). \tag{2}$$

Here $P_j(it)$ is a sequence of polynomials in (it) of the form

$$P_j(it) = \textstyle\sum_{h=1}^{j} c_{hj}(it)^{j+2h},$$

where the constants c_{hj} are polynomials in the moments $\alpha_m = EX_i^m$, $m = 3, 4, \cdots, j + 2$. We have

$$P_0(it) = 1,$$
$$P_1(it) = (1/3!)\alpha_3(it)^3,$$
$$P_2(it) = (1/4!)(\alpha_4 - 3)(it)^4 + (10/6!)\alpha_3^2(it)^6, \qquad \text{etc.}$$

It is easily shown (Cramèr [2]), that

$$|c_{hj}| \leqq c_0 \beta_r^{(j+2h)/r} \tag{3}$$

for $h \leqq j \leqq r$, c_0 being a constant only dependent of r. By estimating the remainder term in (2), we obtain the following lemma (Cramèr [2], pp. 71 and 74, Esseen [4], p. 44).

LEMMA 3. (a) *If* $\beta_r < \infty$, *where* $2 < r \leqq 3$, *then*

$$|f_n(t) - e^{-t^2/2}| \leqq (c_1(r)/n^{(r-2)/2})\beta_r |t|^r e^{-t^2/4}$$

for $|t| \leqq b_1(r)\sqrt{n}/\beta_r^{1/(r-2)}$.

(b) *If* $\beta_r < \infty$, *where* $r > 3$, *then*

$$|f_n(t) - e^{-t^2/2}(1 + \textstyle\sum_{j=1}^{s-2} n^{-j/2} P_j(it))|$$
$$\leqq (c_1(r)/n^{(r-2)/2})\beta_r^{3(s-1)r}(|t|^r + |t|^{3(s-1)})e^{-t^2/4}$$

for $|t| \leqq b_1(r)\sqrt{n}/\beta_r^{3/r}$, *where s is the greatest integer less than r.*

(c) *If* $\beta_k < \infty$ *for an integer* $k \geqq 3$, *then*

$$|f_n(t) - e^{-t^2/2}(1 + \sum_{j=1}^{k-2} n^{-j/2}P_j(it))| \leqq c_2(k)(d(n, t)/n^{(k-2)/2})\beta_k^{3(k-2)/k}(|t|^k + |t|^{3(k-2)})e^{-t^2/4}$$

for $|t| \leqq b_2(k)\sqrt{n}/\beta_k^{3/k}$.

Here the constants $b_1(r)$, $c_1(r)$, $b_2(k)$, $c_2(k)$ depend only on their arguments; $d(n, t)$ is bounded by one for all n and t, and $\lim_{n\to\infty} d(n, t) = \lim_{t\to 0} d(n, t) = 0$.

A proof of this lemma in Case (b) for an integer r is found in Gnedenko-Kolmogoroff [5], pp. 204–208. The other cases can be proved in a similar way.

We now introduce the functions $P_j(-\Phi)(x)$, whose Fourier-Stieltjes transforms are $e^{-t^2/2}P_j(it)$:

$$P_0(-\Phi)(x) = \Phi(x) = \text{the normalised normal distribution function}$$

$$P_1(-\Phi)(x) = (-1/3!)\alpha_3\Phi^{(3)}(x),$$

$$P_2(-\Phi)(x) = (1/4!)(\alpha_4 - 3)\Phi^{(4)}(x) + (10/6!)\alpha_2^3\Phi^{(6)}(x), \qquad \text{etc.}$$

Theorem 1. *Let* $X_1, X_2, \cdots X_n$ *be a sequence of independent identically distributed* r.v.'s *with zero mean and unit standard deviation, and let* Y_n *be defined by* (1). *If* $E|X_i|^k < \infty$ *for an integer* $k \geqq 3$, *then*

$$EY_n^k = \int_{-\infty}^{\infty} x^k\, d\Phi(x) + \sum_{j=1}^{k-2} n^{-j/2}\int_{-\infty}^{\infty} x^k\, dP_j(-\Phi)(x).$$

