ON THE SEMIMARTINGALE CONVERGENCE THEOREM

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0. Summary. This paper consists of three parts: first a new simple proof of the semimartingale theorem of Doob is given, next the limit function is identified as the derivative of a certain σ-additive set function. Finally it is shown how the approach of Sparre-Andersen and Jessen can be generalized to give the convergence and the identification of the limit function for a semimartingale.

1. Introduction. It is well known that the maximal ergodic lemma, asserting that $E(Y_1 \mid \sup_n Y_n > 0) \geqq 0$ for $Y_n = n^{-1}(X_1 + \cdots + X_n)$ where $(X_n, n \geqq 1)$ is stationary, remains true for $(Y_n, n \geqq 1)$ a decreasing martingale sequence and directly implies pointwise convergence in both cases (apply to $Y_n' = (Y_n - b)I_C$ and $Y_n'' = (a - Y_n)I_C$ where $C = [\liminf_n Y_n < a, \limsup_n Y_n > b]$, $a < b$, obtaining $b \leqq E(Y_1 \mid C) \leqq a$ and hence $PC = 0$). This close formal analogy can be exploited to generalize and unify the two convergence theorems (see Tulcea [8]). Furthermore this approach provides a very simple proof of the decreasing martingale (and hence decreasing semimartingale) convergence theorem since the lemma is an immediate consequence of the martingale property (write $[\sup_{1 \leqq i \leqq n} Y_i > 0] = \sum_{i=1}^n A_i$, $A_i = [Y_i > 0, Y_j \leqq 0, i < j \leqq n]$, apply the martingale property, obtaining $E(Y_1 \mid A_i) = E(Y_i \mid A_i) \geqq 0$ and hence $E(Y_1 \mid \sup_{1 \leqq i \leqq n} Y_i > 0) \geqq 0$, and let $n \to \infty$).

In this paper a new simple proof is given of the well known increasing semimartingale theorem of Doob [3], using the same basic ideas and preserving some of the above mentioned formal analogy. The theorem is stated and proved (with no extra difficulty) for $X_n = d\varphi_n/dP$, $n \geqq 1$, where φ_n is a signed measure on the σ-field $\mathfrak{A}_n \uparrow \mathfrak{A}_0$, and $\varphi_n \leqq \varphi_{n+1}$ on $\mathfrak{A}_n$. In this formulation the theorem (1) is also well known (see Krickeberg [6]), and is in fact equivalent to Doob's theorem. In general $(X_n, n \geqq 1)$ is neither a semimartingale nor a lower semimartingale; its convergence can, however, easily be reduced to Doob's theorem under the condition that ensures convergence, namely that $\lim_{n\to\infty} \varphi_n^+ < +\infty$ (remark 2).

In the paper by Andersen and Jessen [1], they prove convergence of $X_n = d\varphi_n/dP$, where $\varphi_n = \varphi$ on $\mathfrak{A}_n$ and φ is σ-additive, and the limit of X_n is dentified as $d\varphi/dP$. In this paper it is shown that in case $\varphi_n \leqq \varphi_{n+1}$ an identification can be made as follows: let $\varphi_0 = \lim_{n\to\infty} \varphi_n$ on $\mathfrak{A}_0$, then $X = d\varphi/dP$, where φ is the σ-additive part of φ_0. For this result, which is new, the set function formulation is a natural one. Finally it is shown how the method of Andersen and Jessen of establishing convergence and identification simultaneously can be generalized to the case where $\varphi_n \leqq \varphi_{n+1}$.

A counter-example is constructed to show that $\varphi_n \to \varphi$ and $X_n \to X$ do not imply $X = d\varphi/dP$ if the "semimartingale" condition $\varphi_n \leqq \varphi_{n+1}$ on $\mathfrak{A}_n$ is dropped.

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2. Preliminaries. Let ψ_0 be a set function defined on a field $\mathfrak{B}_0$ of subsets of a space Ω. We require that all set functions take on at least one finite value. Also notation such as $\psi_0 A$ will imply that A belongs to the domain of definition of ψ_0. Let $\psi_0^+ A = \sup_{A' \subset A} \psi_0 A'$, $A \in \mathfrak{B}_0$, and let $\psi_0^- = (-\psi_0)^+$. We now suppose that ψ_0 is additive; then ψ_0^+ and ψ_0^- are contents, i.e. nonnegative additive set functions.

