ABSTRACTS

(Abstract of a paper presented at the Annual meeting, New Brunswick, New Jersey, August 30–September 2, 1966. Additional papers appeared in earlier issues.)

59. Inference concerning a population mean from a single sample subsequent to an outlier test. FLORENCE G. TETREAULT and T. A. BANCROFT, University of Detroit and Iowa State University.

Let $x_1, x_2, \cdots, x_N$ be a random sample of size N from a normal population with unknown parameters. We consider the following problems: (1) the estimation of the population mean subsequent to a test for an outlying observation, and (2) the size and power of subsequent tests of hypotheses concerning the mean of the population. In the estimation problem the bias and mean square error functions are obtained and tabled. The power and size of subsequent tests of hypotheses are also obtained. Conditions under which Thompson's (1935) criterion for an outlying observation may be substituted for Grubbs' (1950) criterion are discussed. (Received 27 January 1967.)

(Abstract of a paper presented at the European Regional meeting, London, England, September 5–10, 1966. Additional abstracts appeared in earlier issues.)

16. A method of fitting constants for non-orthogonal layouts with interactions and empty cells. KLAUS ABT, U. S. Weapons Laboratory, Dahlgren.

A method is proposed for fitting main effect and interaction constants in analysis of variance models for non-orthogonal n-way layouts where an arbitrary number of cells may be empty. The fitting of each group of constants (main effects, first order interactions, etc.) is performed by visual inspection of the appropriate marginal data classification and is governed by a system of rules derived from the definitions of the various ANOVA effects and from the linear restrictions imposed on the constants which represent the effects. The confounding that possibly exists among the effects (and that is caused by the assumed random occurrence of empty cells) is treated by using the concept of "identities" which is defined. The testability of null hypotheses on main effects and interactions in a step-wise manner is demonstrated assuming that certain groups of constants, representing non-significant effects, are deleted from the model once the test results indicate the justification for doing so. A numerical example is given which illustrates all phases of the proposed method. The application of the method may serve (a) to screen a given body of incomplete and unbalanced data for significant ANOVA effects, (b) to identify confounded effects (if present), and (c) to yield an estimate of the experimental error (if otherwise not obtainable). (Received 13 January 1967.)

(Abstracts of papers presented at the Central Regional meeting, Columbus, Ohio, March 23–25, 1967. Additional abstracts appeared in the February issue and will appear in the June issue.)

3. On selecting the largest category. KHURSHEED ALAM, Indiana University.

Two procedures (I) and (II) are considered for selecting the best cell, that is, the cell with the highest probability from a multinomial population with K cells. According to (I) observations are taken one at a time until the difference between the largest and the next largest cell count is equal to r, a quantity determined such that the probability of a correct selection is at least as large as P*, a specified number. According to (II) observations are taken one at a time until any one cell has m counts in it and the counts in the other cells are each less than or equal to $m - r_m$, where r_m is a decreasing function of m, determined so that the conditional probability of a correct selection, given that the sampling is stopped when the highest cell count is m is at least as large as P^*. Sampling is stopped at the first realization of (m, r_m). The terminal decision for both procedures is to select that cell as the best cell in which the count is largest. The expected sample size for (I) and (II) is compared with and is shown smaller than the expected sample size obtained in the sequential sampling procedure of Cacoullos, T. and Sobel, M. "The effect of inverse sampling on ranking multinomial probabilities." Technical Report No. 55, Dept. of Statistics, University of Minnesota (1965). (Received 25 January 1967.)

4. On a generalized Savage statistic with applications to life testing (preliminary report). A. P. BASU, University of Wisconsin.

