A NOTE ON NONPARAMETRIC TESTS FOR SCALE[1]

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1. Introduction. Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n (m + n = N)$ be two samples from two populations with continuous cumulative distribution functions $F(x)$ and $G(y) = F(\theta y)$ where $\theta > 0$ is a scale parameter. To test the equality of the two populations, i.e., to test the null hypothesis,

$$H_0 : \theta = 1$$

various two-sample nonparametric tests are available. The S-T test proposed by Siegel and Tukey [9], the M-test proposed by Mood [7], the S-test proposed by I. R. Savage [9], the N-S test proposed by Klotz [6], which is asymptotically equivalent to that proposed by Capon [1], are a few among them. When the parent populations are normal the N-S test is known to be asymptotically locally most powerful (ALMP); i.e., its Pitman asymptotic relative efficiency (ARE) with respect to the likelihood ratio test is one. Similarly, when the parent populations are exponential the S-test has been shown in [1] and [4] to be ALMP.

We ask the reader to recall that, under certain conditions, the Pitman ARE of one test with respect to another, against a specific family of alternatives, is given by the ratio of their efficacies (see, e.g., Noether [8] or Chernoff and Savage [2]), against that family of alternatives. In [6], page 501, Klotz asserts that the efficacy of the N-S test against the exponential alternative $G(y) = 1 - e^{-\theta y}$, $\theta \geqq 1$, is infinite. Since the efficacy of the S-test against this alternative is positive and finite (see (3.2) below), this implies that the N-S test is asymptotically better than the S-test, which is impossible, since the S-test is ALMP.

The object of this note is to clarify the above confusion by computing the efficacy of the N-S test against the the exponential alternative. In addition to this we compute ARE's for various pairs of tests against the exponential alternative. In particular we give the correct value of the entries in the first row of Klotz's Table I [6] thereby establishing the superiority of the S-test over the N-S test in the exponential case.

2. General Expressions for Efficacies. The general expressions for the efficacies of the N-S test, M-test and S-T test against the alternative $G(y) = F(\theta y)$ are well known but they are given below for convenience.

	Name of Test	Expression for efficacy
(2.1)	N-S test	$e(N\text{-}S) = (2mn/N)[\int_{-\infty}^{\infty} (\Phi^{-1}(F)/\varphi(\Phi^{-1}(F)))xf^2(x)\,dx]^2$

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$$(2.2)\qquad M\text{-test}\qquad e(M) = (720mn/N)[\int_{-\infty}^{\infty} (F(x) - \tfrac{1}{2})xf^2(x)\,dx]^2$$

$$(2.3)\qquad S\text{-}T\text{ test}\qquad e(S\text{-}T) = (48mn/N)[\int_{\nu}^{\infty} xf^2(x)\,dx - \int_{-\infty}^{\nu} xf^2(x)\,dx]^2,$$

where $\Phi(x)$ and $\varphi(x)$ denote the standard normal cdf and density function, respectively, and ν is any median of F. In this section we shall find the general expression for the efficacy of the S-test.

Denoting by $E_\theta(\cdot)$ the expectation under the hypothesis that θ is the true value, we have from the Chernoff-Savage Theorem [2]

$$(2.4)\qquad E_\theta(S) = \int_{-\infty}^{\infty} \Psi^{-1}[(m/n)F(x) + (n/N)F(\theta x)]\,dF(x)$$

where $\Psi(x) = 1 - e^{-x}$ and $\psi(x) = e^{-x}$, and denoting by σ_N^2 the variance of S under H_0,

$$(2.5)\qquad \lim_{N\to\infty} mN\sigma_N^2/n = 1.$$

From (2.4) and (2.5) the efficacy of the S-test is given by

$$(2.6)\qquad e(S) = [dE_\theta(S)/d\theta\,|_{\theta=1}]^2/\sigma_N^2 \cong (mn/N)[\int_{-\infty}^{\infty} xf^2(x)/\psi(x)\,dx]^2.$$

3. ARE in the exponential case. From the general expressions for efficacies given in the previous section we can easily compute the ARE's of the various tests with respect to the N-S and S-test in the exponential case, for which

$$(3.1)\qquad \begin{aligned} F(x) &= 1 - e^{-x}; \\ f(x) &= e^{-x}. \end{aligned}$$