Let $\mathfrak{B}$ be the σ-field generated by $\mathfrak{B}_0$, and let $\nu^{\pm}$ be the largest measure defined on $\mathfrak{B}$ and bounded by $\psi_0^{\pm}$ on $\mathfrak{B}_0$, see Hewitt and Yosida [9] or Dunford and Schwartz [4]. If $\psi_0^+ < +\infty$ we have $\nu^+ < +\infty$, and we define $\nu = \nu^+ - \nu^-$. We call ν the σ-additive part of ψ_0. We can evaluate ν as follows: $\nu A = \lim \sum_{n=1}^{\infty} \psi_0 A_n$, where $\{A_n, n \geqq 1\}$ is a covering of A and where the limit is taken along the direction determined by refinement of such coverings. If $|\psi_0| < +\infty$ we can characterize ν as the only measure such that $\nu - \psi_0$ is purely finitely additive.

Suppose now we have a probability measure P on $(\Omega, \mathfrak{A})$, $\mathfrak{B} \subset \mathfrak{A}$. Consider again the measure ν on $\mathfrak{B}$. There exists $D \in \mathfrak{B}$ unique up to $(P + \nu)$-null sets such that ν is σ-finite and P-continuous on D and $\nu A = \infty$ if $A \subset \Omega - D$ and $PA > 0$. To establish the uniqueness suppose D and D' are two such sets. If $P(D - D') > 0$ then by σ-finiteness of ν on D there exists $A \subset D - D'$ such that $PA > 0$, $\nu A < \infty$, which is a contradiction, since $A \subset \Omega - D'$; therefore $P(D - D') = 0$ and, by P-continuity of ν on D, $\nu(D - D') = 0$; by the same argument $(P + \nu)(D' - D) = 0$. To prove existence: the collection of sets in $\mathfrak{B}$ on which ν is σ-finite and P-continuous is closed under countable union and therefore the supremum of its P-values is achieved, on D say. Suppose $A \in \mathfrak{B}$, $A \subset \Omega - D$, $PA > 0$, and $\nu A < \infty$. The collection of $\mathfrak{B}$-measurable P-null subsets of A is closed under countable union; let the supremum of its ν-values be achieved on A'. We have $P(A - A') > 0$, $\nu(A - A') < \infty$, and ν is P-continuous on $A - A'$ by definition of A'; but then ν is σ-finite and P-continuous on $D + A - A'$ and $P(D + A - A') > PD$, which contradicts the definition of D. (Note that if ν is finite (or σ-finite) then $\Omega - D$ is P-null and we obtain the usual decomposition of ν into its continuous and singular components.)

Now for the signed measure ψ there exists $E \in B$ such that $\psi \geqq 0$ on E and $\psi \leqq 0$ on $\Omega - E$. Applying the above result to E and $\Omega - E$ separately we get $\Omega = D + D_+ + D_-$, where ψ is σ-finite and P-continuous on D, $\pm\psi \geqq 0$ on $D_{\pm}$, and $\psi A = \pm\infty$ if $A \subset D_{\pm}$ and $PA > 0$. For $A \in \mathfrak{B}$ let $\psi_c A = \psi(AD)$, $\psi_s^+ A = \psi(AD_+)$, and $\psi_s^- A = -\psi(AD_-)$. Also let $Y = d\psi_c/dP$ on D and $\pm\infty$ on $D_{\pm}$. It is easily verified that $Y = d\psi/dP$, that is, Y is $\mathfrak{B}$-measurable and $A \subset [Y < a] \Rightarrow \psi A \leqq aPA$, $A \subset [Y > a] \Rightarrow \psi A \geqq aPA$, a real. Also $d\psi/dP$ is unique up to $(P + \psi^+ + \psi^-)$-null sets: if $Y' = d\psi/dP$ then for $A \subset [Y < a, Y' > a']$, $a < a'$, we obtain $\psi A \leqq aPA$ and $a'PA \leqq \psi A$ so that $PA = 0$ and hence $\psi A = 0$. Therefore $(P + \psi^+ + \psi^-)[Y < a, Y' > a'] = 0$, hence $(P + \psi^+ + \psi^-) \cdot [Y < Y'] = 0$, and by the same argument $(P + \psi^+ + \psi^-)(Y > Y') = 0$. Note that $Y^{\pm} = d\psi^{\pm}/dP$; this is most easily seen from the definition of Y.

