ON RANDOM SUMS OF RANDOM VECTORS

By Henry Teicher

Purdue University

0. Summary. To obtain the limit distribution of a sequence T_n of random vectors, the jth component of T_n being the sum of a random number $N_n^{(j)}$ of jth components of independent, identically distributed chance vectors X_n, it is first necessary to treat the special case where the $N_n^{(j)}$ are degenerate random variables. This is done in Section 2 and generalized to infinitely divisible limits in Section 4. The basic problem is treated in Section 3 and generalizations of theorems of Doeblin [3], Anscombe [1] and Rényi [7] are obtained.

1. Preliminaries. Underlying the succeeding sections will be a sequence

$$X_n = (X_{n1}, X_{n2}, \cdots, X_{nm}), \qquad n = 1, 2, \cdots, \tag{1.1}$$

of independent, identically distributed random vectors defined on some probability space with P the probability measure thereupon. Of especial interest will be the corresponding sequence of partial sums

$$S_n = (S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m)}), \qquad n = 1, 2, \cdots, \tag{1.2}$$

where $S_n^{(j)} = \sum_{i=1}^{n} X_{ij}$, $1 \leqq j \leqq m$. Further,

$$b(n); g(n); n = 1, 2, \cdots \text{ (set } b(0) = 1) \tag{1.3}$$

will signify increasing sequences of positive numbers tending to infinity while

$$N_n = (N_n^{(1)}, N_n^{(2)}, \cdots, N_n^{(m)}), \qquad n = 1, 2, \cdots, \tag{1.4}$$

will constitute a sequence of random vectors whose component random variables are positive integer-valued.

Apropos of terminology, to say that the random vector $Y_n = (Y_{n1}, Y_{n2}, \cdots, Y_{nm})$ converges in probability to a constant (vector) $c = (c_1, c_2, \cdots, c_m)$ signifies that each component random variable Y_{nj} converges in probability (denoted by $\rightarrow_P$) to the corresponding scalar c_j, $1 \leqq j \leqq m$. Convergence of cumulative distribution functions (cdf's) at continuity points of the limit cdf is denoted by $\Rightarrow$. If the original cdf's are attached to random vectors, the latter will be said to converge in distribution.

Throughout Sections 2 and 3, it will be supposed that there exists a cdf $F(x_1, x_2, \cdots, x_m)$ such that as $n \rightarrow \infty$,

$$P\{S_n^{(1)} < x_1 b(n), \cdots, S_n^{(m)} < x_m b(n)\} \Rightarrow F(x_1, \cdots, x_m). \tag{1.5}$$

The characteristic function (cf) of the necessarily stable (see [5], p. 221) cdf F will invariably be denoted by φ.

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The following lemma is an analogue of and may be proved in the same fashion as the corresponding well-known one-dimensional result. The proof of the second lemma is likewise sufficiently simple to omit.

LEMMA 1. *Let* $(X_{n1}, \cdots, X_{nm})$, $(Y_{n1}, \cdots, Y_{n,m})$ *and* $(Z_{n1}, \cdots, Z_{nm})$, $n = 1, 2, \cdots$ *constitute three sequences of random vectors, the first converging in distribution and the latter two converging in probability to constants* $(c_1, \cdots, c_m)$ *and* $(d_1, \cdots, d_m)$ *respectively. Then the characteristic function of the random vector* $(X_{n1}\cdot Y_{n1} + Z_{n1}, \cdots, X_{nm}\cdot Y_{nm} + Z_{nm})$ *converges for all real* $(t_1, \cdots, t_m)$ *to* $\exp\{i\sum_{j=1}^m t_j d_j\}\cdot\varphi(c_1 t_1, \cdots, c_m t_m)$ *where* φ *is the limit* cf *of* $(X_{n1}, \cdots, X_{nm})$.

LEMMA 2. *Let* $\phi(t_1, t_2, \cdots, t_m)$, $\phi_n(t_1, t_2, \cdots, t_m)$, $n \geqq 1$ *be sequence of* cf's *with* $\lim_{n\to\infty}\phi_n(t_1, t_2, \cdots, t_m) = \phi(t_1, t_2, \cdots, t_m)$ *and* $\{d_{kj}, 1 \leqq j \leqq m, k = 1, 2, \cdots\}$ *constants such that* $\lim_{k\to\infty} d_{kj} = 0$, $1 \leqq j \leqq m$, *and* $\{n_k\}$ *a sequence of (not necessarily distinct) positive integers. Then,* $\lim_{k\to\infty}\varphi_{n_k}(d_{k1}t_1, \cdots, d_{km}t_m) = 1$, *all real* $(t_1, \cdots, t_m)$.

2. Some unbalanced limit theorems. In this section, the effect of systematic variation of the number of elements in a component sum is examined. Recall that a function $h(x)$ is slowly varying if $\lim_{x\to\infty} h(cx)/h(x) = 1$ for all $c > 0$.

