A NOTE ON QUEUEING SYSTEMS WITH ERLANGIAN SERVICE TIME DISTRIBUTIONS

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1. Introduction and summary. In [1] Conolly derived an important result (3.19) for the queueing system $GI/E_k/1$. This result has been used by him to obtain the joint distribution of the number of customers served during a busy period initiated by one customer and its length for the queueing system $GI/E_k/1$. In this note we shall derive their steady state properties by using the above result of Conolly. Earlier Wishart [6] and Conolly [2] had studied these features using different methods.

The queueing system $GI/E_k/1$, studied in this note is the one in which

(i) the time intervals between arrivals are independent and are identically distributed according to the law $dA(t)$ $(0 < t < \infty)$, with mean a and Laplace transform $\psi(\theta)$;

(ii) the queue-discipline is "first come, first served";

(iii) there is only one counter and the service-times are independent and identically distributed having a χ^2 distribution, with mean b and $2k$ degrees of freedom i.e.

$$dB(t) = [e^{-kt/b}/(k-1)!(kt/b)^{k-1}k\,dt/b; \text{ and}$$

(iv) the service-times are independent of the inter-arrival times.

Thus we may imagine that the service takes place in k consecutive phases, where the time spent in each phase is distributed negative exponentially with mean b/k.

2. Notations. Let the first customer (who does not have to wait) arrive at the instant $t = 0$.

(i) $df(t; n, m, i)$ $(t \geqq 0, n \geqq 1, 1 \leqq m \leqq n, 1 \leqq i \leqq k)$ is the probability that the $(n + 1)$th customer arrives during $(t, t + dt)$, and finds that the $(n - m + 1)$th customer is receiving service and his service is in the ith phase, under the restriction that the server is always busy during $(0, t)$.

(ii) $df(t; n, m)$ $(t \geqq 0, n \geqq 1, 1 \leqq m \leqq n)$ is the probability that the $(n + 1)$th customer arrivers during $(t, t + dt)$ and finds that the queue length is m (i.e. his arrival makes the queue length $m + 1$), under the restriction that the server is always busy during $(0, t)$. $d\bar{f}(t; n, m)$ $(t \geqq 0, n \geqq 1, 0 \leqq m \leqq n)$ will denote $df(t; n, m)$ without such a restriction.

(iii) $d_{t,x}P(t, x; n)$ $(t \geqq 0, n \geqq 1, x > 0)$ is the probability that the $(n + 1)$th customer arrives during $(t, t + dt)$ and has the waiting time lying in $[x, x + dx)$,

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under the restriction that the server is always busy during $(0, t)$. $d_{t,x}\bar{P}(t, x; n)$ $(t \geqq 0, n \geqq 1, x \geqq 0)$ will denote $d_{t,x}P(t, x; n)$ without such a restriction.

(iv) $$f(n, m)\ (n \geqq 1, 1 \leqq m \leqq n) = \int_{t=0}^{\infty} df(t; n, m).$$

$$\bar{f}(n, m)\ (n \geqq 1, 0 \leqq m \leqq n) = \int_{t=0}^{\infty} d\bar{f}(t; n, m).$$

(v) $$dP(x; n)\ (n \geqq 1, x > 0) = \int_{t=0}^{\infty} d_{t,x}P(t, x; n)$$

$$d\bar{P}(x; n)\ (n \geqq 1, x \geqq 0) = \int_{t=0}^{\infty} d_{t,x}\bar{P}(t, x; n)$$

(vi) $GI/G/1$: (in the present discussion) this system is the generalization of $GI/E_k/1$ to the case where the service-times are distributed according to the law $dF(t)$ $(0 < t < \infty)$, with mean b.

(vii) ρ: the relative traffic intensity, given by the ratio b/a.

(viii) $dP(x)$ $(0 < x < \infty) = \lim_{n\to\infty} d\bar{P}(x, n)$, the limiting probability that a customer has a waiting time lying in $[x, x + dx)$, when $0 < x < \infty$.

(ix) f_m $(0 \leqq m < \infty) = \lim_{n\to\infty} \bar{f}(n, m)$, the limiting probability that an arrival makes the queue length $m + 1$ (at the instant), when $0 \leqq m < \infty$.

It may be noted that in defining the queue length here, the customer presently being served is included. It may also be noted that the limits in (viii) and (ix) exist for the present case (cf. Section 4: discussions following (4.7), (4.9) and (4.11)).

3. Formulae from Conolly's result. From Conolly ([1], (3.19)) we have

$$(3.1)\quad \sum_{n=m}^{\infty} \int_{t=0}^{\infty} e^{-\theta t} df(t; n, m, i) z^{(n+1)k}$$
$$= z \sum_{s=1}^{k} (z\xi_s)^{(m+1)k-i}/H'(\xi_s) (R(\theta) > 0, |z| \leqq 1),$$

where $\xi_1, \cdots, \xi_k$ are the distinct roots in $|x| < 1$ of the equation

$$(3.2)\quad x^k = \psi\{\theta + k(1 - xz)/b\}$$

and

$$(3.3)\quad H(x) = (x - \xi_1) \cdots (x - \xi_k).$$

$H'(\xi_s)$ denotes differentiation of $H(x)$ with respect to x with the subsequent substitution of ξ_s.