Let there be two samples $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n (N = m + n)$ from two populations with continuous cdf's $F(x)$ and $G(y)$. Let the first i ordered observations (out of N combined observations) contain m_i x's and n_i y's $(m_i + n_i = i)$ where m_i and n_i are random numbers. Then to test H_0: $F = G$ against alternatives of the form $G = F^{\Delta}$ or $(1 - G) = (1 - F)^{\Delta}$ we propose the statistic $S_r^{(N)} = \sum_{i=1}^{r} a_i z_i + [(m - m_r)/(N - r)](\sum_{r+1}^{N} a_i) - (m + n)/2$ based on the first r ordered observations only where $a_i = \sum_{j=N-i+1}^{N} 1/j$ and $z_i = 1$ if the ith ordered observation is an x_i and 0 otherwise. The statistic is the asymptotically most powerful rank test for censored data and is equivalent to the Savage statistic (*Ann. Math. Statist* **27** (1956) 590–615) when $r = N$. Exact and large sample properties of $S_r^{(N)}$ are studied and a possible k-sample extension of it is considered.
Suitability of the statistic in life testing problems is also discussed. (Received 13 January 1967.)

5. Transient behavior of the queue $GI/M/S$. U. N. BHAT, Case Institute of Technology.

The multi-server queueing system with general independent arrivals, negative exponential service times and $s(\geqq 1)$ servers is studied in terms of the time dependent behavior of the queue length process. Let $Q(t_n)$ and $S(t_n)$ be the actual number waitng and the number of customers in service respectively just before the nth arrival. Let $P_{lm}^{(n)}(i, j, t) = \Pr\{Q(t_n) = j, S(t_n) = m; t_n \leqq t \mid Q(t_0) = i, S(t_0) = l\}$ (for $i > 0$, $l = s$ and $j > 0$, $m = s$). Writing down the recurrence relations for $P_{ss}^{(n)}$ $(0, j, t)$ its transform can be obtained in terms of the transforms of $P_{ls}^{(n)}$ $(0, j, t)$ $(l \leqq s)$. The latter transforms are determined by a set of equations which can be solved recursively. One of these equations involves the unique root in $|z| < 1$ of the equation $z = \psi(\theta + s\mu - s\mu z)$ where $\psi(\theta)$ is the Laplace transform of the interarrival time distribution and μ^{-1} is the service rate. (Received 26 January 1967.)

6. Testing hypotheses in randomized factorial experiments. S. EHRENFELD and S. ZACKS, New York University, Bronx, and Kansas State University.

The objective of the present study is to find conditions on the nuisance parameters, under which the F-tests for testing hypotheses concerning the pre-assigned parameters in randomized fractional replication designs R.P.I. (see Ehrenfeld & Zacks, *Amer. Math. Soc.* **32** (1961) 270–297) are optimal. The search for these conditions is concentrated on the conditional bias functions of the pre-assigned parameters. These linear combinations of alias parameters can be represented under R.P.I. as sums of random variables which constitute *martingale* systems. Thus, we study the conditions which guarantee that the asymptotic distributions of these sums, as the number of terms increase to infinity, are normal.

It is proven that when the distributions of the conditional bias functions under R.P.I. are normal with zero mean then the F-tests of significance of the pre-assigned parameters are uniformly most powerful among all unbiased invariant tests. It is also shown that under any distribution of the conditional bias functions the F-tests employed are minimax (with respect to the expected loss in power). Let $\beta_1, \cdots, \beta_{2^t-1}$ designate the nuisance parameters, and let $B_t^2 = \sum_{j=1}^{2^t-1} \beta_j^2$. Sufficient conditions for the asymptotic normality of the conditional bias functions divided by B_t, as $t \to \infty$, are under R.P.I.: (i) $\lim B_t^2 = \infty$, (ii) $\sup_{1 \leqq k \leqq t} \sum_{j=2^{k-2}}^{2^{k-1}} |\beta_j| B_t^{-1} = o(1)$, as $t \to \infty$. (Received 27 January 1967.)

7. Percentage points of Weibull test statistics. H. LEON HARTER, Aerospace Research Laboratories, Wright-Patterson Air Force Base.

