ON THE GENERALIZED MELLIN TRANSFORM OF A COMPLEX RANDOM VARIABLE AND ITS APPLICATIONS

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1. Introduction. The Mellin transform

$$h(s) = E[X^s] \tag{1.1}$$

of a real positive random variable X is a useful tool to treat products

$$Y = A \cdot X_1 \cdots X_n \tag{1.2}$$

of independent positive random variables $X_1, X_2, \cdots, X_n$, A being a positive constant. It can also be used to treat products of powers

$$W = A \cdot X_1^{a_1} \cdot X_2^{a_1} \cdots X_n^{a_n} \tag{1.3}$$

where $a_1, a_2, \cdots, a_n$ are real (see [2], [3], [5], [6], [9]).

This Mellin transform is not as useful in cases for which X_k take both positive and negative values or complex values. W. M. Zolotariow [10] has given a tool to treat products of real (not necessary positive) random variables; this tool is not useful in cases when the factors are complex. P. Lévy has given a tool to treat products (1.2) of complex random variables (see [7]); this tool is not as useful for products (1.3) with a_k real.

In this paper a generalization of the Mellin transform (1.1) is given in such a way that it will be useful to treat products (1.3) where $X_1, X_2, \cdots, X_n$ are complex random variables for which $P\{X_k = 0\} = 0$, i.e. taking values in the set G^* of non-zero complex numbers, and a_k being real.

Under multiplication (1.2) the set G^* of non-zero complex numbers is an Abelian locally compact group isomorphic to the direct product $\mathcal{R} \times T$, where $\mathcal{R}$ is the multiplicative group of positive real numbers, which is isomorphic to the additive group of real numbers, and T denotes the additive group of real numbers modulo 2π. Given this structure of G^* the natural transform of a complex random variable $Z = R \cdot e^{i\Phi}$ on G^* would be

$$h(t, n) = E[R^{it}e^{in\Phi}], \quad -\infty < t < +\infty, n = \cdots, -1, 0, 1, \cdots. \tag{1.4}$$

(On this subject see [8], p. 141, [1], p. 73, [4], pp. 166–167).

The integral transform (1.4) does not suffice in cases where products (1.3) are treated with a_k being real but not necessary integer. In such a case it is more convenient to treat probability distributions on the set G being the Riemann surface of the function $w = \log z$. Under multiplication (1.3) the set G is isomorphic to the direct product $\mathcal{R} \times \mathcal{R}$, and that is why the natural transform of a probability distribution on G would be

$$h(t, v) = E[R^{it}e^{iv\Phi}], \qquad -\infty < t < +\infty, -\infty < v < +\infty. \tag{1.5}$$

Received 7 August 1964; revised 28 April 1965.

The way from a distribution on G to the corresponding distribution on G^* should be made by the suitable projection of the Riemann surface of the function $w = \log z$ on the non-zero complex plane G^*.

In this paper we shall take the transform (1.5) for t and v complex, the case t and v real will be a particular one.

The generalized Mellin transform may also be used to obtain the distribution of the scalar product $X_1X_2 + Y_1Y_2$ of two bivariate independent random vectors $[X_1, Y_1]$, $[X_2, Y_2]$ as well as the distribution of the determinant $\begin{vmatrix} X_1, & Y_1 \\ X_2, & Y_2 \end{vmatrix}$ (see Section 4.4.).

2. The definition of the generalized Mellin transform of a complex random variable and its properties. Let us consider a bivariate random variable (R, Φ) taking values (r, φ) on the half plane

$$0 < r < \infty, \quad -\infty < \varphi < +\infty. \tag{2.1}$$

Denote Φ^* the principal value of Φ, i.e.