Proof. Setting

$$h(t) = f_n(t) - e^{-t^2/2}(1 + \sum_{j=1}^{k-2} n^{-j/2}P_j(it))$$

and

$$H(x) = F_n(x) - \Phi(x) - \sum_{j=1}^{k-2} n^{-j/2}P_j(-\Phi)(x),$$

where $F_n(x)$ is the distribution function (d.f.) of Y_n, we have from Lemma 3(c), $h(t) = o(t^k)$, and by Lemma 1(b), $\gamma_k = \int_{-\infty}^{\infty} x^k\, dH(x) = 0$, which proves the assertion.

Remark. Because of symmetry properties of the functions $P_j(-\Phi)$, it is easily seen, that the sum will only contain terms of even degree in $1/\sqrt{n}$ if k is even, and of odd degree in $1/\sqrt{n}$ if k is odd.

In order to be able to prove the corresponding result for absolute moments, we must express absolute moments in terms of c.f.'s. We start by evaluating the following integral:

$$J(\nu) = \int_0^{\infty} \{[e^{iu} - \sum_{j=0}^{m} ((iu)^j/j!)]/u^{\nu+1}\}\, du,$$

where ν is a non-integer positive number: $\nu = m + \delta$, m integer $\geqq 0$, $0 < \delta < 1$. Integrating by parts, we obtain for every $\nu > 1$, $J(\nu) = (i/\nu)J(\nu - 1)$, and by induction

$$J(\nu) = [i^{m+1}/\nu(\nu - 1)\cdot \cdots \cdot\delta]\int_0^{\infty} e^{iu}u^{-\delta}\, du. \tag{4}$$

By changing variable and by the Cauchy integration theorem, we get

$$\int_0^\infty e^{iu}u^{-\delta}\,du = i^{1-\delta}\int_0^\infty e^{-z}z^{-\delta}\,dz = i^{1-\delta}\Gamma(1-\delta) = i^{1-\delta}\pi/\Gamma(\delta)\sin\pi\delta,$$

and thus from (4)

$$J(\nu) = (i^{m+2-\delta}\pi/\nu(\nu-1)\cdot\,\cdots\,\cdot\delta\Gamma(\delta)\sin\pi\delta) = -(i^{-\nu}\pi/\Gamma(\nu+1)\sin\nu\pi).$$

Taking real parts, we get

$$\int_0^\infty \{\cos u - \textstyle\sum_{j=0}^{\lambda}[(-1)^j u^{2j}/(2j)!]/u^{\nu+1}\}\,du$$
$$= (\pi/2\Gamma(\nu+1)\cos(\nu+1)(\pi/2))$$

where 2λ is the greatest even integer contained in ν. By putting $u = |x|t$ and extending the integration to the whole real axis, we finally get

$$(5)\quad |x|^\nu = [\Gamma(\nu+1)/\pi]\cos(\nu+1)(\pi/2)$$
$$\int_{-\infty}^{\infty}\{\cos xt - \textstyle\sum_{j=0}^{\lambda}[(-1)^j(xt)^{2j}/(2j)!]/|t|^{\nu+1}\}dt$$

and this formula evidently holds for every positive ν, which is not an even integer, and for all x. We now state the following lemma:

LEMMA 4. *Let $H(x)$ be a function of bounded variation on $(-\infty,\infty)$ with the finite absolute moment*

$$\delta_\nu = \int_{-\infty}^{\infty}|x|^\nu|dH(x)| < \infty, \qquad \nu > 0,\ \nu \neq \text{even integer},$$

and with monents

$$\gamma_j = \int_{-\infty}^{\infty}x^j\,dH(x), \qquad j = 0, 1, \cdots, j \leqq \nu,$$

and let $h(t)$ be the corresponding Fourier-Stieltjes transform. Then

$$\int_{-\infty}^{\infty}|x|^\nu dH(x) = (\Gamma(\nu+1)/\pi)\cos(\nu+1)(\pi/2)$$
$$\int_{-\infty}^{\infty}\{Rh(t) - \textstyle\sum_{j=0}^{\lambda}[(-1)^j\gamma_{2j}t^{2j}/(2j)!]/|t|^{\nu+1}\}\,dt,$$

where R stands for real part, and $\lambda = [\nu/2]$.

