SOME GENERALIZATIONS OF DISTINCT REPRESENTATIVES WITH APPLICATIONS TO STATISTICAL DESIGNS[1]

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1. Introduction. If $S_1, S_2, \cdots, S_n$ are n sub-sets of a given finite set S, then we say that $(a_1, a_2, \cdots, a_n)$ is a system of distinct representatives (SDR) for the sets $S_1, S_2, \cdots, S_n$ if a_i belongs to S_i and all a_i's are distinct. The necessary and sufficient condition in order that the sets $S_1, S_2, \cdots, S_n$ possess an SDR is that the union of any k of the sets contain at least k distinct elements ([6], [8]). The concept of distinct representatives has been generalized in various directions with a wide field of applications ([5], [8], [9]). In this paper some further generalizations are given with applications to design of experiments.

2. Generalization.

DEFINITION 2.1. If $S_1, S_2, \cdots, S_n$ are the n sub-sets of a given finite set S, then $(O_1, O_2, \cdots, O_n)$ will be called a $(m_1, m_2, \cdots, m_n)$ SDR if

(i) $O_i \subseteq S_i$,

(ii) $n(O_i) = m_i$, and

(iii) $O_i \cap O_j = \emptyset, i \neq j, = 1, 2, \cdots, n$,

where $n(O_i)$ is the number of elements in the set O_i.

If $m_1 = m_2 = \cdots = m_n = m$, the sets will be said to possess an m-ple SDR.

We can prove the following theorem on similar lines as Theorem 2.1 of [8].

THEOREM 2.1. *A necessary and sufficient condition in order that* $S_1, S_2, \cdots, S_n$ *may possess a* $(m_1, m_2, \cdots, m_n)$ SDR *is that*

$$n(S_{i_1} \cup S_{i_2} \cup S_{i_3} \cup \cdots \cup S_{i_k}) \geqq \textstyle\sum_{j=1}^{k} m_{i_j},$$

$$1 \leqq i_1 < i_2 < \cdots < i_k \leqq n; \quad 1 \leqq k \leqq n.$$

3. Applications.

LEMMA 3.1. *Given positive integers v, b, r and k such that $bk = vr$ and $v > k$ then there exists an equi-replicate binary incomplete block design in v treatments each replicated r times in b blocks of constant block size k.*

THEOREM 3.1. *In every binary equi-replicate design (with column as blocks) of constant block size k such that $bk = vr$ and $b = mv$, the treatments can be rearranged into blocks, so that every treatment occurs in a row m times.*

PROOF. Form the sets $S_1, S_2, \cdots, S_v$ where S_i is the set of all block numbers containing the treatment i. Now,

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$$n(S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_u}) \geqq ur/k = umk/k = um,$$
$$1 \leqq i_1 < \cdots < i_k \leqq v; \quad 1 \leqq k \leqq v;$$

and hence by Theorem 2.1 we can choose an m-ple SDR, say $(O_1, O_2, \cdots, O_v)$.

Let $\bar{S}_i = S_i - \{O_i\}, i = 1, 2, \cdots, v$.

Now every $\bar{S}_i$ contains $m(k - 1)$ different block numbers and each block number is replicated $(k - 1)$ times.

$$n(\bar{S}_{i_1} \cup \bar{S}_{i_2} \cup \cdots \cup \bar{S}_{i_{u'}}) \geqq u'm(k - 1)/(k - 1) = u'm,$$
$$1 \leqq i_1 < i_2 \cdots < i_{u'} \leqq v; \quad 1 \leqq u' \leqq v;$$

and hence there exists a second SDR. It is important to note that the second SDR is such that if the representations of S_i are $O_i^{(1)}$ and $O_i^{(2)}$ then $O_i^{(1)}$ and $O_i^{(2)}$ are disjoint. Evidently the process can be continued to get k SDR's which may be written as 1st, 2nd, $\cdots$, kth row. Replace the block numbers in the row by the set number which they represent. We will find that each treatment occurs in a row m times.

Example. Let us take a BIB design of [2] page 471 with parameters $v = 5$, $k = 3, r = 6, b = 10, \lambda = 3$.