3. Main results. Let $\varphi_1, \varphi_2, \cdots$ be signed measures on σ-fields $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}$ such that $\varphi_n \leqq \varphi_{n+1}$ on $\mathfrak{A}_n$ (and hence $\varphi_n^+ \leqq \varphi_{n+1}^+$ on $\mathfrak{A}_n$) and let $X_n =$

$d\varphi_n/dP$. Let $B_{ij} = \bigcup_{n=i}^{j} [X_n \leqq b] \varepsilon \mathfrak{A}_j$, $C_{ij} = \bigcup_{n=i}^{j} [X_n \geqq c] \varepsilon \mathfrak{A}_j$, b, c real ($B_{ij} = C_{ij} = \emptyset$ if $i > j$); also let $B = \lim_i \lim_j B_{ij} \supset [\liminf X_n < b]$, $C = \lim_i \lim_j C_{ij} \supset [\limsup X_n > c]$. The first two inequalities below are standard; the third, which is a slight variation of the second, will not be used in proving a.s. convergence.

$$(1) \qquad \varphi_n^+(AC_{ij}) \geqq cP(AC_{ij}) \quad \text{for} \quad A \,\varepsilon\, \mathfrak{A}_i, \qquad c \geqq 0, i \leqq j \leqq n.$$

$$(2) \qquad \varphi_n^+(AB_{ij}) \leqq bP(AB_{ij}) + \varphi_n^+\Omega - \varphi_i^+\Omega \quad \text{for} \quad A \,\varepsilon\, \mathfrak{A}_i, \qquad b \geqq 0, i \leqq j \leqq n, \text{ if } \varphi_n^+ < \infty.$$

$$(3) \qquad \varphi_n^-(AC_{ij}) \leqq -cP(AC_{ij}) + \varphi_n^+\Omega - \varphi_i^+\Omega \quad \text{for} \quad A \,\varepsilon\, \mathfrak{A}_i, \qquad c \leqq 0, i \leqq j \leqq n, \text{ if } \varphi_n^+ < \infty.$$

PROOFS. (1) Let $A_m = A(C_{im} - C_{i,m-1})$, $m = i, \cdots, j$, then $[X_m^+ \geqq c] = [X_m \geqq c] \supset A_m \,\varepsilon\, \mathfrak{A}_m$ so that $\varphi_n^+A_m \geqq \varphi_m^+A_m \geqq cPA_m$. Sum over m.

(2) Let $A_m = A(B_{im} - B_{i,m-1})$, $m = i, \cdots, j$, then $\varphi_n^+A_m = \varphi_m^+A_m + \varphi_n^+A_m - \varphi_m^+A_m \leqq bP(A_m) + \sum_{r=m}^{n-1} (\varphi_{r+1}^+A_m - \varphi_r^+A_m)$ and hence, summing over m,

$$\varphi_n^+(AB_{ij}) \leqq bP(AB_{ij}) + \sum_{r=i}^{n-1} (\varphi_{r+1}^+A_r' - \varphi_r^+A_r') \leqq bP(AB_{ij}) + \sum_{r=i}^{n-1} (\varphi_{r+1}^+\Omega - \varphi_r^+\Omega),$$

where $A_r' \,\varepsilon\, \mathfrak{A}_r$.

(3) Define A_m as in (1); from $\varphi_m A_m \leqq \varphi_n A_m$, that is, $(\varphi_m^+ - \varphi_m^-)A_m \leqq (\varphi_n^+ - \varphi_n^-)A_m$, we obtain $\varphi_n^-A_m \leqq \varphi_m^-A_m + \varphi_n^+A_m - \varphi_m^+A_m$, and since $A_m \subset [X_m \geqq c] = [X_m^- \leqq -c]$ we have $\varphi_m^-A_m \leqq -cP(A_m)$. The rest of the argument is exactly as in (2).