THEOREM 1. *Set* $b(n) = n^{1/\alpha}h(n)$, $0 < \alpha \leqq 2$, *where* $h(n)$ *is a slowly varying function and let* X_n, S_n, $b(n)$ *be as described in* (1.1), (1.2), (1.3). *Suppose that* (1.5) *holds for some* cdf $F(x_1, x_2, \cdots, x_m)$ *whose* cf *is* $\phi(t_1, t_2, \cdots, t_m)$ *and let* $\{n_{kj}, 1 \leqq j \leqq m, k = 1, 2, \cdots\}$ *constitute* m *increasing sequences of positive integers for which*

$$\lim_{k\to\infty} n_{k,j-1}/n_{k,j} = \bar{c}_j < \infty, \qquad 2 \leqq j \leqq m. \tag{2.0}$$

(*Without essential loss of generality* $\bar{c}_j$, $2 \leqq j \leqq m$ *will be supposed bounded by unity*[1] *and concomitantly* $n_{k,j-1}$ *will be supposed at most equal to* n_{kj}, $2 \leqq j \leqq m$, $k = 1, 2, \cdots$). *Define* $c_j = (\bar{c}_2\cdot\bar{c}_3\cdots\bar{c}_j)^{1/\alpha}$ *and* $d_j^\alpha = 1 - \bar{c}_j$, $2 \leqq j \leqq m$. *Then as* $k \to \infty$,

$$\begin{aligned} P\{S_{n_{k1}}^{(1)} < x_1\cdot b(n_{k1}), \cdots, S_{n_{km}}^{(m)} < x_m\cdot b(n_{km})\} \\ \Rightarrow G_\alpha(x_1, \cdots, x_m; c_2, \cdots, c_m) \end{aligned} \tag{2.1}$$

where G_α *is a* cdf *with* cf *given by*[2]

$$\begin{aligned} \phi(t_1, c_2t_2, \cdots, c_mt_m)\cdot\phi(0, d_2t_2, d_2c_3t_3/c_2, \cdots, d_2c_mt_m/c_2) \\ \cdots\phi(0, \cdots, 0, d_{m-1}t_{m-1}, d_{m-1}c_mt_m/c_{m-1})\cdot\phi(0, \cdots, 0, d_mt_m). \end{aligned} \tag{2.2}$$

PROOF. Note that $0 < \bar{c}_j \leqq 1$, $2 \leqq j \leqq m$ is tantamount to $1, c_2, c_3, \cdots, c_m$ forming a non-increasing sequence. If, at the outset, $\bar{c}_j$ is not bounded above by unity for all $2 \leqq j \leqq m$, this can be achieved by a permutation of the components of X_n, and we therefore suppose it to be true initially. Thus, for all sufficiently

[1] A method for obtaining the limit distribution when some $\bar{c}_j > 1$ will be found in the "Prescription" following Theorem 1.

[2] 0/0 is to be interpreted as 0.

large k, $n_{k,j-1} < n_{kj}$ if $c_j < c_{j-1}$ and otherwise (as has been supposed for convenience) $n_{k,j-1} \leqq n_{kj}$.

As a consequence of (2.0), (1.3) and the slowly varying property,

$$\lim_{k\to\infty} [b(n_{k1})/b(n_{kj})] = c_j, \qquad 2 \leqq j \leqq m,$$

$$(2.3) \quad \lim_{k\to\infty} [b(n_{kj} - n_{k,j-1})/b(n_{kj})] = d_j = (1 - c_j^{\alpha}/c_{j-1}^{\alpha})^{1/\alpha} \quad \text{if } c_{j-1} > 0, \quad 2 \leqq j \leqq m,$$
$$= d_j = 1 \quad \text{if } c_{j-1} = c_j = 0.$$

Thus, denoting $\sum_{i=h+1}^{n} X_{ij}$ by $S_{h,n}^{(j)}$, independence implies that

$$(2.4) \quad \begin{aligned} &E[\exp\{i\textstyle\sum_{j=1}^{m} t_j S_{n_{kj}}^{(j)}/b(n_{kj})\}] \\ &= E[\exp\{i\textstyle\sum_{j=1}^{m} t_j[b(n_{k1})/b(n_{kj})][S_{n_{k1}}^{(j)}/b(n_{k1})]\}] \\ &\cdot E[\exp\{i\textstyle\sum_{j=2}^{m} t_j[b(n_{k2} - n_{k1})/b(n_{kj})][S_{n_{k1},n_{k2}}^{(j)}/b(n_{k2} - n_{k1})]\}] \\ &\cdots E[\exp\{it_m[b(n_{km} - n_{k,m-1})/b(n_{km})][S_{n_{k,m-1},n_{km}}^{(m)}/b(n_{km} - n_{k,m-1})]\}]. \end{aligned}$$

If $d_h > 0$, $b(n_{kh} - n_{k,h-1})$ and therefore $n_{kh} - n_{k,h-1}$ converges to infinity by (2.3). Consequently, by Lemma 1, the factor of (2.4) in which $b(n_{kh} - n_{k,h-1})$ appears, approaches its correspondent in (2.2). On the other hand, if $d_h = 0$, $n_{kh} - n_{k,h-1}$ need not approach infinity. Nonetheless, applying Lemma 2, the limit of the appropriate factor in (2.4) will be unity, exactly the value assigned by $\phi(0, 0, \cdots, 0)$. Thus, the limit of (2.4) exists and is given by (2.2).