$$\Phi^* \equiv \Phi \bmod 2\pi, \qquad -\pi < \Phi^* \leqq \pi. \tag{2.2}$$

The bivariate random variable (R, Φ^*) takes its values (r, φ^*) on the half strip

$$0 < r < \infty, \quad -\pi < \varphi^* \leqq \pi. \tag{2.3}$$

For a given distribution $P\{R \leqq r, \Phi \leqq \varphi\}$ of (R, Φ), the distribution of (R, Φ^*) may be easily found by the projecting formula

$$P(R \leqq r, \Phi^* \leqq \varphi^*) = \sum_{k=-\infty}^{+\infty} P\{R \leqq r, k\cdot 2\pi - \pi < \Phi \leqq k\cdot 2\pi + \varphi^*\}, \quad 0 < r < \infty, -\pi < \varphi^* \leqq \pi. \tag{2.4}$$

Denote

$$Z = R\cdot e^{i\Phi}. \tag{2.5}$$

We obtain a complex random variable which does not meet the zero value. The function (2.5) is one-to-one for Φ taking values $-\pi < \varphi \leqq \pi$, but is not otherwise. Using the periodicity of the exponential function, we may also write (2.5) in form

$$Z = R\cdot e^{i\Phi^*}. \tag{2.6}$$

This function is one-to-one.

Now we define the generalized Mellin transform of the complex random variable (2.5). It is given by the formula

$$h(u, v) = E[R^u e^{iv\Phi}], \tag{2.7}$$

where u and v are complex variables. It is easy to see that the transform (2.7) is for $u = it$, the characteristic function of the bivariate random variable $(\log R, \Phi)$

$$h(it, v) = E[R^{it}e^{iv\Phi}] = E[\exp\,(i(t\cdot\log R + v\cdot\Phi))] = \psi_{(\log R,\Phi)}(t, v). \tag{2.8}$$

From the known properties of characteristic functions it follows that the transform (2.7) is well defined in some pair of strips

$$u_1 \leqq \operatorname{Re} u \leqq u_2\,, \quad v_1 \leqq \operatorname{Im} v \leqq v_2\,, \tag{2.9}$$

where u_1, u_2, v_1, v_2 are real and satisfy the conditions $u_1 \leqq 0 \leqq u_2$, $v_1 \leqq 0 \leqq v_2$.

It is enough to take the transform (2.7) for $u = it$, t and v being real. In such a case the inequalities (2.9) should be omitted. In this paper it is more convenient to take u and v complex.

It should be easily seen that $h(u, 0)$ is the Mellin transform of the positive random variable R, and $h(0, v)$ is the characteristic function of the random variable Φ. The random variables R and Φ are independent if and only if

$$h(u, v) = h(u, 0)\cdot h(0, v). \tag{2.10}$$

From the properties of characteristic functions it follows that $h(u, v)$ is continuous in the pair of strips (2.9), that $h(u, v)$ defines the distribution of (R, Φ) uniquely, and $h(0, 0) = 1$.

Further for t and v real there is

$$|h(it, v)| \leqq 1, \tag{2.11}$$

$$h(-it, -v) = \overline{h(it, v)}, \tag{2.12}$$

and $h(it, v)$ is a positive definite function.

From the uniqueness property it follows that $h(u, v)$ defines also the distribution of (X, Y), where

$$X = R\cos\Phi, \quad Y = R\sin\Phi, \tag{2.13}$$

uniquely. The distribution of (X, Y) may be found by the suitable projecting.

Let us consider n independent complex random variables

$$Z_k = R_k\cdot e^{i\Phi_k}, \qquad k = 1, 2, \cdots, n, \tag{2.14}$$

satisfying the conditions $P\{Z_k = 0\} = 0$, and having their generalized Mellin transforms

$$h_k(u, v) = E[R_k^{\,u}e^{iv\Phi_k}]. \tag{2.15}$$

Let $A = A_0e^{i\varphi_0}$ be a free non-zero complex number, and let $a_1, a_2, \cdots, a_n$ be free real numbers. Denote