The lemma immediately follows from the Formula (5) and from the definition of ch.f. and of the moments, after change of the order of integration, which is allowed since the integrand in (5) has a constant sign.

THEOREM 2. *Let $X_1, X_2, \cdots X_n$ be independent identically distributed* r.v.'s *with zero mean and unit standard deviation and let Y_n be defined by* (1). *If $\beta_r = E|X|^r < \infty$, $r > 2$, we have for every positive $\nu \leqq r$:*

$$|E|Y_n|^\nu - \int_{-\infty}^{\infty}|x|^\nu d\Phi(x) - \textstyle\sum_{j=1}^{[r/2]-1}n^{-j}\int_{-\infty}^{\infty}|x|^\nu dP_{2j}(-\Phi)(x)|$$
$$\leqq C[(\beta_r^2/n^{(r-2)/2}) + (\beta_r^{(\nu+1)/(r-2)}/n^{(\nu+1)/2})], \qquad \text{if } 2 < r < 3$$
$$\leqq C[(\beta_r^3/n^{(r-2)/2}) + (\beta_r^{3(\nu+1)/r}/n^{(\nu+1)/2})], \qquad \text{if } 3 \leqq r < 4$$
$$\leqq C[(\beta_r^3/n^{(r-2)/2}) + (\beta_r^{3(\nu+1)/r}/n^{(\nu+1)/2}) + (\beta_r^{3(\nu+r)/r}/n^{(\nu+r)/2})], \qquad \text{if } r \geqq 4,$$

where C is a finite constant only depending on r.

PROOF. We restrict ourselves to studying the case $r \geqq 4$. The other cases are treated in similar ways. Let X_i and Y_n have the ch.f.'s $f(t)$ and $f_n(t) = f^n(t/\sqrt{n})$ and the d.f.'s $F(x)$ and $F_n(x)$ respectively. Putting

$$g_n(t) = e^{-t^2/2}(1 + \sum_{j=1}^{[r]-2} n^{-j/2}P_j(it)),$$
$$G_n(x) = \Phi(x) + \sum_{j=1}^{[r]-2} n^{-j/2}P_j(-\Phi)(x),$$
$$h_n(t) = f_n(t) - g_n(t), \qquad H_n(x) = F_n(x) - G_n(x),$$

we shall estimate the expression $\int_{-\infty}^{\infty} |x|^\nu dH_n(x)$. If ν is an even integer, we see by Lemma 3(c) if r is an integer, and by Lemma 3(b) if r is not, that $h_n(t) = o(t^\nu)$, and accordingly we have

$$\int_{-\infty}^{\infty} |x|^\nu dH_n(x) = 0$$

in the same way as in Theorem 1.

Thus, in the sequel we assume that $\nu = 2(\lambda + \vartheta)$, where λ is an integer and $0 < \vartheta < 1$. Now, for $0 \leqq 2j \leqq 2\lambda < \nu$, we have by Lemma 3(b) $h_n(t) = o(t^{2j})$ and consequently

$$\gamma_{2j} = \int_{-\infty}^{\infty} x^{2j} dH_n(x) = 0.$$

Hence by Lemma 4, we obtain

$$\int_{-\infty}^{\infty} |x|^\nu dH_n(x) = (\Gamma(\nu + 1)/\pi) \cos (\nu + 1)(\pi/2) \int_{-\infty}^{\infty} Rh_n(t)/|t|^{\nu+1}\, dt, \tag{6}$$

which is the starting point of our estimations.