THEOREM 1. *If* $\lim \varphi_n^+ < \infty$ *then* X_n *converges* a.s.

PROOF. It suffices to show that $P(BC) = 0$ for $b < c$. We can take $b = 0$ since $d(\varphi_n - bP)/dP = X_n - b$ and $(\varphi_n - bP)^+\Omega \leqq \varphi_n^+\Omega + |b|$. For $p \leqq q \leqq r$ we have

$$cP(B_{pq}C_{qr}) \leqq \varphi_r^+(B_{pq}C_{qr}) \leqq \varphi_r^+B_{pq} \leqq \varphi_r^+\Omega - \varphi_p^+\Omega,$$

where we have applied (1) with $A = B_{pq}$, $i = q, j = n = r$, and (2) with $i = p$, $j = q$, $n = r$. Taking $\lim_p \lim_q \lim_r$ the result follows.

Let $\mathfrak{A}_0 = \cup\, \mathfrak{A}_n$; we can assume that the field $\mathfrak{A}_0$ generates $\mathfrak{A}$. Let $\varphi_0 = \lim \varphi_n$ on $\mathfrak{A}_0$, that is, for $A \,\varepsilon\, \mathfrak{A}_0$, $A \,\varepsilon\, \mathfrak{A}_r$ say, $\varphi_0 A = \lim_{n \geqq r} \varphi_n A$. Now $\varphi_0(A + A') = \varphi_0 A + \varphi_0 A'$ if $\{\varphi_0 A, \varphi_0 A'\} \neq \{\infty, -\infty\}$; it easily follows that φ_0^+ and φ_0^- are contents. Also $\varphi_0^+ = \lim \varphi_n^+$, for $\varphi_n^+ \leqq \varphi_0^+$ on $\mathfrak{A}_n$ so that $\lim \varphi_n^+ \leqq \varphi_0^+$, and if $A' \subset A$, A', $A \,\varepsilon\, \mathfrak{A}_r$, then $\varphi_0 A' = \lim_{n \geqq r} \varphi_n A' \leqq \lim_{n \geqq r} \varphi_n^+ A = (\lim \varphi_n^+)A$ so that, taking supremum over $A' \subset A$, $\varphi_0^+A \leqq (\lim \varphi_n^+)A$. Further, if $\varphi_0 A < \infty$ then $\varphi_0^-A = \varphi_0^+A - \varphi_0 A = (\lim \varphi_n^+)A - (\lim \varphi_n)A = (\lim \varphi_n^-)A$. Let $\mu_\pm$ be the largest measure on $\mathfrak{A}$ bounded above by $\varphi_0^\pm$ on $\mathfrak{A}_0$, and let $X = \liminf X_n^+ - \liminf X_n^-$ so that $X^\pm = \liminf X_n^\pm$. (Theorem 1 asserts that $X_n \to X$ a.s.

if $\varphi_0^+ < \infty$, this fact will not be used, however, in the lemma and Theorem 2 below.)

LEMMA. *If* $\varphi_0^+ < \infty$ *then* $\mu^\pm A \leqq bPA$ *for* $A \subset [X^\pm < b]$.

PROOF. Let $A \varepsilon \mathfrak{A}_0$, $A \varepsilon \mathfrak{A}_r$ say; then for $r \leqq i \leqq j, b \geqq 0$ we have $\mu_+(AB_{ij}) \leqq \varphi_0^+(AB_{ij}) \leqq bP(AB_{ij}) + \varphi_0^+\Omega - \varphi_i^+\Omega$ by letting $n \to \infty$ in (2). Taking $\lim_i \lim_j$ we obtain $\mu_+(AB) \leqq bP(AB)$. By the monotone class argument this holds for $A \varepsilon \mathfrak{A}$; now take $A \subset [X^+ < b] \subset B$. A parallel argument, using (3), applies to μ_-, X^-.