$$W = A\cdot Z_1^{\,a_1}\cdot Z_2^{\,a_2}\cdots Z_n^{\,a_n}. \tag{2.16}$$

This product can also be written in form

$$|W| = A_0\cdot R_1^{\,a_1}\cdot R_2^{\,a_2}\cdots R_n^{\,a_n}; \tag{2.16'}$$

$$\arg W = \varphi_0 + a_1\Phi_1 + a_2\Phi_2 + \cdots + a_n\Phi_n\,.$$

Then the generalized Mellin transform of the random variable W is

$$\begin{aligned} h_W(u, v) &= E[|W|^u e^{iv \arg W}] \\ &= E[(A_0 R_1^{a_1} R_2^{a_2} \cdots R_n^{a_n})^u \\ &\quad \cdot \exp [(iv(\varphi_0 + a_1\Phi_1 + a_2\Phi_2 + \cdots + a_n\Phi_n))] \\ &= A_0^u \cdot e^{i\varphi_0 v} \cdot E[R_1^{a_1 u} e^{iva_1\Phi_1}] \cdot E[R_2^{a_2 u} e^{iva_2\Phi_2}] \cdots E[R_n^{a_n u} e^{iva_n\Phi_n}] \\ &= A_0^u \cdot e^{i\varphi_0 v} \cdot h_1(a_1 u, a_1 v) \cdot h_2(a_2 u, a_2 v) \cdots h_n(a_n u, a_n v). \end{aligned} \tag{2.17}$$

In particular cases we have

(2.18.1) $h_W(u, v) = h_Z(au, av),$ for $W = Z^a$;

(2.18.2) $h_W(u, v) = h_Z(-u, -v),$ for $W = 1/Z$;

(2.18.3) $h_W(u, v) = h_{Z_1}(u, v) \cdot h_{Z_2}(u, v),$ for $W = Z_1 \cdot Z_2$;

(2.18.4) $h_W(u, v) = h_{Z_1}(u, v) \cdot h_{Z_2}(-u, -v),$ for $W = Z_1/Z_2$.

Further we have

(2.19) $h_W(u, v) = h_Z(u, -v),$ for $W = \overline{Z}.$

3. Particular cases.

3.1. Let the complex random variable (2.5) have the uniform distribution on the arc $r = \alpha$, $\beta - \gamma < \varphi < \beta + \gamma$. This distribution is defined by the density

$$\begin{aligned} g(\varphi) &= 1/(2\gamma), && \text{for } \beta - \gamma < \varphi < \beta + \gamma, \\ &= 0, && \text{otherwise.} \end{aligned} \tag{3.1.1}$$

The generalized Mellin transform is in this case

$$h(u, v) = E[R^u e^{iv\Phi}] = \alpha^u \int_{\beta-\gamma}^{\beta+\gamma} e^{iv\varphi}(1/(2\gamma))\, d\varphi = \alpha^u e^{i\beta v}[(\sin \gamma v)/\gamma v] \tag{3.1.2}$$

Taking $\alpha = 1$, $\beta = 0$, $\gamma = \pi$, we obtain the uniform distribution on the unity circle, having density

$$\begin{aligned} g(\varphi) &= 1/2\pi, && \text{for } -\pi < \varphi < +\pi, \\ &= 0, && \text{otherwise,} \end{aligned} \tag{3.1.3}$$

and the generalized Mellin transform

$$h(u, v) = (\sin \pi v)/\pi v. \tag{3.1.4}$$

3.2. Let the complex random variable (2.5) have the uniform distribution inside the sector of the circle $0 < r < \alpha$, $\beta - \gamma < \varphi < \beta + \gamma$, where

$$-\pi \leqq \beta - \gamma < \beta + \gamma \leqq \pi. \tag{3.2.1}$$

This distribution is given by the density

$$\begin{aligned} f(x, y) &= 1/\gamma\alpha^2, && \text{for } (0 < r < \alpha, \beta - \gamma < \varphi < \beta + \gamma), \\ &= 0, && \text{otherwise.} \end{aligned} \tag{3.2.2}$$