We define two quantities,

$$T = b\sqrt{n}/\beta_r^{3/r}, \qquad T_1 = bn^{(r-2)/2r}/\beta_r^{1/r},$$

and assume at first $T \geqq 1$, $T_1 \geqq 1$; b is a constant only depending on r, to be determined later. We divide the right hand side integral I in (6) into three parts

$$I = \int_{|t| \leqq 1} + \int_{1 < |t| \leqq T} + \int_{|t| > T} = I_1 + I_2 + I_3 .$$

Estimation of I_1 : Introducting the functions $g(t) = \sum_{j=0}^{[r]} [\alpha_j (it)^j/j!]$, where $\alpha_j = EX^j$, and $h(t) = f(t) - g(t)$, we put

$$h_n(t) = h_{n1}(t) + h_{n2}(t), \tag{7}$$

where $h_{n1}(t) = f^n(t/\sqrt{n}) - g^n(t/\sqrt{n})$ and $h_{n2}(t) = g^n(t/\sqrt{n}) - g_n(t)$, and divide I_1 into two parts corresponding to the division (7) of the integrand:

$$I_1 = I_{11} + I_{12} .$$

Here

$$I_{11} = \int_{|t| \leqq 1} (Rh_{n1}(t)/|t|^{\nu+1})\, dt$$
$$= (1/n^{\nu/2}) \int_{|t| \leqq (1/\sqrt{n})} R(f^n(t) - g^n(t))(1/|t|^{\nu+1})\, dt.$$

Now

$$f^n - g^n = nhg^{n-1} + Z, \tag{8}$$

where

$$|Z| < (|g| + |h|)^n - |g|^n - n|h||g|^{n-1} \leqq n^2|h|^2|g|^{n-2} \tag{9}$$

if $n\ |h/g| \leqq 1$. We must examine this condition.

From Lemma 1a we get $|h(t)| \leqq c_1\beta_r|t|^r$. (Here and in the sequel, we denote by $c_1, c_2, \cdots$, finite constants only depending on r). From the definition of $g(t)$, we have, since $|\alpha_j| \leqq \beta_r^{j/r}$,

$$1 - (\beta_r^{1/r}|t|)^2 \leqq |g(t)| \leqq 1 - (t^2/2) + \tfrac{1}{2}(\beta_r^{1/r}|t|)^3$$

if $\beta_r^{1/r}|t| \leqq 1$. Now $|t| \leqq 1/\sqrt{n} \leqq T/\sqrt{n}$ and $\leqq T_1/\sqrt{n}$, and consequently

$$0,9 \leqq |g(t)| \leqq 1 \quad \text{and} \quad n|h(t)/g(t)| \leqq 1 \tag{10}$$

if we choose the value of the constant b sufficiently small. We now get according to (8), (9) and (10)

$$R(f^n(t) - g^n(t)) = nRh(t)Rg^{n-1}(t) - nIh(t)Ig^{n-1}(t) + \theta_1(t)n^2|h(t)|^2|g(t)|^{n-2},$$

where $|\theta_1(t)| \leqq 1$ and I stands for imaginary part. We divide the integral $n^{\nu/2}I_{11}$ accordingly:

$$n^{\nu/2}I_{11} = I_{111} - I_{112} + I_{113}\,.$$
$$I_{111} = \int_{|t| \leqq 1/\sqrt{n}} nRh(t)Rg^{n-1}(t)(1/|t|^{\nu+1})\,dt.$$

Now

$$Rh(t) = \int_{-\infty}^{\infty} \{\cos xt - \textstyle\sum_{j=0}^{l} [(-1)^j(xt)^{2j}]/(2j)!\}\ dF(x), \qquad (l = [r/2]),$$

and consequently we get, after changing the order of integration

$$|I_{111}| \leqq n\int_{-\infty}^{\infty}|x|^\nu\, dF(x)$$
$$\cdot\int_{|u|<|x|/\sqrt{n}} \{[|\cos u - \textstyle\sum_{j=0}^{l} ((-1)^j u^{2j}/(2j)!)|]/|u|^{\nu+1}\}\ du.$$

An elementary estimation of the inner integrand $a(u)$ gives

$$a(u) \leqq c_2|u|^{2l-\nu+1} \qquad \text{for } |u| \leqq 1$$
$$\leqq c_2|u|^{2l-\nu-1} \qquad \text{for } |u| \geqq 1.$$