It will be necessary in Section 3 to stipulate the limiting distribution when the $\bar{c}_j$'s are positive but not necessarily bounded above by unity. To this end, we give the following

PRESCRIPTION. If, in Theorem 1, $\lim_k n_{k,j}^{-1} n_{k,j-1} = \bar{c}_j'$, $2 \leqq j \leqq m$ with the $\bar{c}_j'$ positive but not all bounded above by unity, let π be a permutation of $(1, 2, \cdots, m)$ which when applied to the entries of $(1, \bar{c}_2', \bar{c}_2'\bar{c}_3', \cdots, \bar{c}_2'\bar{c}_3'\cdots\bar{c}_m')$ arranges these in non-increasing order. Denote by $\pi(j)$ the image of j, $1 \leqq j \leqq m$ and set $\tilde{X}_{i,\pi(j)} = X_{ij}$, $\tilde{S}_n^{(j)} = \sum_{i=1}^{n} \tilde{X}_{ij}$ and $\tilde{n}_{k,\pi(j)} = n_{k,j}$. Then, clearly $\lim_{k\to\infty} \tilde{n}_{kj}^{-1}\tilde{n}_{k,j-1} = \bar{c}_j$ exists and $\bar{c}_j \leqq 1$, $2 \leqq j \leqq m$. Again, setting $c_j = (\bar{c}_2 \cdot \bar{c}_3 \cdots \bar{c}_j)^{1/\alpha}$, the limiting joint cdf of $(\tilde{S}_{\tilde{n}_{k1}}^{(1)}/b(\tilde{n}_{k1}), \cdots, \tilde{S}_{\tilde{n}_{km}}^{(m)}/b(\tilde{n}_{km})$ is given by (2.2). Now, in *each* factor of (2.2), apply π^{-1} to the *arguments* of the m entries of ϕ and then relabel the t's (only) to agree with the positions they occupy. The resulting cf will be the limit cf of $(S_{n_{k1}}^{(1)}/b(n_{k1}), \cdots, S_{n_{km}}^{(m)}/b(n_{km}))$. Note that $c_j = (c'_{\pi^{-1}(j)})/(c'_{\pi^{-1}(1)})$, $1 \leqq j \leqq m$ where $c_j' = (\bar{c}_2'\bar{c}_3' \cdots \bar{c}_j')^{1/\alpha}$.

In Theorem 1, $b(n)$ may vary with j if $n_{k,j-1} < n_{kj}$ and

$$\lim_{k\to\infty} [b_j(n_{k,j-1})/b_j(n_{k,j})] = \bar{c}_j < \infty, \quad \lim_{k\to\infty} [b_j(n_{kj} - n_{k,j-1})/b_j(n_{kj})] = d_j.$$

THEOREM 2. *Let X_n, S_n be as in* (1.1), (1.2) *with* (1.5) *holding for $b(n) = n^{\frac{1}{2}}$ and $F(x_1, x_2, \cdots, x_m) = \Phi_\Sigma(x_1, x_2, \cdots, x_m)$, where the* cf *of Φ_Σ is $\phi_\Sigma(t_1, t_2, \cdots, t_m) = \exp\{-\frac{1}{2}\sum_{i,j=1}^{m} \sigma_{ij}t_it_j\}$. If $\{n_{kj}, 1 \leqq j \leqq m, k = 1, 2, \cdots\}$ are m increasing sequences of positive integers with $\lim_{k\to\infty}(n_{k,j-1}/n_{kj}) = \bar{c}_j < \infty$, where without* *essential loss of generality, it will be supposed that* $\bar{c}_j \leqq 1$, $2 \leqq j \leqq m$, *then as* $k \to \infty$,

$$(2.5) \qquad P\{S_{n_{k1}}^{(1)} < x_1 n_{k1}^{\frac{1}{2}}, \cdots, S_{n_{k,m}}^{(m)} < x_m n_{km}^{\frac{1}{2}}\} \Rightarrow \Phi_{\Sigma_c}(x_1, \cdots, x_m)$$

where, setting $c_1 = 1$, *the* cf *of* Φ_{Σ_c} *is*[3]

$$(2.6) \quad \phi_{\Sigma_c}(t_1, t_2, \cdots, t_m) = \exp\{-\tfrac{1}{2}[\textstyle\sum_{i=1}^m \sigma_{ii} t_i^2 + 2\sum_{i=1}^m \sum_{j>i}(c_j/c_i)\sigma_{ij} t_i t_j]\},$$

and $c_j = \prod_{i=2}^j (\bar{c}_j)^{\frac{1}{2}}$. *Moreover, if* Σ *is positive definite, so is* Σ_c *for all* $1 \geqq c_2 \geqq c_3 \geqq \cdots \geqq c_m \geqq 0$; *if* $\sigma_{ii} > 0$, $1 \leqq i \leqq m$ *but* Σ *is singular, then* Σ_c *is positive definite for* $1 > c_2 > c_3 > \cdots > c_m > 0$.