The corresponding density for (R, Φ) is

$$(3.2.3)\quad g(r, \varphi) = (1/\gamma\alpha^2)\cdot r, \quad \text{for} \quad (0 < r < \alpha, \beta - \gamma < \varphi < \beta + \gamma),$$
$$= 0, \quad \text{otherwise.}$$

The integral transform is in this case

$$(3.2.4)\quad h(u, v) = E[R^u e^{iv\Phi}] = \int_{\beta-\gamma}^{\beta+\gamma} d\varphi \int_0^\alpha e^{iv\varphi} r^u (1/\gamma\alpha^2) r\, dr$$
$$= [2/(2 + u)]\alpha^u e^{i\beta v}[(\sin \gamma v)/\gamma v].$$

(The restriction (3.2.1.) may be omitted in Formulae (3.2.3), (3.2.4).)

Taking $\alpha = 1$, $\beta = 0$, $\gamma = \pi$, we obtain the uniform distribution inside the unity circle, having for (X, Y) the density

$$(3.2.5)\quad f(x, y) = 1/\pi, \quad \text{for} \quad x^2 + y^2 < 1,$$
$$= 0, \quad \text{otherwise.}$$

The corresponding density of (R, Φ) is

$$(3.2.6)\quad g(r, \varphi) = (1/\pi)r, \quad \text{for} \quad 0 < r < 1, -\pi < \varphi < +\pi,$$
$$= 0, \quad \text{otherwise.}$$

The corresponding integral transform is

$$(3.2.7)\quad h(u, v) = [2/(2 + u)]\cdot[(\sin \pi v)/\pi v].$$

3.3. Let the complex random variable (2.5) have its distribution given by the density

$$(3.3.1)\quad g(r, \varphi) = (1/2\gamma)g_0(r), \quad \text{for} \quad (0 < r < \infty, \beta - \gamma < \varphi < \beta + \gamma),$$
$$= 0, \quad \text{otherwise.}$$

The integral transform is in this case

$$(3.3.2)\quad h(u, v) = E[R^u e^{iv\Phi}] = (1/2\gamma) \int_0^\infty r^u g_0(r)\, dr \int_{\beta-\gamma}^{\beta+\gamma} e^{iv\varphi}\, d\varphi.$$

Denoting $h_0(u) = \int_0^\infty r^u g_0(r)\, dr$, we obtain

$$(3.3.3)\quad h(u, v) = h_0(u) e^{i\beta v}[(\sin \gamma v)/\gamma v].$$

Taking

$$(3.3.4)\quad g_0(r) = [|q| a^{p/q}/\Gamma(p/q)] r^{p-1} e^{-ar^q}, \qquad q \neq 0,\ a > 0,\ p/q > 0,$$

we obtain the corresponding integral transform

$$(3.3.5)\quad h(u, v) = a^{-u/q}[\Gamma((p + u)/q)/\Gamma(p/q)] e^{i\beta v}[(\sin \gamma v)/\gamma v].$$

Taking

$$(3.3.6)\quad g_0(r) = [|q|\Gamma(a)/\Gamma(p/q)\Gamma(a - (p/q))][r^{p-1}/(1 + r^q)^a],$$
$$q \neq 0,\ p/q > 0,\ a - (p/q) > 0,$$

we obtain the corresponding integral transform

$$(3.3.7)\quad h(u, v) = e^{i\beta v}[(\sin \gamma v)/\gamma v][\Gamma((p + u)/q)/\Gamma(p/q)] \cdot[\Gamma(a - (p + u)/q)/\Gamma(a - (p/q))].$$

3.4. Let the complex random variable (2.5) take its values on the sides of the angle $\varphi = \alpha$, $\varphi = \beta$, and the densities of R on these sides are $g_\alpha(r)$, $g_\beta(r)$, $0 < r < \infty$. The integral transform is in this case

$$(3.4.1)\quad h(u, v) = E[R^u e^{iv\Phi}] = e^{i\alpha v} \int_0^\infty r^u g_\alpha(r)\, dr + e^{i\beta v} \int_0^\infty r^u g_\beta(r)\, dr.$$