For the inner integral $b(|x|/\sqrt{n})$, we get

$$b(|x|/\sqrt{n}) \leqq [2c_2/(2l + 2 - \nu)](|x|/\sqrt{n})^{2l+2-\nu}$$
$$\leqq [c_2/(1 - \vartheta)](|x|/\sqrt{n})^{r-\nu} \quad \text{for } |x|/\sqrt{n} \leqq 1.$$

For $|x|/\sqrt{n} \geqq 1$ we must consider separately the two cases $\nu < 2l$ and $\nu > 2l$. The case $\nu = 2l$ has been treated previously.

$$b(|x|/\sqrt{n})$$
$$\leqq b(1) + [2c_2/(2l - \nu)](|x|/\sqrt{n})^{2l-\nu} \leqq [2c_2/(1 - \vartheta)](|x|/\sqrt{n})^{r-\nu} \quad \text{if } \nu < 2l$$

$b(|x|/\sqrt{n})$

$\leqq b(1) + [2c_2/(\nu - 2l) \leqq [c_2/(1 - \vartheta)] + c_2/\vartheta = c_2/\vartheta(1 - \vartheta) \qquad \text{if } \nu > 2l.$

Summing up we get

$$b(|x|/\sqrt{n}) \leqq [2c_2/\vartheta(1 - \vartheta)](|x|/\sqrt{n})^{r-\nu}, \qquad \text{for all} \quad |x|/\sqrt{n},$$

which immediately gives

$$|I_{111}| \leqq [c_3/\vartheta(1 - \vartheta)][\beta_r/n^{(r-\nu-2)/2}].$$

$$I_{112} = \int_{|t| \leqq 1/\sqrt{n}} nIh(t)Ig^{n-1}(t)(1/|t|^{\nu+1})\, dt.$$

Now $|Ig^{n-1}(t)| \leqq n|Ig(t)| \leqq nc_4(\beta_r^{1/r}|t|)^3$, and thus

$$|I_{112}| \leqq n\int_{|t| \leqq 1/\sqrt{n}} c_1\beta_r|t|^r nc_4\beta_r^{3/r}|t|^3(1/|t|^{\nu+1})\, dt = (c_5/n^{(r-\nu-1)/2})\beta_r^{1+(3/r)}.$$

$$|I_{113}| = |\int_{|t| \leqq 1/\sqrt{n}} \theta_1(t)n^2|h(t)|^2|g(t)|^{n-2}(1/|t|^{\nu+1})\, dt|$$

$$\leqq \int_{|t| \leqq 1/\sqrt{n}} n^2c_1^2\beta_r^2t^{2r}(1/|t|^{\nu+1})\, dt \leqq (c_6\beta_r^2/n^{(r-\nu-2)/2}).$$

The estimations of I_{111}, I_{112} and I_{113} give

$$|I_{11}| \leqq [c_7/\vartheta(1 - \vartheta)][\beta_r^3/n^{(r-2)/2}].$$

We now consider the integral

$$I_{12} = \int_{|t| \leqq 1}(Rh_{n2}(t)/|t|^{\nu+1})\, dt.$$

Using Lemma 3(b) for the "characteristic function $g(t)$" or, which is more adequate, developing the function $e^{t^2/2}g^n(t/\sqrt{n})$ in the same way as in (2) and estimating the remainder, we get

$$|g^n(t/\sqrt{n}) - e^{-t^2/2}(1 + \textstyle\sum_{j=1}^{2l-1} n^{-j/2}P_j(it))| \leqq (c_8/n^l)\beta_r^{6l/r}(|t|^{2l+2} + |t|^{6l})e^{-t^2/4}$$

for $|t| \leqq T$. Now, if $[r]$ is an odd integer, the left hand side equals $|h_{n2}(t)|$, but if $[r]$ is an even integer, the sum contains a new polynomial $P_{2l-1}(it)$, generated by the "moments" α_3, α_4, $\cdots$ $\alpha_{[r]}$ and 0. This polynomial, however, is purely imaginary, so the inequality above holds in all cases for $|Rh_{n2}(t)|$. We thus get

$$|I_{12}| < \int_{|t| \leqq 1} \{(c_8/n^l)\beta_r^{6l/r}(t^{2l+2} + t^{6l})e^{-t^2/4}/|t|^{\nu+1}\}\, dt$$

$$\leqq [c_9\beta_r^{6l/r}/(2l + 2 - \nu)n^l] \leqq [c_9\beta_r^{6l/r}/2(1 - \vartheta)n^l].$$