PROOF. To obtain (2.6), apply Theorem 1 with $b(n) = n^{\frac{1}{2}}$, noting that $d_j^2 = 1 - c_j^2/c_{j-1}^2$, $2 \leqq j \leqq m$. Next, write $a_j = \bar{c}_j$ and $\Sigma_c = \Sigma(a_2, a_3, \cdots, a_m)$. Even if Σ is positive definite, from Theorem 1 it follows merely that $\Sigma(a_2, a_3, \cdots, a_m)$ is positive semi-definite. On the other hand, positive definiteness of $\Sigma = \Sigma(1, 1, \cdots, 1)$ readily implies that of $\Sigma(a_2, a_3, \cdots, a_m)$ for $a_j = 0$ or 1, $2 \leqq j \leqq m$. Further, for any integer j in $[2, m]$ and any real λ in $(0, 1)$,

$$\begin{aligned}\Sigma(a_2, \cdots, \lambda a_j' + (1 - \lambda)a_j'', \cdots, a_m) \\ = \lambda\Sigma(a_2, \cdots, a_j', \cdots, a_m) + (1 - \lambda)\Sigma(a_2, \cdots, a_j'', \cdots, a_m).\end{aligned}$$

Since a convex linear combination of positive definite matrices is positive definite, $\Sigma(a_2, a_3, \cdots, a_m)$ is positive definite throughout the cube $0 \leqq a_j \leqq 1, 2 \leqq j \leqq m$ which is tantamount to the statement immediately following (2.6). On the other hand, if Σ is singular but $\sigma_{ii} > 0$, $1 \leqq i \leqq m$, the positive definiteness of $\Sigma(0, 0, \cdots, 0)$ together with the fact that a convex linear combination (with $0 < \lambda < 1$) of a positive semi-definite and positive definite matrix is positive definite, insures the positive definiteness of $\Sigma(a_2, a_3, \cdots, a_m)$ at all interior points of the cube.

If $m = 2$, $n_{k1} = r_k$, $n_{k2} = n_k$, $\lim_{k\to\infty} r_k/n_k = c = 0$ and $\sigma_{12} \neq 0$, the random variables $r_k^{-\frac{1}{2}} S_{r_k}^{(1)}$ and $n_k^{-\frac{1}{2}} S_{n_k}^{(2)}$ are asymptotically independent, even though $n^{-\frac{1}{2}} S_n^{(1)}$ and $n^{-\frac{1}{2}} S_n^{(2)}$ are not.

3. Random sums of random vectors. The question of the limit distribution of the sum of a random number of random variables has sought and received attention in the literature. Let $\{N_n, n = 1, 2, \cdots\}$ denote a sequence of positive integer-valued random variables and $\{X_i, i = 1, 2, \cdots\}$ a sequence of independent identically distributed random variables. In [8] (see also [4]), independence of N_n and X_i was postulated, the (normalized) N_n were assumed to converge in distribution and the limit distribution of $\sum_{i=1}^{N_n} X_i$ was determined. In more recent works, the independence assumption has been abandoned at the small cost of strengthening the convergence of the N_n. Anscombe, [1], considers the case of a randomly selected sequence (specializing to a sum) and postulates that $N_n \to_P$ constant. In [7], Rényi demonstrates that if $EX_i = 0$, $EX_i^2 = 1$,

[3] 0/0 is to be interpreted as 0.

a central limit theorem continues to hold for $N_n^{-\frac{1}{2}}\sum_{i=1}^{N_n} X_i$ provided that $N_n/n \to_P Y$ where Y is a positive discrete random variable. The special case of Y = constant was implicitly dealt with by Doeblin [3] in his work on Markov Chains. Extension to the case of a positive (but non-discrete) random variable Y, conjectured in [7], was achieved by Blum, Hanson and J. Rosenblatt in [2] and Mogyorodi [6].

The following theorem generalizes some of the preceding to stable limits and higher dimensions. Analogous results for $m = 1$ appear in [11].