Denoting

$$(3.4.2)\qquad h_\alpha(u) = \int_0^\infty r^u h_\alpha(r)\, dr, \qquad h_\beta(u) = \int_0^\infty r^u g_\beta(r)\, dr,$$

we obtain

$$(3.4.3)\qquad h(u, v) = e^{i\alpha v}\cdot h_\alpha(u) + e^{i\beta v}\cdot h_\beta(u).$$

Taking

$$(3.4.4)\quad \alpha = 0,\ \beta = \pi,\ g_\alpha(r) = g_\beta(r) = (1/\sqrt{\pi})e^{-r^2},\ (0 < r < \infty),$$

we obtain Z as a real random variable taking positive values as well as negative; its integral transform is

$$(3.4.5)\quad h(u, v) = (1 + e^{i\pi v}) \int_0^\infty r^u(1/\sqrt{\pi})e^{-r^2}\, dr = e^{i\frac{1}{2}\pi v} \cos \tfrac{1}{2}\pi v(1/\sqrt{\pi})\Gamma(\tfrac{1}{2}(1 + u)).$$

3.5. Let the complex random variable (2.5) have R and Φ independent and their distributions given by densities $g_0(r)$ and $g_1(\varphi)$ where

$$(3.5.1)\qquad \begin{aligned} g_1(\varphi) &= 0, && \text{for } \varphi \leqq \alpha - \beta,\ \varphi \geqq \alpha + \beta, \\ &= [\beta + (\varphi - \alpha)]/\beta^2, && \text{for } \alpha - \beta < \varphi < \alpha, \\ &= [\beta - (\varphi - \alpha)]/\beta^2, && \text{for } \alpha < \varphi < \alpha + \beta; \end{aligned}$$

Φ has in this case the triangular distribution. The integral transform is in this case

$$(3.5.2)\qquad \begin{aligned} h(u, v) &= E[R^u e^{iv\Phi}] = E[R^u]\cdot E[e^{iv\Phi}] \\ &= \int_0^\infty r^u g_0(r)\, dr[\textstyle\int_{\alpha-\beta}^{\alpha} e^{iv\varphi}[(\beta + (\varphi - \alpha))/\beta^2]\, d\varphi \\ &\quad + \textstyle\int_\alpha^{\alpha+\beta} e^{iv\varphi}[(\beta - (\varphi - \alpha)]/\beta^2]\, d\varphi] \\ &= h_0(u)e^{i\alpha v}[(\sin \tfrac{1}{2}\beta v)/\tfrac{1}{2}\beta v]^2 \end{aligned}$$

where $h_0(u) = \int_0^\infty r^u g_0(r)\, dr$.

Taking for instance, $\alpha = 0$, $\beta = 2\pi$, we obtain

$$(3.5.3)\qquad \begin{aligned} g_1(\varphi) &= 0, && \text{for } \varphi \leqq -2\pi,\ \varphi \geqq 2\pi \\ &= (2\pi + \varphi)/4\pi^2, && \text{for } -2\pi < \varphi < 0 \\ &= (2\pi - \varphi)/4\pi^2, && \text{for } 0 < \varphi < 2\pi \end{aligned}$$

and

(3.5.4) $$h(u, v) = h_0(u) \cdot [(\sin \pi v)/\pi v]^2.$$

Now let us project the Riemann surface of the function $w = \log z$ ($0 < r < \infty$, $-2\pi < \varphi < 2\pi$) on the complex plane ($0 < r < \infty$, $-\pi < \varphi \leqq \pi$). It can be easy seen that the triangular distribution (3.5.1) on the interval $(-2\pi, 2\pi)$ becomes rectangular distribution on the interval $(-\pi, \pi)$. That is why the distribution of $Z^* = R \cdot e^{i\Phi^*}$, where Φ^* is the principal value of Φ, is given by the integral transform