We can easily see that

$$(3.4)\quad df(t; n, m) = \sum_{i=1}^{k} df(t; n, m, i)$$

and

$$(3.5)\quad d_{t,x}P(t, x; n)$$
$$= \sum_{m=1}^{n} \sum_{i=1}^{k} e^{-kx/b}[(kx/b)^{mk-i}/(mk - i)!]\, df(t; n, m, i)k\, dx/b$$

Hence from (3.1), (3.4) and (3.5) we get

$$(3.6)\quad \sum_{n=m}^{\infty} \int_{t=0}^{\infty} e^{-\theta t} df(t; n, m) z^{(n+1)k}$$
$$= z \sum_{s=1}^{k} (z\xi_s)^{mk}\{1 - (z\xi_s)^k\}/[H'(\xi_s)(1 - z\xi_s)], \quad (R(\theta) > 0, |z| \leqq 1)$$

and

$$\sum_{n=1}^{\infty} \int_{t=0}^{\infty} e^{-\theta t}\, d_{t,x}P(t, x; n) z^{(n+1)k}$$

(3.7) $$= z \sum_{s=1}^{k} \exp\left(-kx(1 - z\xi_s)/b\right)[(z\xi_s)^k/H'(\xi_s)]k\, dx/b$$

$$(R(\theta) > 0, |z| \leqq 1).$$

4. Some steady state results. From the total probability law the following theorem will be proved.

THEOREM. *For the queueing system GI/G/1*

(4.1) $$f_m = 0 \qquad (1 \leqq m < \infty)\ (\rho \geqq 1)$$
$$= f_0 \sum_{n=m}^{\infty} f(n, m) \qquad (1 \leqq m < \infty)\ (\rho < 1)$$

and

(4.2) $$dP(x) = 0 \qquad (0 < x < \infty)\ (\rho \geqq 1)$$
$$= P(0) \sum_{n=1}^{\infty} dP(x; n) \qquad (0 < x < \infty)\ (\rho < 1),$$

where

$$f_0 = P(0) = [1 + \sum_{n=1}^{\infty} \int_{y=0+}^{\infty} dP(y; n)]^{-1} > 0 \qquad (\rho < 1)$$
$$= 0 \qquad (\rho \geqq 1)$$

is the limiting probability that a customer does not have to wait.

PROOF. Let z_α be the service time of the αth customer, R_α be the inter-arrival time between the arrival of the αth and the $(\alpha + 1)$th customer, and

$$S_n = \sum_{\alpha=1}^{n} (z_\alpha - R_\alpha).$$

Then

$$dP(x; n) = \Pr(S_r > 0, r = 1, 2, \cdots, n; x \leqq S_n < x + dx)$$
$$\leqq \Pr(S_r > 0, r = 1, 2, \cdots, n).$$

From Wald ([5], A. 19), we have

(4.3) $$\lim_{n\to\infty} \Pr(S_r > 0, r = 1, 2, \cdots, n) = 0$$

when $E(z_\alpha - R_\alpha) \leqq 0$ i.e. $\rho \leqq 1$. Hence

(4.4) $$\lim_{n\to\infty} dP(x; n) = 0 \qquad \text{when } \rho \leqq 1.$$

It will now be shown that the restriction $\rho \leqq 1$ in (4.4) is not necessary. To do this we proceed as follows: Let $\bar{S}_r = S_{n-r} - S_n$, then

$$dP(x; n) = \Pr(S_r > 0, r = 1, 2, \cdots, n; x \leqq S_n < x + dx)$$

(4.4)' $$= \Pr(x + \bar{S}_r > 0, r = 0, 1, \cdots, n-1; x \leqq S_n < x + dx)$$
$$\leqq \Pr(x + \bar{S}_r > 0, r = 0, 1, \cdots, n-1).$$

As in the previous case from A. 19 in Wald [5], we have $\lim_{n\to\infty} \Pr (x + \bar{S}_r > 0, r = 0, 1, \cdots, n-1) = 0$ when $E(-z_\alpha + R_\alpha) \leqq 0$ i.e. $\rho \geqq 1$. This gives

$$\lim_{n\to\infty} dP(x; n) = 0 \qquad \text{when } \rho > 1. \tag{4.5}$$

Let $P(0; n)$ be the probability that the $(n + 1)$th customer is the first amongst the customers who do not have to wait after $t = 0$. Then using A. 19, A. 71 and A. 97 from Wald [5] we can see that

$$\begin{aligned} \sum_{n=1}^{\infty} nP(0; n) &= \infty \qquad \text{when } \rho = 1 \\ &< \infty \qquad \text{when } \rho < 1 \end{aligned} \tag{4.6}$$

and using the expression A. 19 in Wald [5], we see that

$$\begin{aligned} \sum_{n=1}^{\infty} P(0; n) &= 1 \qquad \text{if } \rho \leqq 1 \\ &< 1 \qquad \text{if } \rho > 1. \end{aligned} \tag{4.7}$$