Theorem 3. *Let X_n, S_n, $b(n)$, $g(n)$ and N_n be as described in* (1.1), (1.2), (1.3), (1.4) *and suppose that* (1.5) *obtains with $b(n) = n^{1/\alpha}h(n)$, $0 < \alpha \leqq 2$, where $h(n)$ is a slowly varying function. Further, let a_j be positive constants* (*without loss of generality, take $a_1 \leqq a_2 \leqq \cdots \leqq a_m$*) *such that*

$$N_n^{(j)}/g(n) \to_P a_j, \qquad 1 \leqq j \leqq m. \tag{3.1}$$

Then

$$P\{\textstyle\bigcap_{j=1}^{m}[S_{N_n^{(j)}}^{(j)} < x_j(N_n^{(j)})^{1/\alpha}h(N_n^{(j)})]\} \Rightarrow G_\alpha(x_1, \cdots, x_m; c_2, \cdots, c_m) \tag{3.2}$$

where the cf *corresponding to G_α is given by* (2.2) *with $c_j = (a_1/a_j)^{1/\alpha}$ and $d_j = (1 - a_{j-1}/a_j)^{1/\alpha}$, $2 \leqq j \leqq m$.*

Proof. Set $n_{kj} = [a_j g(k)]$ where $[y]$ signifies the largest integer $\leqq y$ and observe that with $b(n) = n^{1/\alpha}h(n)$, $c_j = (a_1/a_j)^{1/\alpha}$, $d_j = (1 - a_{j-1}/a_j)^{1/\alpha}$, all the conditions of Theorem 1 are met. Consequently, writing $q(n;\, a_j) = ([a_j g(n)])^{1/\alpha}h([a_j g(n)])$,

$$S_{[a_j g(n)]}^{(j)}/q(n;\, a_j), \qquad 1 \leqq j \leqq m, \tag{3.3}$$

have the limiting cdf G_α whose cf is exhibited in (2.2).

On the other hand, we have for $1 \leqq j \leqq m$ (paralleling Rényi [7])

$$\begin{aligned} S_{N_n^{(j)}}^{(j)}/q(N_n^{(j)};\, 1) &= S_{[a_j g(n)]}^{(j)}/q(n;\, a_j) \\ &\quad + [q(n;\, a_j)/q(N_n^{(j)};\, 1)][(S_{N_n^{(j)}}^{(j)} - S_{[a_j g(n)]}^{(j)})/q(n;\, a_j)] \\ &\quad + [S_{[a_j g(n)]}^{(j)}/q(n;\, a_j)][q(n;\, a_j)/q(N_n^{(j)};\, 1) - 1]. \end{aligned} \tag{3.4}$$

In view of (3.1), (3.3) and the slowly varying property, the last factor of the right hand side converges in probability to zero for $1 \leqq j \leqq m$.

Once it is shown that

$$(S_{N_n^{(j)}}^{(j)} - S_{[a_j g(n)]}^{(j)})/q(n;\, a_j) \to_P 0, \qquad 1 \leqq j \leqq m, \tag{3.5}$$

the second factor of the right hand side of (3.4) will likewise converge in probability to zero for $1 \leqq j \leqq m$. Thus, according to Lemma 1 the limit distributions of the two random vectors whose jth components are respectively the first terms of each side of the equality (3.4) will be identical and the theorem will have been proved.

To validate (3.5), we modify somewhat an argument of Rényi (and Doeblin). Let $\delta > 0$ and define

$$B_{nj}(\delta) = \{|N_n^{(j)} - a_j \cdot g(n)| < \delta g(n)\},$$
$$A_{nj}(\epsilon) = \{|S_{N_n^{(j)}}^{(j)} - S_{[a_j g(n)]}^{(j)}| > \epsilon q(n; a_j)\}.$$

Now, for any $\epsilon > 0$, omitting the component index j, denoting the complement of a set B by B', and setting $r_n = [\delta \cdot g(n)]$,

$$\begin{aligned} P\{|S_{N_n} - S_{[ag(n)]}| > \epsilon q(n; a)\} &\leqq P\{A_n(\epsilon) \cdot B_n(\delta)\} + P\{B_n'(\delta)\} \\ &\leqq P\{\max_{|i-ag(n)|<\delta g(n)} |S_i - S_{[ag(n)]}| > \epsilon q(n; a)\} + P\{B_n'(\delta)\} \\ &\leqq 2P\{\max_{1 \leqq i \leqq r_n} |S_i| > \epsilon(a \cdot r_n/2\delta)^{1/\alpha} \cdot h([a \cdot r_n/2\delta])\} + P\{B_n'(\delta)\}. \end{aligned}$$

Since ([9], [10]) the limiting distribution, say $Q(\cdot)$, of $(r_n^{1/\alpha} h(r_n))^{-1} \cdot \max_{1 \leqq i \leqq r_n} |S_i|$ exists and is identical with that of $\sup_{0<\tau<1} |X_\alpha(\tau)|$ where $X_\alpha(\tau)$ is a stable process of index α, the first term of the right hand side differs (for sufficiently large n) by an arbitrarily small amount from $2(1 - Q(\epsilon(a/2\delta)^{1/\alpha}))$. But for sufficiently small positive δ, the latter is, in turn, arbitrarily small while for any fixed $\delta > 0$, the second term $P\{B_n'(\delta)\}$ is less than any preassigned quantity for all but a finite number of values of n. Thus, (3.5) and hence the theorem holds.