The estimation of I_1 is now completed:

$$|I_1| \leqq [c_{10}/\vartheta(1 - \vartheta)][\beta_r^3/n^{(r-2)/2}].$$

Estimation of I_2 : From Lemma 3(c) if r is an integer, and from Lemma 3(b) if r is not, we have

$$|h_n(t)| \leqq (c_{11}/n^{(r-2)/2})\beta_r^{3([r]-1)/r}(|t|^r + |t|^{3([r]-1)})e^{-t^2/4}$$

for $|t| \leqq T$, and accordingly

$$|I_2| = |\int_{1<|t| \leqq T} (Rh_n(t)/|t|^{\nu+1})\, dt| \leqq (c_{12}/n^{(r-2)/2})\beta_r^3.$$

Estimation of I_3:

$$I_3 = \int_{|t|>T} (Rh_n(t)/|t|^{\nu+1})\, dt$$
$$= \int_{|t|>T} (Rf^n(t/\sqrt{n})/|t|^{\nu+1})\, dt - \int_{|t|>T} (Rg_n(t)/|t|^{\nu+1})\, dt = I_{31} - I_{32}.$$
$$|I_{31}| \leqq (1/n^{\nu/2}) \int_{|t|>T/\sqrt{n}} (|f(t)|^n/|t|^{\nu+1})\, dt$$
$$\leqq (2/n^{\nu/2}) \sum_{j=0}^{\infty} [1/(K + jd)^{\nu+1}] \int_{K+jd}^{K+(j+1)d} |f(t)|^n\, dt,$$

where $K = T/\sqrt{n}$ and $d = c'/\beta_3 \geqq c'/\beta_r^{3/r}$. By Lemma 2, every integral is less than $c''/\sqrt{n}$, and an elementary estimation of the sum gives

$$|I_{31}| < (c_{13}/\vartheta n^{(\nu+1)/2})\beta_r^{3(\nu+1)/r}.$$
$$I_{32} = \int_{|t|>T} (Rg_n(t)/|t|^{\nu+1})\, dt.$$

We have

$$Rg_n(t) = e^{-t^2/2}(1 + \sum_{j=1}^{l-1} n^{-j} P_{2j}(it)).$$

By the Inequality (3), we get

$$|Rg_n(t)| \leqq c_{14}[1 + (\beta_r^{3/r}/\sqrt{n})^{2(l-1)}](1 + t^{6(l-1)})e^{-t^2/2}.$$

Now $T \geqq 1$ implies $\beta_r^{2/r}/\sqrt{n} \leqq b$, and thus

$$|I_{32}| \leqq c_{15} \int_T^{\infty} (1 + t^{6(l-1)})e^{-t^2/4}(1/t^{\nu+1})\, dt \leqq c_{16}/T^{\nu+1}$$
$$= (c_{17}/n^{(\nu+1)/2})\beta_r^{3(\nu+1)/r}.$$

Summing up all estimations, we get

$$\left|\int_{-\infty}^{\infty} |x|^\nu dH_n(x)\right| \leqq [\Gamma(\nu + 1)/\pi]|\sin(\nu\pi/2)|[c_{18}/\vartheta(1 - \vartheta)][(\beta_r^3/n^{(r-2)/2}) + (\beta_r^{3(\nu+1)/r}/n^{(\nu+1)/2})].$$

But $|\sin(\nu\pi/2)/\vartheta(1 - \vartheta)| = \sin \vartheta\pi/\vartheta(1 - \vartheta) \leqq 4$, and this completes the proof in the case $1 \leqq T$, $1 \leqq T_1$.