(3.5.5) $$h^*(u, v) = h_0(u) \cdot [(\sin \pi v)/\pi v].$$

4. Applications.

4.1. Let the complex random variable,

(4.1.1) $$Z = X + iY = R \cdot e^{i\Phi},$$

have the bivariate distribution given by the density

(4.1.2) $$f(x, y) = (1/\pi) \exp[-(x^2 + y^2)], \quad -\infty < x < +\infty, -\infty < y < +\infty.$$

The density of the bivariate random variable (R, Φ) is

(4.1.3) $$g(r, \varphi) = (1/\pi) re^{-r^2} \qquad 0 < r < \infty, -\pi < \varphi < \pi.$$

The integral transform of this random variable is (see Formulae (3.3.4) and (3.3.5))

(4.1.4) $$h(u, v) = [(\sin \pi v)/\pi v]\Gamma(1 + \tfrac{1}{2}u).$$

The reciprocal $1/Z$ has the integral transform (see Formula (2.18.2))

(4.1.5) $$h_1(u, v) = h(-u, -v) = [(\sin \pi v)/\pi v]\Gamma(1 - \tfrac{1}{2}u).$$

The distribution of the modulus and the argument of the reciprocal is given by the corresponding to (4.1.5) density (see Formulae (3.3.4), (3.3.5)

(4.1.6) $$g_1(r, \varphi) = (1/\pi) r^{-3} e^{-r^{-2}} \qquad 0 < r < \infty, -\pi < \varphi < \pi.$$

Thus the distribution of the real and imaginary parts of the reciprocal is

(4.1.7) $$f_1(x, y) = (1/\pi)[1/(x^2 + y^2)^2] \cdot \exp[-1/(x^2 + y^2)], \quad > \infty < x < +\infty, -\infty < y < +\infty.$$

4.2. Let us have two independent complex random variables Z_1, Z_2 having identical normal distributions given by density (4.1.2). We shall find the distribution of the quotient of these variables

(4.2.1) $$W = Z_1/Z_2.$$

The integral transform of W (see Formulae (2.18.4), (4.1.4)) is

(4.2.2) $$h_1(u, v) = h(u, v) \cdot h(-u, -v) = [(\sin \pi v)/\pi v]^2\Gamma(1 + \tfrac{1}{2}u)\Gamma(1 - \tfrac{1}{2}u).$$

The density of the modulus and the argument of W is given by

$$(4.2.3) \qquad g_1(r, \varphi) = [2r/(1 + r^2)^2]\cdot g_2(\varphi), \qquad 0 < r < \infty, -2\pi < \varphi < 2\pi$$

where $g_2(\varphi)$ is given by Formula (3.5.3) (see also Formula (3.5.4)).

Projecting the Riemann surface of the function $w = \log z$ $(0 < r < \infty, -2\pi < \varphi < 2\pi)$ on the complex plane $(0 < r < \infty, -\pi < \varphi < \pi)$ we see that the triangular distribution of the argument W becomes rectangular distribution on the interval $(-\pi, +\pi)$. That is why the density of the modulus and the principal value of the argument is

$$(4.2.4) \quad g_1^*(r, \varphi) = (1/2\pi)[2r/(1 + r^2)^2] = (1/\pi)[r/(1 + r^2)^2], \quad 0 < r < \infty, -\pi < \varphi < +\pi.$$

Hence the distribution of the real and imaginary parts of W is given by density

$$(4.2.5) \quad f_1(x, y) = (1/\pi)[1/(1 + x^2 + y^2)^2], \quad -\infty < x < +\infty, -\infty < y < +\infty.$$

4.3. Let us have two independent real random variables Z_1, Z_2 having identical normal distribution given by density $f(x) = (1/\pi)e^{-x^2}$, $-\infty < x < +\infty$. The integral transform of this random variable (see Formulae (3.4.4), (3.4.5)) is

$$(4.3.1) \qquad h(u, v) = e^{i\frac{1}{2}\pi v} \cos \tfrac{1}{2}\pi v(1/\sqrt{\pi})\Gamma(\tfrac{1}{2}(1 + u)).$$