COROLLARY 1. *Let X_n, S_n, N_n, $g(n)$ and a_j be as described in* (1.1), (1.2), (1.4), (3.1) *and suppose that* (1.5) *obtains with $F(x_1, x_2, \cdots, x_m) = \Phi_\Sigma(x_1, x_2, \cdots, x_m)$ and $b(n) = n^{1/2}$. Then* (3.2) *holds, the* cf *of G_α being given by* (2.6) *with $c_j = (a_1/a_j)^{1/2}$, $2 \leqq j \leqq m$.*

COROLLARY 2. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables and $h(n)$ a slowly varying function with $P\{S_n < xn^{1/\alpha}h(n)\} \Rightarrow G_\alpha(x)$, a stable* cdf *of characteristic exponent α $(0 < \alpha \leqq 2)$. Further, let N_n be a sequence of positive integer-valued random variables, $g(n)$ as in* (1.3) *and $N_n/g(n) \rightarrow_P a > 0$. Then $P\{S_{N_n} < x(N_n)^{1/\alpha}h(N_n)\} \Rightarrow G_\alpha(x)$.*

THEOREM 4. *Let X_n, S_n, $g(n)$, $b(n)$, N_n be as in* (1.1)–(1.4) *and suppose that* (1.5) *obtains with $b(n) = n^{1/\alpha}h(n)$, $0 < \alpha \leqq 2$ with h slowly varying; let $Y^{(j)}$ be positive discrete random variables such that $N_n^{(j)}/g(n) \rightarrow_P Y^{(j)}$, $1 \leqq j \leqq m$. Then*

$$P\{\textstyle\bigcap_{j=1}^m [S_{N_n^{(j)}}^{(j)} < x(N_n^{(j)})^{1/\alpha}h(N_n^{(j)})]\}$$
$$\Rightarrow \textstyle\sum_{y_2 \cdots y_m} G_\alpha(x_1, \cdots, x_m; y_2, \cdots, y_m) \cdot Q(y_2, \cdots, y_m)$$

where $Q(y_2, \cdots, y_m) = P\{\bigcap_{j=2}^m [Y^{(j)}/Y^{(1)} = y_j]\}$ and G_α is the cdf *of* (2.1) *with $c_j^\alpha = y_j^{-1}$, $d_j^\alpha = 1 - (y_{j-1}/y_j)$ when $1 \geqq y_m \geqq \cdots \geqq y_2 > 0$ and otherwise G_α is the modification of the prescription following Theorem* 1.

THEOREM 5. *Let X_n, S_n, $b(n)$, N_n be as in* (1.1)–(1.4) *and suppose that* (1.5) *obtains for non-singular F, where $b(n) = n^{1/\alpha}h(n)$, $0 < \alpha \leqq 2$, with h slowly varying; let $Y^{(j)}$ be* positive random variables such that $N_n^{(j)}/n \rightarrow_P Y^{(j)}$, $1 \leqq j \leqq$

m. Then if $R^+ = \{(y_2, \cdots, y_m): y_j > 0, 2 \leqq j \leqq m\}$,

$$(3.6)\quad P\{\bigcap_{j=1}^{m} [S_{N_n^{(j)}}^{(j)} < x_j(N_n^{(j)})^{1/\alpha}h(N_n^{(j)})]\} \Rightarrow \int_{R^+} G_\alpha(x_1, \cdots, x_m; y_2, \cdots, y_m)\, dQ(y_2, \cdots, y_m)$$

where $Q(y_2, \cdots, y_m) = P\{\bigcap_{j=2}^{m} [Y^{(j)}/Y^{(1)} < y_j]\}$ *and* G_α *is the* cdf *of* (2.1) *for* $1 \geqq y_m \geqq \cdots \geqq y_2 > 0$ *and the modification of the Prescription otherwise. Again,* $y_j^{-1} = c_j^{\alpha}, d_j^{\alpha} = 1 - y_{j-1}/y_j$.

The proof of Theorem 5 is the m-dimensional analogue of [2], modified (as in Theorem 3) by the employment of a limiting stable process rather than Kolmogoroff's inequality. Similarly, the proof of Theorem 4 is essentially that of Rényi, again circumventing Kolmogoroff's inequality as in Theorem 3.