The test statistics considered are the Weibull-Z, Weibull-T, and Weibull-V statistics introduced by Dubey [*Ann. Math. Statist.* **35** (1964), 1391], which are analogues of the normal-z, Student-t, and (chi square)/(degrees of freedom) statistics, respectively, the difference being that the underlying population is Weibull rather than normal. The exact distributions of these statistics are not known. Good approximations to the percentage points of the Weibull-Z statistic have been obtained by use of the Cornish-Fisher expansion. This method does not yield accurate results for the Weibull-T and Weibull-V statistics, so approximate percentage points of these statistics have been found by means of a Monte Carlo simulation. The percentage points of all three statistics have been tabulated for all combinations of values of shape parameter $m = 1.1(0.1)10.0$; sample size $n = 2(1)40$, 48, 60, 80, 120, 240, ∞; and cumulative probability $P = 0.005$, 0.01, 0.025, 0.05, 0.1, 0.25, 0.75, 0.9, 0.95, 0.975, 0.99, 0.995. This paper gives details of the method of computation. The theory of testing hypotheses concerning the mean and the variance of a Weibull population, the complete tables, and examples illustrating their use are given by Harter and Dubey in a forthcoming ARL technical report. (Received 26 January 1967.)

8. Test for possible changes in the parameter at unknown time points. CHANDAN K. MUSTAFI, Columbia University.

Let x_i $(i = 1, 2, \cdots, n)$ be independent observations of a random variable X taken at n successive time points. Under the assumption that the distribution of X belongs to the one parameter exponential family, we consider the problem of testing the equality of these n parameters against the alternative that the parameter has changed r times at some unknown points where r is some finite positive integer less than n. We derive a test procedure by generalizing an approach made by Kander and Zacks (*Ann. Math. Statist.* **37** (1966) 1196–1210). The test statistic is shown to be asymptotically normal both under the null and the alternative hypothesis. For $r = 1$, the procedure reduces to the one obtained in the reference mentioned before. (Received 13 January 1967.)

9. Nonparametric estimation in Markov processes. GEORGE G. ROUSSAS, University of Wisconsin.

For $i = 1, 2$, let K_i be bounded, continuous probability densities defined on the i-dimensional Euclidean spaces $(\mathcal{E}_i, \mathcal{B}^{(i)})$ and satisfying the conditions: $\|z\|^i K_i(z) \to 0$, as $\|z\| \to \infty$, $z \varepsilon \mathcal{E}_i$, and $K_1(z) > 0$, $z \varepsilon \mathcal{E}_1$. Let $\{X_n\}$, $n \geqq 1$ be a Markov process having initial, 2-dimensional, and transition densities denoted by p, q, and t, respectively, and satisfying some additional regularity conditions. For two sequences of positive constants $\{h_i(n)\}$, $n \geqq 1$ with the property that: $h_i(n) = h_i \to 0$, and $n^i h_i \to \infty$, as $n \to \infty$, we set: $p_n(x) = (nh_1)^{-1} \cdot \sum_{j=1}^n K_1[(x - X_j)h_1^{-1}]$, $q_n(y) = (nh_2^2)^{-1} \sum_{j=1}^n K_2[(y - Y_j)h_2^{-1}]$, $t_n(x' \mid x) = q_n(y)/p_n(x)$, where $x, x' \varepsilon \mathcal{E}_1$, $y = (x, x')$, $Y_j = (X_j, X_{j+1})$. Then the following theorems are proved:

THEOREM 1. *The random variables* $p_n(x)$ *and* $q_n(y)$ *are asymptotically unbiased estimates of* $p(x)$ *and* $q(y)$, *respectively.* THEOREM 2. *The random variables* $p_n(x)$, $q_n(y)$, *and* $t_n(x' \mid x)$ *are consistent estimates of* $p(x)$, $q(y)$, *and* $t(x' \mid x)$, *respectively, the first two in quadratic mean and the last one in probability.* THEOREM 3. *All three estimates in Theorem* 2, *properly normalized, are asymptotically normal.* (Received 27 January 1967.)

10. Investigations on the basic theory of 2^m3^n fractional factorial designs of resolution V and related orthogonal arrays. J. N. SRIVASTAVA, Colorado State University.