The main difference in the other cases occurs when $T < 1$ and $r > 4$ in the estimation of I_{32}. Now $T < 1$ implies $\beta_r^{3/r}/\sqrt{n} \geqq b$, and thus

$$|I_{32}| \leqq c_{19}(\beta_r^{3/r}/\sqrt{n})^{2(l-1)} \int_T^{\infty} (1 + t^{6(l-1)})e^{-t^2/2}(1/t^{\nu+1})\, dt$$
$$\leqq c_{20}(\beta_r^{3/r}/\sqrt{n})^{2(l-1)}(1/\nu T^\nu) \leqq (c_{21}/\vartheta)(\beta_r^{3(r+\nu)/r}/n^{(r+\nu)/2}),$$

which causes the appearance of the third term in the stated inequality.

4. General case. We now drop the assumption of equally distributed r.v.'s, and assume instead that each X_i has the d.f. $F_i(x)$, the ch.f. $f_i(t)$, the moments

$$\alpha_{ji} = EX_i^j = \int_{-\infty}^{\infty} x^j\, dF_i(x),$$

and the absolute moments

$$\beta_{ri} = E|X_i|^r = \int_{-\infty}^{\infty} |x|^r\, dF_i(x).$$

We have

$$EX_i = 0, \qquad EX_i^2 = \sigma_i^2, \qquad 1 \leqq i \leqq n, \tag{11}$$

and put

$$Y_n = (X_1 + \cdots + X_n)/s_n, \qquad \text{where} \quad s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2, \tag{12}$$

and we shall investigate the convergence of the moments and absolute moments of Y_n towards the corresponding moments of $\Phi(x)$. We start by introducing the semi-invariants κ_{ji} of the d.f. $F_i(x)$, defined by the equation

$$\log f_i(t) = \sum_{j=1}^{r} (\kappa_{ji}(it)^j/j!) + o(t^r), \qquad \text{if} \quad \beta_{ri} < \infty,$$

and hence from (11)

$$\kappa_{1i} = 0; \qquad \kappa_{2i} = \sigma_i^2.$$

Following Cramèr [2], we put

$$K_{jn} = (1/n)\sum_{i=1}^{n} \kappa_{ji} \quad \text{and} \quad \lambda_{jn} = K_{jn}/(K_{2n})^{j/2}$$

and obtain in the same way as in (2)

$$\begin{aligned} e^{t^2/2}\bar{f}_n(t) &= \exp\left[\sum_{i=3}^{r} (\lambda_{jn}(it)^j/j!n^{(j-2)/2}) + o(t)^r\right] \\ &= 1 + \sum_{j=1}^{r-2} n^{-j/2}P_{jn}(it) + o(1/n^{(r-2)/2}). \end{aligned} \tag{13}$$

Here $P_{jn}(it)$ are polynomials in (it), whose coefficients are polynomials in the λ_{ij}. Defining the functions $P_{jn}(-\Phi)(x)$, as those who have the Fourier-Stieltjes transforms $e^{-t^2/2}P_{jn}(it)$, we can prove in the same way as in Theorem 1 the following theorem.

THEOREM 3. *Let $X_1, X_2, \cdots X_n$ be a sequence of independent* r.v.'s *with zero mean and let Y_n be defined by* (12) *and* (11). *If $E|X_i|^k < \infty$, $1 \leqq i \leqq n$ for an integer $k > 3$, then*

$$EY_n^k = \int_{-\infty}^{\infty} x^k\, d\Phi(x) + \sum_{j=1}^{k-2} n^{-j/2}\int_{-\infty}^{\infty} x^k\, dP_{jn}(-\Phi)(x).$$

By estimating the remainder term in (13), we obtain an analogue of Lemma 3, with β_k replaced by ρ_{kn}, defined by

$$\rho_{kn} = B_{kn}/(B_{2n})^{k/2} \quad \text{and} \quad B_{kn} = (1/n)\sum_{i=1}^{n} \beta_{ki}\,.$$