It is trivial to reformulate Rényi's basic result on mixing sequences in a manner suitable for use in Theorem 4:

LEMMA 3. *Let* $A_{k_{1n}, k_{2n}, \cdots, k_{mn}}$ *be events of positive probability where the* k_{jn} *are non-negative integers with* $\lim_{n\to\infty} k_{jn} = \infty$, $1 \leqq j \leqq m$. *Then a necessary and sufficient condition that* $\lim_{n\to\infty} P\{B \cdot A_{k_{1n}, \cdots, k_{mn}}\} = \alpha P\{B\}$ *for all events* B(*where* $0 < \alpha < 1$) *is that* $\lim_{n\to\infty} P\{A_{k_{1n}, \cdots, k_{mn}} \cdot A_{k_{1i}, \cdots, k_{mi}}\} = \alpha \cdot P\{A_{k_{1i}, \cdots, k_{mi}}\}$ *and* $\lim_{n\to\infty} P\{A_{k_{1n}, \cdots, k_{mn}}\} = \alpha$, *all* $i = 1, 2, \cdots$.

However, it is not quite so straightforward to verify that the condition (and hence conclusion) of the lemma holds for

$$A_{k_{1n}, \cdots, k_{mn}} = \{S_{k_{1n}}^{(1)}/b(k_{1n}) < x_1, \cdots, S_{k_{mn}}^{(m)}/b(k_{mn}) < x_m\}, \qquad n = 1, 2, \cdots.$$

Let I_A be the indicator (set characteristic function) of the event A and set $k_{0i} = \max_{1 \leqq j \leqq m} k_{ji}$ and $J_i = \{j: k_{ji} < k_{0i}\}$. Then denoting conditional probabilities and expectations by $P\{\cdot \mid \cdot\}$ and $E\{\cdot \mid \cdot\}$,

$$\begin{aligned}
&\lim_{n\to\infty} P\{\bigcap_{j=1}^{m} [S_{k_{jn}}^{(j)} < x_j b(k_{jn})] \mid I_{[S_{k_{ji}}^{(j)} < x_j b(k_{ji})]}, 1 \leqq j \leqq m; S_{k_{0i}}^{(j)}, j \in J_i\} \\
&\quad = \lim_{n\to\infty} P\{\bigcap_{j=1}^{m} [S_{k_{jn}}^{(j)} - S_{k_{0i}}^{(j)} < x_j b(k_{jn})] \mid I_{[S_{k_{ji}}^{(j)} < x_j b(k_{ji})]}, 1 \leqq j \leqq m; \\
&\qquad\qquad S_{k_{0i}}^{(j)}, j \in J_i\} \\
&\quad = \lim_{n\to\infty} P\{\bigcap_{j=1}^{m} [(S_{k_{jn}}^{(j)} - S_{k_{0i}}^{(j)})/b(k_{jn} - k_{0i}) < x_j b(k_{jn})/b(k_{jn} - k_{0i})]\} \\
&\quad = G_\alpha(x_1, x_2, \cdots, x_m)
\end{aligned}$$

if $b(n) = n^{1/\alpha}h(n)$ with h slowly varying and the hypotheses of Theorem 1 hold. From the relation

$$\begin{aligned}
&P\{\bigcap_{j=1}^{m} [S_{k_{jn}}^{(j)} < x_j b(k_{jn})] \mid I_{[S_{k_{ji}}^{(j)} < x_j b(k_{ji})]}, 1 \leqq j \leqq m\} \\
&\quad = E[P\{\bigcap_{j=1}^{m} [S_{k_{jn}}^{(j)} < x_j b(k_{jn})] \mid I_{[S_{k_{ji}}^{(j)} < x_j b(k_{ji})]}, 1 \leqq j \leqq m; S_{k_{0i}}^{(j)}, j \in J_i\} \\
&\qquad\qquad \cdot \mid I_{[S_{k_{ji}}^{(j)} < x_j b(k_{ji})]}, 1 \leqq j \leqq m]
\end{aligned}$$

and the dominated convergence theorem, it follows that

$$\lim_{n\to\infty} P\{A_{k_{1n}, \cdots, k_{mn}} A_{k_{1i}, \cdots, k_{mi}}\} = G(x_1, \cdots, x_m)$$

and consequently $\lim_{n\to\infty} P\{A_{k_{1n},\cdots,k_{mn}}\cdot B\} = G(x_1, \cdots, x_m)P\{B\}$ for $0 < G(x_1, \cdots, x_m) < 1$.