(4.6) and (4.7) are due to Finch [3]. It is known that $\lim_{n\to\infty} \bar{f}(n, 0)$ exists (cf. Lindley [4]). Hence we have, using Abel's theorem and the total probability theorem

$$P(0) = f_0 = \lim_{z\to 1-} (1 - z)\phi(z)\{1 - \phi(z)\}^{-1}, \tag{4.8}$$

where $\phi(z) = \sum_{n=1}^{\infty} z^n P(0; n)$, $|z| \leqq 1$. From (4.8), (4.7) and (4.6), after some easy mathematics, we get

$$\begin{aligned} P(0) = f_0 &= 0 \qquad \text{if } \rho \geqq 1 \\ &= [1 + \textstyle\sum_{n=1}^{\infty} \int_{y=0+}^{\infty} dP(y; n)]^{-1} > 0 \qquad \text{if } \rho < 1. \end{aligned}$$

(4.4)′ and A. 71 (Wald [5]) give, for $\rho > 1$, that $\sum_{n=1}^{\infty} dP(x; n) < \infty$. We have from the law of total probability

$$d\bar{P}(x; n) = dP(x; n) + \sum_{s=1}^{n-1} \bar{f}(n - s, 0) dP(x; s) \qquad (n \geqq 1), \tag{4.9}$$

where $\sum_{s=1}^{n-1}$ is zero when $n = 1$. By taking the limit of (4.9) for $\rho \neq 1$ as $n \to \infty$, we get (4.2). (As the right hand side has a limit, the left hand side must have one. When $\rho = 1$ also $\lim_{n\to\infty} d\bar{P}(x; n)$ exists and equals zero (cf. Lindley [4]).)

Now it remains to establish (4.1). This may be done as follows: If the queue length just after the present arrival is $m + 1$ and if 'the mth previous customer' arrived z units of time earlier and if his waiting time was $y (< z)$, then his service. time (i.e. the service time of 'the mth previous customer') must be $> z - y$ Hence

$$\begin{aligned} f(n, m) &= \int_{z=0+}^{\infty} \int_{y=0+}^{z} dA_m(z) dP(y; n - m)(1 - F(z - y)) \\ &\qquad (n > m) \\ &= \int_{z=0}^{\infty} dA_m(z)(1 - F(z))(n = m) \end{aligned} \tag{4.10}$$

and

$$\bar{f}(n, m) = \int_{z=0^+}^{\infty} \int_{y=0^+}^{z} dA_m(z) d\bar{P}(y; n - m)(1 - F(z - y))$$

$$(4.11) \qquad + \int_{z=0}^{\infty} dA_m(z)(1 - F(z))\bar{f}(n - m, 0)(n > m)$$

$$= \int_{z=0}^{\infty} dA_m(z)(1 - F(z))(n = m),$$

where $A_m(z)$ is the m-fold convolution of $A(z)$ with itself. Taking the limit of $\bar{f}(n, m)$ in (4.11) as $n \to \infty$ ($\lim_{n\to\infty} \bar{f}(n, m)$ exists because $\lim_{n\to\infty} d\bar{P}(x; n)$ and $\lim_{n\to\infty} \bar{f}(n, 0)$ exist), and using (4.2) and (4.10) we get (4.1).

The theorem also shows that the steady state distribution exists when $\rho < 1$ (cf. Lindley [4]). Thus, if $\rho < 1$ from (3.6), (3.7), (4.1) and (4.2), in the case of the queueing system $GI/E_k/1$ we have

$$(4.12) \qquad f_m = f_0 \textstyle\sum_{s=1}^{k} \eta_s^{mk} (1 - \eta_s^{\,k})/\bar{H}'(\eta_s)(1 - \eta_s)(1 \leqq m < \infty)$$

$$(4.13) \quad dP(x) = P(0) \textstyle\sum_{s=1}^{k} \exp(-kx(1 - \eta_s)/b)\eta_s^{\,k} k \, dx/\bar{H}'(\eta_s) b \qquad (0 < x < \infty),$$

where $f_0 = P(0) = \bar{H}(1)$, $\eta_1, \cdots, \eta_k$ are the distinct roots of $x^k = \psi(k(1 - x)/b)$ in $|x| < 1$ and $\bar{H}(x) = (x - \eta_1)\cdots(x - \eta_k)$, $\bar{H}'(\eta_s)$ being defined similarly, as $H'(\xi_s)$. The results (4.12) and (4.13) are the same as those derived by Wishart [5].

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