Using (3.1), (2.6), (2.3) and (2.2) we easily see after some computation that in the exponential case

$$(3.2)\qquad e(S) = mn/N,$$

$$(3.3)\qquad e(S\text{-}T) = \tfrac{3}{4}(2\ln 2\text{-}1)^2 mn/N \cong .11192mn/N,$$

$$(3.4)\qquad e(M) = (5/36)mn/N \cong .13889mn/N.$$

From (2.1) the efficacy of the N-S test in the exponential case is given by

$$(3.5)\qquad e(N\text{-}S) = (2mn/N)[\int_0^{\infty} \Phi^{-1}(1 - e^{-x})/\varphi(\Phi^{-1}(1 - e^{-x}))xe^{-2x}\,dx]^2.$$

Klotz ([6], page 501), states that this expression is infinite; we now show that it is, in fact, finite and give its approximate value.

It follows easily from the Feller-Laplace expansion of Mills' ratio ([3], page 179, eqn. (6.1)) that

$$(3.6)\qquad \begin{aligned} |\Phi^{-1}(z)|/\varphi(\Phi^{-1}(z)) &\leqq 1/z & 0 < z \leqq \tfrac{1}{2} \\ &\leqq 1/1 - z & \tfrac{1}{2} \leqq z < 1. \end{aligned}$$

Thus

$$\begin{aligned}(3.7)\qquad &|\textstyle\int_0^\infty \Phi^{-1}(1-e^{-x})/\varphi(\Phi^{-1}(1-e^{-x}))xe^{-2x}\,dx| \\ &\quad\leqq \textstyle\int_0^{\ln(2)} xe^{-2x}/1-e^{-x}\,dx + \int_{\ln(2)}^\infty xe^{-x}\,dx \\ &\quad\leqq \textstyle\int_0^\infty x(x+1)e^{-x}\,dx < \infty.\end{aligned}$$

We now derive several alternative expressions for the integral in (3.5). Making use of the fact that $\Phi^{-1}(z) = -\Phi^{-1}(1-z)$ and using the transformation $z = \Phi^{-1}(e^{-x})$, so that $-e^{-x}\,dx = \varphi(z)\,dz$, we obtain

$$\begin{aligned}(3.8)\qquad &\textstyle\int_0^\infty [\Phi^{-1}(1-e^{-x})/\varphi(\Phi^{-1}(1-e^{-x}))]xe^{-2x}\,dx \\ &\quad= -\textstyle\int_0^\infty [\Phi^{-1}(e^{-x})/\varphi(\Phi^{-1}(e^{-x}))]xe^{-2x}\,dx \\ &\quad= \textstyle\int_{-\infty}^\infty z\ln(\Phi(z))\Phi(z)\,dz.\end{aligned}$$

After repeated integrations by parts we find that the last integral above is equal to

$$(3.9)\qquad \tfrac{1}{4}\int_{-\infty}^\infty \varphi^3(z)/\Phi^2(z)\,dz = (4\pi^{\frac{1}{2}})^{-1}\int_{-\infty}^\infty (\varphi(2^{\frac{1}{2}}z)/\Phi(2^{\frac{1}{2}}z))^2 e^{-z^2}\,dz \doteq .29782.$$

The second integral in (3.9) was evaluated using the 40-point Gauss-Hermite quadrature formula;[4] however, we have not investigated its accuracy. Thus, the efficacy of the N-S statistic in the exponential case is given, approximatey, by

$$(3.10)\qquad e(N\text{-}S) = (.17739)mn/N.$$

From (3.2), (3.3), (3.4) and (3.10) we can now easily compute the ARE of one test with respect to another. Denoting by $e(T_1, T_2)$ the ARE of test T_1 with respect to T_2, we find

$$\begin{aligned} e(N\text{-}S/S) &= .17739 \\ e(S\text{-}T/S) &= .11192 \\ e(M/S) &= .13889 \\ e(S\text{-}T/N\text{-}S) &= .63093 \\ e(M/N\text{-}S) &= .78296 \\ e(S\text{-}T/M) &= .80582.\end{aligned}$$

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[4] See, e.g., Hildebrand [5], page 327; the zeros and weight factors of the 40th degree Hermite polynomial were obtained from an unpublished paper of Baber, Krishnaiah and Armitage.

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