4. Double sequences. Let

$$\begin{array}{l} X_{11} \\ X_{21}, X_{22} \\ \vdots \\ X_{n1}, X_{n2}, \cdots, X_{nn} \\ \vdots \end{array} \tag{4.1}$$

constitute a double sequence of row-wise independent, identically distributed random *vectors*, say $X_{ni} = (X_{ni}^{(1)}, X_{ni}^{(2)}, \cdots, X_{ni}^{(m)})$, $1 \leqq i \leqq n$, $n = 1, 2, \cdots$. Consider analogously the row sum vectors

$$S_n = (S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m)}), \qquad n = 1, 2, \cdots, \tag{4.2}$$

where $S_n^{(j)} = \sum_{i=1}^n X_{ni}^{(j)}$ and suppose that for $b(n)$ as in (1.3), there exists a (necessarily infinitely divisible) cdf $F(x_1, x_2, \cdots, x_m)$ with cf $\phi(t_1, t_2, \cdots, t_m)$ such that

$$P\{S_n^{(1)} < x_1 b(n), \cdots, S_n^{(m)} < x_m b(n)\} \Rightarrow F(x_1, x_2, \cdots, x_m). \tag{4.3}$$

Then, we have

THEOREM 6. *Let X_{ni}, $1 \leqq i \leqq n$, S_n, $b(n)$, $F(x_1, x_2, \cdots, x_m)$ be as described in* (4.1), (4.2), (1.3) *and* (4.3). *Further, let $\{k_{jn}, 1 \leqq j \leqq m, n = 1, 2, \cdots\}$ represent m increasing sequences of positive integers such that*

$$\begin{aligned} &\text{(i) } k_{j,n} \leqq n, \qquad \lim_{n\to\infty} k_{j,n}/n = \delta_j \text{ where } 0 < \delta_1 < \cdots < \delta_m \leqq 1, \\ &\text{(ii) } \lim_{n\to\infty} b(n)/b(k_{jn}) = \beta_j, 1 \leqq j \leqq m. \end{aligned} \tag{4.4}$$

Then

$$P\{S_{k_{1n}}^{(1)}/b(k_{1n}) < x_1, \cdots, S_{k_{mn}}^{(m)}/b(k_{mn}) < x_m\} \Rightarrow G(x_1, x_2, \cdots, x_m) \tag{4.5}$$

where the Fourier transform of the cdf *G is given by*

$$\phi^{\delta_1}(\beta_1 t_1, \beta_2 t_2, \cdots, \beta_m t_m)\cdot\phi^{\delta_2-\delta_1}(0, \beta_2 t_2, \cdots, \beta_m t_m) \cdots \phi^{\delta_m-\delta_{m-1}}(0, 0, \cdots, 0, \beta_m t_m). \tag{4.6}$$

PROOF. Let $\phi_n(t_1, t_2, \cdots, t_m)$ denote the cf of X_{n1}. The hypothesis (4.3) implies that for all real $t = (t_1, t_2, \cdots, t_m)$,

$$\lim_{n\to\infty} \phi_n{}^n(t_1/b(n), \cdots, t_m/b(n)) = \phi(t_1, t_2, \cdots, t_m).$$

Since ϕ is infinitely divisible and hence non-vanishing for real t, inside any fixed but arbitrary cube $C = \{|t_i| \leqq T, 1 \leqq i \leqq m\}$ the cf's ϕ_n are non-vanishing for real t for all sufficiently large n. Thus, setting, for all n, $k_{0n} \equiv 0 = \delta_0$, (4.7) $\lim_{n\to\infty} \phi^{k_{jn}-k_{j-1,n}}(t_1/b(n), \cdots, t_m/b(n)) = \phi^{\delta_j-\delta_{j-1}}(t_1, t_2, \cdots, t_m)$

uniformly in C for $1 \leqq j \leqq m$.

Denote $S_n^{(j)}$ by $S^{(j)}(n)$ and $\sum_{i=k_{un}+1}^{k_{u+1,n}} X_{ni}^{(j)}$ by $S^{(j)}(k_{un}, k_{u+1,n})$. For all sufficiently large n, (4.4)(i) implies

$$E[\exp\{i\sum_{j=1}^m t_j S^{(j)}(k_{j,n})/b(k_{jn})\}]$$
$$= E[\exp\{i\sum_{j=1}^m t_j[b(n)/b(k_{jn})][S^{(j)}(k_{1n})/b(n)]\}]$$
$$\cdot E[\exp\{i\sum_{j=2}^m t_j[b(n)/b(k_{jn})][S^{(j)}(k_{1n}, k_{2n})/b(n)]\}]$$
$$\cdots E[\exp\{it_m[b(n)/b(k_{m,n})][S^{(m)}(k_{m-1,n}, k_{mn})/b(n)]\}].$$

Consequently, employing (4.7) and (ii) of (4.4)

$$\lim_{n\to\infty} E[\exp\{i\sum_{j=1}^m t_j S^{(j)}(k_{jn})/b(k_{jn})\}]$$
$$= \phi^{\delta_1}(\beta_1 t_1, \beta_2 t_2, \cdots, \beta_m t_m)\cdot\phi^{\delta_2-\delta_1}(0, \beta_2 t_2, \cdots, \beta_m t_m)$$
$$\cdots \phi^{\delta_m-\delta_{m-1}}(0, \cdots, 0, \beta_m t_m).$$

It is clear that extensions to the case of randomly chosen indices can be given under suitable assumptions.

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