Suppose $\varphi_0^+ < \infty$ and let $\varphi = \mu_+ - \mu_-$ (if also $\varphi_0^- < \infty$ then, as stated earlier, the signed measure φ can be characterized by the fact that $\varphi_0 - \varphi$ is purely finitely additive). From the lemma $\mu_\pm[X^\pm = 0] = 0$ so that $\varphi^\pm = \mu_\pm$. Also by the lemma φ is P-continuous and σ-finite on $[X$ finite$]$ so that $\varphi_s^\pm$ is concentrated on $[X = \pm\infty] = [X_n \to \pm\infty]$, that is, $X_n \to \pm\infty$ a.e. $(\varphi_s^\pm)$. Combining this with Theorem 1 gives

THEOREM 1′. *If* $\varphi_0^+ < \infty$ *then* $X_n \to X$ a. e. $(P + \varphi^+ + \varphi^-)$.

THEOREM 2. *If* $\varphi_0^+ < \infty$ *then* $X = d\varphi/dP$.

PROOF. In view of the lemma it suffices to show $\mu_\pm A \geqq bPA$ for $A \subset [X^\pm > b]$. For $A \varepsilon \mathfrak{A}_0$, $A \varepsilon \mathfrak{A}_r$ say, let $\mu_\pm' A = \int_A X^\pm \, dP$; by Fatou's lemma $\mu_\pm' A \leqq \liminf \int_A X_n^\pm \, dP \leqq \lim_{n \geqq r} \varphi_n^\pm A = \varphi_0^\pm A$ so that $\mu_\pm \geqq \mu_\pm'$, by the maximality of $\mu_\pm$. If $A \subset [X^\pm > b]$ then $\mu_\pm A \geqq \mu_\pm' A \geqq bPA$.

By using the explicit evaluation of φ as stated in the preliminaries it is possible to extend the method of Andersen and Jessen [1] to prove the convergence of $X_n = d\varphi_n/dP$ and the identification of $\lim_{n\to\infty} X_n$ simultaneously as follows: Let $D_{ij} = [X_j \leqq b, X_n > b, i \leqq n < j] \varepsilon \mathfrak{A}_j$, $E_{ij} = [X_j \geqq c, X_n < c; i \leqq n < j] \varepsilon \mathfrak{A}_j$, $i \leqq j$, and let $\underline{X} = \liminf X_n$, $\bar{X} = \limsup X_n$. Suppose $A \subset [\underline{X} < b]$, $\varphi A > -\infty$, and let $(A_n, n \geqq 1)$ be a disjoint covering of A by sets of $\mathfrak{A}_0$. We can take $\varphi_0 A_n > -\infty$ and choose i_n so large that $A_n \varepsilon \mathfrak{A}_{i_n}$ and $\varphi_0 A_n - \varphi_{i_n} A_n < \epsilon/2^n$ where $\epsilon > 0$. The refinement $(A_n D_{i_n j}, n \geqq 1, j \geqq i_n)$ of $(A_n, n \geqq 1)$ is also a covering of A since $[\underline{X} < b] \subset \bigcup_{j \geqq i} [X_j \leqq b] = \sum_{j \geqq i} D_{ij}$. We have $\sum_{n,j} \varphi_0(A_n D_{i_n j}) = \sum_{n,j} \varphi_j(A_n D_{i_n j}) + \sum_{n,j} \sum_{r \geqq j} (\varphi_{r+1} A_n D_{i_n j} - \varphi_r A_n D_{i_n j}) \leqq \sum_{n,j} bP(A_n D_{i_n j}) + \sum_n \sum_{r \geqq i_n} (\varphi_{r+1} A_n - \varphi_r A_n)$. Note that the last term is bounded by ϵ. Passing to the limit along the disjoint coverings and letting $\epsilon \to 0$ yields $\varphi A \leqq bPA$. If $\varphi A = -\infty$ the inequality is trivially true. If $A \subset [\bar{X} > c]$, $A \varepsilon \mathfrak{A}$, and $(A_n, n \geqq 1)$ is a disjoint covering of A with $A_n \varepsilon \mathfrak{A}_{i_n}$ say, then $\sum_{nj} \varphi_0(A_n E_{i_n j}) \geqq \sum_{nj} \varphi_j(A_n E_{i_n j}) \geqq c \sum_{nj} P(A_n E_{i_n j})$ and passing to the limit we obtain $\varphi A \geqq cPA$. Since $A \subset [\underline{X} > c] \subset [\bar{X} > c] \Rightarrow \varphi A \geqq cPA$ and $A \subset [\bar{X} < b] \subset [\underline{X} < b] \Rightarrow \varphi A \leqq bPA$ we have $\underline{X} = d\varphi/dP = \bar{X}$ so that $\bar{X} = \underline{X}$ a.e. $(P + \varphi^+ + \varphi^-)$. Note that the assumption $\varphi_0^+ < \infty$ enters in the existence of φ.