Elsewhere, (e.g. *Ann. Math. Statist.* **37** 1865; *Proc. Internat. Statist. Inst., Belgrade*, (1965), paper 58), the author has developed the theory of nonsingular (i.e. resolution V) fractions T of 3^n series, where T consists of assemblies $\mathbf{y}$ satisfying any one of the equations $B\mathbf{y} = \mathbf{C}_i$, $(i = 1, \cdots, f)$ over $GF(3)$. In this paper a parallel theory for fractions T of 2^m3^n series, given as the set of all assemblies $(\mathbf{x}', \mathbf{y}')'$ satisfying one of $A\mathbf{x} + B\mathbf{y} = \mathbf{C}_i$, is developed, where the elements of $\mathbf{x}$ $(m \times 1)$ take the values 2 and 0 only. (See Method II in Bose and Srivastava: *Bull. Intern. Statist. Inst.*, 34th session, Ottawa (1963), p. 789). Also discussed are fractions (in the Connor and Young form) obtained by associating orthogonal arrays (of low strength) over $GF(2)$ to those over $GF(3)$. For both methods, a general mathematical theory is established, studying the nonsingularity and orthogonality properties of T in terms of the defining matrix A, B, and C. This includes as intermediate steps introduction of a new geometric definition of mixed interactions, expressions of M (the matrix occurring in the normal equations) in terms of A, B, and C, and obtaining conditions in terms of C (A and B being in a canonical form), for nonsingularity of M and its decomposition into a direct sum. Studies are made on the reduction of (tr M^{-1}) by suitable choice of C. As a by-product, a new method of construction of orthogonal arrays (strength $\leqq 4$) of 2^m3^n series $(m, n \geqq 0)$ in $k \cdot 2^r3^s$ assemblies is obtained. (Received 17 January 1967.)

11. Some sequential tests for Student's hypothesis (preliminary report). M. S. SRIVASTAVA, University of Toronto.

For testing the hypothesis that the mean θ of a normal population is equal to θ_0 against the alternative that $\theta = \theta_1$, with prescribed error probabilities α and β, we propose the following two sequential procedures as an alternative to Stein's two-stage procedure: (1) *Chow and Robbins' procedure* (*Ann. Math. Statist.* (1965)). An extension of Chow and Robbins' results give the following procedure for student's hypothesis: (I) Sample one observation at a time and stop at $N = n$ where n is the smallest integer for which $S_n \leqq n(\theta_1 - \theta_0)^2/(t_{n-1} + t'_{n-1})^2$; S_n is the sample variance, and t_{n-1} and t'_{n-1} are the upper α and β percentage points of t-distribution with $n - 1$ df. (II) When sampling is stopped at $N = n$, accept or reject the hypothesis according as $\bar{x}_n \lessgtr ad/(a + a')$, where a and a' are the upper α and β percentage points of the standard normal distribution. $\bar{x}_n = (\sum_{i=1}^{n} x_i/n)$. Some numerical comparisons and an extension to a slippage problem has been carried out. (2) *Conditional* SPRT. This procedure is the SPRT with σ^2 replaced by its estimate. After this result has been obtained, the author learned that it has been proposed by Hall (*Biometrika* (1962)). However, in this paper we have obtained more explicit expressions for OC and ASN functions. (Received 3 January 1967.)

12. The inadmissibility of confidence interval of location parameters. S. K. PERNG, Michigan State University.

In a recent paper (*Amer. Math. Soc.* **37** 626–637) by V. M. Joshi, a set of sufficient conditions was presented for the admissibility of confidence intervals. Here admissibility is defined in an unusual way by Joshi. Joshi proved a more general theorem than he stated. The observation may be a vector of non independent random variables provided the parameter θ is a location parameter for the sufficient statistic. We give here a probability density and a confidence interval which satisfy all the conditions mentioned in Joshi's paper except the moment condition. Then the given confidence interval is inadmissible. Let $x = (x_1, x_2)$ and consider the following probability density and confidence interval for $\theta f(x, \theta) = f_1(x_1, \theta)f_2(x_2) = [1/2(1 + |x_1 - \theta|)^2]\cdot[3/2(1 + |x_2|)^4]$, for $-\infty < x_1, x_2 < \infty$; $= 0$, otherwise; for $-\infty < \theta < \infty$. $I(x) = (x_1 - |x_2|, x_1 + |x_2|)$. Clearly $\int |x_1|^{\alpha} f_1(x_1, \theta)\, dx_1 < \infty$, for $0 \leqq \alpha < 1$ and $= \infty$, for $\alpha \geqq 1$. Hence the moment condition of the sufficient statistic mentioned in Joshi's paper is not satisfied. Let $I^*(x) = I(x)$ if $0 \not\varepsilon I(x)$ or $x_1 = 0$; $= (x_1 - |x_2| - \alpha(x_1 + |x_2|), x_1 + |x_2| - \alpha(x_1 + |x_2|))$, if $0 \varepsilon I(x)$ and $x_1 < 0$; $= (x_1 = |x_2| - \alpha(x_1 - |x_2|), x_1 + |x_2| - \alpha(x_1 - |x_2|))$ if $0 \varepsilon I(x)$ and $x_1 > 0$; where $0 < \alpha < 1$. Then we can show that $I^*(x)$ dominates $I(x)$ for sufficient small α. (Received 25 January 1967.)