We shall find the distribution of the quotient (4.2.1). Its integral transform is (see Formulae (2.18.4), (4.3.1))

$$(4.3.2) \quad h_1(u, v) = h(u, v)\cdot h(-u, -v) = \cos^2 \tfrac{1}{2}\pi v\cdot (1/\pi)\Gamma(\tfrac{1}{2}(1 + u))\Gamma(\tfrac{1}{2}(1 - u)).$$

Hence we see that the modulus and the argument of the quotient W are independent, the argument being distributed according to the characteristic function $h_1(0, v) = \cos^2 \frac{1}{2}\pi v$.

Then we see that the argument of W takes values $-\pi$, 0, $+\pi$ with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ respectively. Projecting the half line $\varphi = -\pi$ on the half line $\varphi = \pi$ we see that the principal argument of W takes values 0 and π with probabilities $\frac{1}{2}$, $\frac{1}{2}$.

The modulus of W is distributed according to the Mellin transform

$$(4.3.3) \qquad h_1(u, 0) = (1/\pi)\Gamma(\tfrac{1}{2}(1 + u))\Gamma(\tfrac{1}{2}(1 - u)) = 1/\cos \tfrac{1}{2}\pi u.$$

That is why the distribution of W is on the real line, it is given by the density

$$(4.3.4) \quad g_0(r) = (1/\pi)\cdot[1/(1 + r^2)], \qquad g_\pi(r) = (1/\pi)\cdot[1/(1 + r^2)], \quad 0 < r < \infty.$$

From the Formula (4.3.4) it follows that W is distributed on the whole real line according to the Cauchy law, $f_1(x) = (1/\pi)\cdot[1/(1 + x^2)]$, $-\infty < x < +\infty$.

4.4. Let us have two independent real bivariate random vectors $Q_1 = [X_1, Y_1]$, $Q_2 = [X_2, Y_2]$. We shall find the distribution of the scalar product $U = X_1\cdot X_2 + Y_1\cdot Y_2$, and the determinant $V = \begin{vmatrix} X_1, & Y_1 \\ X_2, & Y_2 \end{vmatrix}$.

Denote

$$Z_1 = X_1 + iY_1, \quad Z_2 = X_2 + iY_2, \quad W = U + iV. \tag{4.4.1}$$

The complex random variables Z_1, Z_2 are connected with U, V by the formula

$$W = U + iV = \overline{Z_1} \cdot Z_2. \tag{4.4.2}$$

Taking the integral transform of (4.4.2) we obtain

$$h_W(u, v) = h_{Z_1}(u, -v) \cdot h_{Z_2}(u, v), \tag{4.4.3}$$

(see Formulae (2.18.3), (2.19)).

Let, for example, Q_1 be normal according to the density (4.1.2) and Q_2 be reciprocal normal according to the density (4.1.7). Then $h_{Z_1}(u, v)$ and $h_{Z_2}(u, v)$ are given by Formulae (4.1.4) and (4.1.5).

Using Formula (4.4.3) we obtain

$$\begin{aligned} h_W(u, v) &= h_{Z_1}(u, -v) \cdot h_{Z_2}(u, v) \\ &= [(\sin \pi v)/\pi v]\Gamma(1 + \tfrac{1}{2}u)\Gamma(1 - \tfrac{1}{2}u). \end{aligned} \tag{4.4.4}$$

The density of W corresponding to (4.4.4) is

$$\begin{aligned} &f_W(x, y) \\ &\quad = (1/\pi)[1/(1 + x^2 + y^2)^2], \quad -\infty < x < +\infty, -\infty < y < +\infty, \end{aligned} \tag{4.4.5}$$

(see Section 4.2). From this density we see that U and V have in this case both the same marginal distributions given by density

$$f_U(x) = f_V(x) = 1/[2(1 + x^2)^{3/2}], \qquad -\infty < x < +\infty. \tag{4.4.6}$$

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