4. Remarks. 1. In Theorem 1 the assumption $\varphi_0^+ < \infty$ can be replaced by φ_0^+ σ-finite and Theorem 2 still holds if $\varphi_0^+\Omega = \infty$ but φ_0^+ is σ-finite and $\varphi_0^- < \infty$ (if $\Omega = \sum A_m$, $A_m \varepsilon \mathfrak{A}_{i_m}$, $\varphi_0^+ A_m < +\infty$, let $\varphi_{nm} A = \varphi_n(AA_m)$, $A \varepsilon \mathfrak{A}_n$, $n \geqq i_m$ and treat the sequence $(\varphi_{nm}, n \geqq i_m)$).

2. Theorem 1 can be reduced to Doob's theorem as follows: Suppose $\varphi_0^+ < \infty$

and in addition $\varphi_n \geqq 0$. Let $\tilde{\varphi}_{i+1}$ denote φ_{i+1} contracted to $\mathfrak{A}_i$ and define $X_1'' = 0$, $X_n'' = \sum_{i=1}^{n-1} d(\tilde{\varphi}_{i+1} - \varphi_i)/dP$, $n > 1$, and $X_n' = X_n'' - X_n$ so that X_n'' is nondecreasing and X_n' is $\mathfrak{A}_n$-measurable. Further X_n' is a semimartingale, for $X'_{n+1} - X_n' = d(\tilde{\varphi}_{n+1} - \varphi_n)/dP - (X_{n+1} - X_n) = d\tilde{\varphi}^c_{n+1}/dP + d\tilde{\varphi}^s_{n+1}/dP - X_{n+1}$ so that $\int_A (X'_{n+1} - X_n')\, dP = \varphi^c_{n+1}A + (\tilde{\varphi}^s_{n+1})^c A - \varphi^c_{n+1}A \geqq 0$, $A \in \mathfrak{A}_n$. Since $E(X_n')^+ \leqq EX_n'' \leqq \varphi_n\Omega - \varphi_1\Omega \leqq \varphi_0\Omega - \varphi_1\Omega < \infty$, we have $\lim X_n''$ finite and, by the semimartingale convergence theorem (see Loève [6]), a.s. convergence of X_n' so that X_n converges a.s. We now drop the assumption $\varphi_n \geqq 0$. Since $(\varphi_n + mP)^+ \leqq \varphi_n^+ + m \leqq \varphi_0^+ + m < \infty$ (where $m \geqq 0$) we have, by the above result, a.s. convergence of $d(\varphi + mP)^+/dP = (X_n + m)^+$ that is, a.s. convergence of X_n on [lim sup $X_n > -m$], and since m is arbitrary, on [lim sup $X_n > -\infty$], and hence on Ω since $\lim X_n = -\infty$ on [lim sup $X_n = -\infty$].

3. We here construct a counter example to show that the identification given in Theorem 2 is not simply a consequence of the convergence of the set functions and their derivatives (dropping the "semimartingale" condition). Let $\Omega = (0, 1]$, $\mathfrak{A} =$ (Borel sets) $P =$ Lebesgue measure, let $\mathfrak{B}_n$ be generated by $((j - 1/s_n, j/s_n]$, $j = 1, \cdots, s_n)$, where s_{n-1} divides s_n, and let ν_n assign mass $1/s_{n-1}$ to each $\mathfrak{B}_{n-1}$-atom, concentrating it on a single $\mathfrak{B}_n$-atom. We have $\nu_n \to P$ on $\mathfrak{B}_0 = \bigcup \mathfrak{B}_n$ and $P[d\nu_n/dP > 0] = s_{n-1}/s_n = 1/2^n$. (Choosing $s_n = 2^{n(n+1)/2}$) so that, by the Borel-Cantelli lemma, $d\nu_n/dP \to 0$ a.s.

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