							1st SDR	2nd SDR	3rd SDR
S_1	1,	2,	3,	6,	7,	8	1, 2	7, 6	3, 8
S_2	1,	2,	4,	6,	9,	10	9, 10	1, 2	4, 6
S_3	1,	4,	5,	7,	8,	9	7, 8	5, 4	1, 9
S_4	3,	4,	5,	6,	7,	10	6, 4	10, 3	7, 5
S_5	2,	3,	5,	8,	9,	10	3, 5	8, 9	2, 10

Hence the required design is as follows:

Rows	Block No.									
	1	2	3	4	5	6	7	8	9	10
1	1	1	5	4	5	4	3	3	2	2
2	2	2	4	3	3	1	1	5	5	4
3	3	5	1	2	4	2	4	1	3	5

Theorem 3.2. *In every binary equi-replicate incomplete block design (with blocks as columns) of constant block size k such that $bk = vr$ and $r = mk + t$ (m is a positive integer or zero and $0 < t \leqq k - 1$), the treatments can be rearranged into blocks so that every treatment occurs m or $(m + 1)$ times in a row.*

Proof. $b = vr/k = v(mk + t)/k = vm + tv/k$.

Add $(k - t)$ dummy replications of the v treatments in s blocks of constant block size k clearly $s = [(k - t)/k]v$. Our Lemma 3.1 asserts that this can always be done. Let the new blocks be numbered $(b + 1), (b + 2), \cdots, (b + s)$.

As in Theorem 3.1, form the sets $S_1, S_2, \cdots, S_v$ of $(b + s)$ block numbers where S_i is the set of all block numbers containing treatment i.

Clearly $b + s = (m + 1)v$, and applying Theorem 2.1 there exist k SDR's. Write them as k rows. Delete the dummy block numbers and replace the block numbers by the set they represent. We find that every treatment occurs either m or $(m + 1)$ times in a row.

EXAMPLE. Take the PBIB design SR-4 in [1] with parameters $v = 6$, $r = 6$, $k = 4$, $b = 9$, $m = 2$, $n = 3$. Introduce 3 dummy blocks say (10, 11 and 12) and form the sets $S_1, S_2, \cdots, S_6$.

									1st SDR	2nd SDR	3rd SDR	4th SDR
S_1	1,	2,	3,	4,	5,	6,	10,	11	1, 10	4, 2	6, 11	3, 5
S_2	1,	2,	4,	6,	7,	9,	11,	12	3, 11	6, 7	9, 12	4, 1
S_3	1,	2,	3,	7,	8,	9,	12,	10	2, 12	8, 3	1, 10	7, 9
S_4	1,	2,	4,	5,	7,	8,	10,	11	4, 5	1, 10	7, 8	2, 11
S_5	4,	5,	6,	7,	8,	9,	11,	12	6, 7	9, 11	4, 5	8, 12
S_6	2,	3	5,	6,	8,	9,	12,	10	8, 9	5, 12	2, 3	6, 10

Delete the dummy block numbers (10, 11 and 12) and we get the required design.

Rows	Block No.								
	1	2	3	4	5	6	7	8	9
1	1	3	2	4	4	5	5	6	6
2	4	1	3	1	6	2	2	3	5
3	3	6	6	5	5	1	4	4	2
4	2	4	1	2	1	6	3	5	3

COROLLARY. *Let n_{ij} denote the number of times that the* ith *treatment occurs in the* jth *row.*

CASE (1). When $t = 1$, then in each row v/k treatments occur $(m + 1)$ times each and the others m times each. Each treatment appears $(m + 1)$ times in some row and m times in the other rows. Define the ith and uth treatments to be first associates if there is a row in which they both occur $(m + 1)$ times each, and to be second associates otherwise. Then

$$\textstyle\sum_j n_{ij}^2 = m^2k + 2m + 1, \qquad i = 1, 2, \cdots, v,$$

$$\textstyle\sum_j n_{ij}n_{uj} = m^2k + 2m + 1, \qquad \text{if } i \text{ and } u \text{ are the first associates,}$$

$$= m^2k + 2m, \qquad \text{if they are second associates.}$$

The row association scheme is that of a group divisible design with k groups of v/k treatments each.

CASE (2). When $t = k - 1$ each treatment appears exactly m times in some row and $(m + 1)$ times in each of the other rows. In this case define the ith and uth treatments to be first associates if there is a row in which they both appear exactly m times each. Then

$$\sum_j n_{ij}^2 = k(m + 1)^2 - 2(m + 1) + 1, \qquad i = 1, 2, \cdots, v,$$

$$\sum_j n_{ij}n_{uj} = k(m + 1)^2 - 2(m + 1) + 1, \quad \text{if } i \text{ and } u \text{ are first associates,}$$

$$= k(m + 1)^2 - 2(m + 1), \qquad \text{if } i \text{ and } u \text{ are second associates.}$$

The row association scheme is again that of a group divisible design.

Thus, if the original design is a balanced incomplete block design with $r = mk \pm 1$, we obtain a partially balanced design for two-way elimination of heterogeneity [7].

4. Concluding remarks. An algorithm to obtain an m-ple SDR and further applications will be given in a subsequent paper.

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