This result enables us to prove the following theorem:

THEOREM 4. *Let $X_1, X_2, \cdots X_n$ be a sequence of independent* r.v.'s *with* zero *mean, and let Y_n be defined by* (11). *If $\beta_{ri} = E|X_i|^r < \infty$, $1 \leqq i \leqq n$, $r > n, 2$, we have for every positive $\nu \leqq r$*

$$|E|Y_n|^\nu - \int_{-\infty}^{\infty} |x|^\nu\, d\Phi(x) - \sum_{j=1}^{[r/2]-1} n^{-j}\int_{-\infty}^{\infty} |x|^\nu\, dP_{2j,n}(-\dot{\Phi})(x)| \leqq CR(\nu, r)$$

where

$$R(\nu, r) = (\rho_{rn}^2/n^{(r-2)/2}) + (\rho_{rn}^{\nu(+1)/(r-2)}/n^{(\nu+1)/2}) \exp [c\sqrt{n} \sum_{i=1}^{n} (\sigma_i/s_n)^3 (\beta_{ri}/\rho_{rn}\sigma_i^r)^{1/(r-2)}] \quad \text{if } 2 < r < 3$$

$$= (\rho_{rn}^{3+1/r}/n^{(r-2)/2}) + (\rho_{rn}^{3(\nu+1)/r}/n^{(\nu+1)/2}) \quad \text{if } 3 \leqq r < 4$$

$$= (\rho_{rn}^{3+1/r}/n^{(r-2)/2}) + (\rho_{rn}^{3(\nu+1)/r}/n^{(\nu+1)/2}) + (\rho_{rn}^{3(\nu+r)/r}/n^{(\nu+r)/2}) \quad \text{if } r \geqq 4.$$

C and c are finite constants only depending on r.

REMARK. If the r.v.'s satisfy the conditions of Theorem 2, the right hand side of the inequality above will agree with the corresponding expression in Theorem 2, except for the exponent $3 + 1/r$ of ρ_{rn}. The main difference between the proofs of Theorem 2 and Theorem 4 lies in the treatment of the integral I_{31}. Here we have

$$I_{31} = \int_{|t|>T} [R \prod_{i=1}^n f_i(t/s_n)/|t|^{\nu+1}] \, dt = (1/s_n^{\nu}) \int_{|t|>T/s_n} [R \prod_{i=1}^n f_i(t)/|t|^{\nu+1}] \, dt,$$

and by a generalisation of Hölder's inequality

$$|I_{31}| \leqq (1/s_n^{\nu}) \prod_{i=1}^n \{\int_{|t|>T/s_n} (|f_i(t)|^{s_n^2/\sigma_i^2}/|t|^{\nu+1}) \, dt\}^{\sigma_i^2/s_n^2}.$$

Now each integral in the product is treated in the same way as I_{31} in the previous proof. This product is the root of the more complicated expression appearing in the case $2 < r < 3$.

5. Application. D. Brillinger [1] has showed for independent equally distributed r.v.'s and J. L. Doob ([3], p. 225) for a class of Markov chains, that

$$E|X_1 + X_2 + \cdots + X_n|^r \leqq Kn^{r/2}, \qquad r \geqq 2,$$

where K is a finite constant depending on the actual distributions. We easily see that if the r.v.'s satisfy the conditions of Theorem 4, then

$$E|X_1 + X_2 + \cdots + X_n|^r \leqq C_1 s_n^r(1 + R(r, r)), \qquad r > 2,$$

where C_1 is a finite constant only depending on r.

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REFERENCES

[1] BRILLINGER, D. (1962). A note on the rate of convergence of a mean. *Biometrika* **49** 574–576.

[2] CRAMÉR, H. (1937). *Random Variables and Probability Distributions*. Cambridge Tracts No. 36.

[3] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.
[4] Esseen, C.-G. (1945). Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. *Acta Math.* **77** 1–125.
[5] Gnedenko, B. V. and Kolmogoroff, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Reading.
6] Loève, M. (1960). *Probability Theory*. Van Nostrand, Princeton.