13. On an inverse Gaussian process. M. T. WASAN, Queen's University, Kingston.

An inverse Gaussian process which is stationary and has independent increment has been defined. A covariance function, stochastic integral, conditional density function, distribution function with a condition on the process, density function of time when variate first attains real positive value a and the distribution function of the supremum of the process over a finite time interval have been investigated. A multivariate inverse Gaussian density is obtained and conditions are obtained when it becomes a multivariate Gaussian density function. An upper and lower bound for the probability $\Pr\{X > C\}$ (where X is inverse Gaussian variate and c is real positive value) have been obtained. Let $X_1, X_2, \cdots, X_n, \cdots$ be independent random variables with common inverse Gaussian density and $Z_n = \max(X_1 \cdots X_n)$ then it is proved that $\lim_{n\to\infty} \Pr[2 \log n + \epsilon(2 \log n)^{\frac{1}{2}} \geqq Z_n \geqq 2 \log n - \epsilon(2 \log n)^{\frac{1}{2}}] = 1$. A few other distributions of functions of the inverse Gaussian variates are also obtained. (Received 16 January 1967.)

(Abstracts of papers presented at the Eastern Regional meeting, Atlanta, Georgia, April 3–5, 1967. Additional abstracts appeared in the February issue and will appear in the June issue.)

3. Estimation of two ordered translation parameters. SAUL BLUMENTHAL and ARTHUR COHEN, New York University, Bronx, and Rutgers—The State University.

Let X_{ij} $(i = 1, 2; j = 1, 2, \cdots, n)$ be two sets of independent random variables, having for fixed i, a common density function $f(x - \theta_i)(i = 1, 2)$, centered so that $EX_{ij} = \theta_i$. Let $\theta_2 \geqq \theta_1$. The problem is to estimate both θ_1 and θ_2 with sum of squared errors as the loss function. Define X_i to be the usual Pitman estimator of θ_i and $Y_i = (Y_{i1}, \cdots, Y_{i,n-1})$ where $Y_{ij} = X_{i,j+1} - X_{i1}$. That is, X_i is the *a posteriori* expected value of θ_i, given $(X_{i1}, \cdots, X_{in})$, and given θ_i has the uniform prior on the real line. Let $p(x, y)$ be the conditional density of X_i given Y_i, when $\theta_i = 0$, and let $P(x, y)$ be the cumulative distribution function corresponding to $p(x, y)$. Let δ_i, $i = 1, 2$, be the Pitman estimator for this problem. That is, δ_i is the *a posteriori* expected value of θ_i, given $(X_{11}, X_{12}, \cdots, X_{2n})$, and given (θ_1, θ_2) has the uniform prior over the half plane $\theta_2 \geqq \theta_1$. The following results are obtained. (a) If $EE[(X_1^2 + X_2^2) \mid Y_1, Y_2] < \infty$ and if $p(x, y) = p(-x, y)$, then the Pitman estimator $\delta = (\delta_1, \delta_2)$ is minimax. The normal and uniform densities are examples of when this condition is satisfied. (b)

If $EE[(X_1^2 + X_2^2) \mid Y_1, Y_2] < \infty$ and if $p(x, y)$ is such that for each y, $p(x, y)/[1 - P(x, y)]$ increases in x (i.e. increasing hazard rate) and $p(x, y)/P(x, y)$ decreases with x, then the Pitman estimator is minimax. The gamma family of densities, for suitable values of the appropriate parameter, are examples of when this condition is satisfied. (c) An example is given which indicates that in general the Pitman estimator is not minimax. The example is justified by a computation performed by numerical integration which shows that the risk of the Pitman estimator exceeds the risk of an estimator known to be minimax. The results (b) and (c) indicate that whereas in a related one dimensional problem, (see Farrell (1964)), the Pitman estimator is always minimax (save for moment and continuity conditions), the same is not true for this two dimensional problem. (d) Let

$$\rho(y) = \max \{\sup_{-\infty<x<-1} [\int_{-\infty}^{x} v\, dv \int_{-\infty}^{\infty} p(u - v, y_1)p(u + v, y_2)\, du/x \cdot \int_{-\infty}^{x} \int_{-\infty}^{\infty} p(u - v, y_1)p(u + v, y_2)\, du\, dv], 2\}.$$

If $E\rho^2(y)E[(X_1^4 + X_2^4)(1 + |\log(X_1^2 + X_2^2)|^\beta) \mid Y_1, Y_2] < \infty$, for some $\beta > 0$, then the Pitman estimator is admissible. The normal density is an example for which this condition holds. Whereas Katz (1963) stated the admissibility result for the normal case, the proof there was not adequate. The proof of the minimax result uses the method of Farrell (1964) and the proof of admissibility uses the results of Stein (1959), (1961).

4. Estimation of the larger translation parameter. SAUL BLUMENTHAL and ARTHUR COHEN, New York University, Bronx, and Rutgers—The State University.

Let X_{ij} $(i = 1, 2; j = 1, \cdots, n)$ be two sets of independent random variables, having for fixed i, a common density function $f(x - \theta_i)$ $(i = 1, 2)$, centered so that $E(X_{ij}) = \theta_i$. Let $\varphi(\theta_1, \theta_2) = \text{maximum } (\theta_1, \theta_2)$. The problem is to estimate $\varphi(\theta_1, \theta_2)$ with squared error loss, so that the risk of an estimator $\delta(\cdot)$ is $R(\delta, \theta_1, \theta_2) = E[\delta(X_{11}, \cdots, X_{2n}) - \varphi(\theta_1, \theta_2)]^2$. Define X_i to be the usual Pitman estimator of θ_i and $Y_i = (Y_{i1}, \cdots, Y_{i,n-1})$ where $Y_{ij} = X_{i,j+1} - X_{i1}$. Let $p(x, y)$ be the conditional density of X_i given Y_i when $\theta_i = 0$. We consider two estimators, (i) $\varphi(X_1, X_2) = \max(X_1, X_2)$ and (ii) the Pitman-like estimator $\delta^*(X_1, X_2, Y_1, Y_2) = \int\int \varphi(\theta_1, \theta_2)p(X_1 - \theta_1, Y_1)p(X_2 - \theta_2, Y_2)\, d\theta_1\, d\theta_2$. The following results are obtained (a) when $p(x, y) = p(-x, y)$, and $EE[(X_1^3 + X_2^3) \mid Y_1, Y_2] < \infty$ then $\varphi(X_1, X_2)$ is minimax, but $\delta^*(\cdot)$ is not minimax; (b) when $p(x, y) \neq p(-x, y)$, an example is given showing that $\varphi(\cdot)$ is not minimax; (c) when $f(\cdot)$ is the normal distribution, $\varphi(\cdot)$ is not admissible; (d) if $E\{E[(X_1^2 + X_2^2)\, |\log(X_1^2 + X_2^2)|^\beta \mid Y_1, Y_2]\}^2 < \infty$ for some $\beta > 0$, then $\delta^*(\cdot)$ is admissible. We also give a general discussion of invariance conditions and their consequences, maximum likelihood estimation, and unbiased estimation (e.g., no unbiased estimates exist for the normal distribution). Finally, we consider related formulations of the estimation problem and discuss the practicality of the two estimators considered above. The proof of the minimax property of $\varphi(\cdot)$ depends somewhat on the method of Farrell (1964). The inadmissibility of $\varphi(\cdot)$ follows essentially from a theorem of Sacks (1963) and the admissibility of δ^* is proved using results of Stein (1959), (1961). (Received 16 December 1966.)

5. Operating characteristics of some sequential-design rules. ROBERT BOHRER, Research Triangle Institute.

Wald's exact evaluation of operating characteristics for some sequential, non-design rules (Appendix 4 of his sequential analysis book) is extended to the case of some sequential-design rules. Exact values of average sample numbers and error probabilities, as well as the average number of trials using the "wrong" experiment, are derived for the rules considered by Chernoff [*Ann. Math. Statist.*, (1959)] and by Bohrer [*Biometrika*, (1966)]. Applications of the work are presented. (Received 16 January 1967.)

6. Order statistics for exchangeable variates. H. A. DAVID and P. C. JOSHI, University of North Carolina.

Let $X_{i:n}$ $(i = 1, 2, \cdots, n)$ be the order statistics obtained by re-arranging in non-decreasing order of magnitude the variates X_i having common marginal cdf $P(x)$. Denote by $F_{i:n}(x)$ and $\mu_{i:n}$ the cdf and expected value of $X_{i:n}$. Recurrence relations for moments and other functions of the $X_{i:n}$ have been derived by many authors, the simplest result being for $r = 1, 2, \cdots, n - 1$,

$$n\mu_{r:n-1} = r\mu_{r+1:n} + (n - r)\mu_{r:n}. \tag{1}$$

However, almost all such relations are proved under the assumption that the X_i are independent variates, either of specific distributional form (usually normal) or absolutely continuous. It is shown that (1) and many other recurrence relations continue to hold when the X_i are exchangeable, continuous or discrete variates, i.e., if their joint cdf Pr $\{X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n\}$ is symmetric in $x_1, x_2, \cdots, x_n$. One of the results is applied to tabulation of the upper 5 and 1% points of $(Y_{n-1:n} - Y_0)/2^{\frac{1}{2}}\sigma$ with the help of tables of the cdf of $(Y_{n:n} - Y_0)/2^{\frac{1}{2}}\sigma$, where $Y_0, Y_1, \cdots, Y_n$, are independent normal $N(\mu, \sigma^2)$ variates and $Y_{n:n}$, $Y_{n-1:n}$ are respectively the largest and second largest of Y_1, $Y_2, \cdots, Y_n$. These statistics arise in multiple comparisons of n "treatment" means with a "control" mean. (Received 17 January 1967.)

7. A class of tests based on U statistics for the several sample problem. JAYANT V. DESHPANDÉ, University of North Carolina.

c samples consisting of $n_1, \cdots, n_c$ independent real observations are drawn from populations with cdf's $F_1, \cdots, F_c$ respectively. c-plets are formed by taking one observation from each sample. Define v_{ij} as the number of c-plets in which the observation from the ith sample is larger than exactly $(j - 1)$ observations and $u_{ij} = v_{ij}/\prod n_i$. Let $L_i = \sum_{j=1}^{c} a_j u_{ij}$ for $i = 1, \cdots c$ where a_j are real constants, not all equal. Then as $n \to \infty$ with $n_i = ns_i$, s_i being fixed positive integers, $N = \sum n_i$ and $p_i = n_i/N$,

$$\mathfrak{L}(a_1, \cdots, a_c) = [N(c - 1)^2/Ac^2]\,[\textstyle\sum_{i=1}^{c} p_i L_i^2 - (\sum_{i=1}^{c} p_i L_i)^2]$$

where

$$A = \textstyle\sum_{j=1}^{c} \sum_{l=1}^{c} a_j a_l[\binom{c-1}{j-1}\binom{c-1}{l-1}/\binom{2c-2}{j\,l-2}(2c - 1) - c^{-2}]$$

has, in the limit, χ^2 distribution with $c - 1$ df under $H_0: F_1 = \cdots = F_c$. Under the alternative hypotheses of shift and different scales it has under certain conditions, limiting noncentral χ^2 distribution with $c - 1$ df a_j, depending on F, are obtained which maximize the noncentrality parameter for the given alternative hypothesis, thus obtaining from the class of tests using $\mathfrak{L}(a_1, \cdots, a_c)$ as the test statistic, the test which has maximum asymptotic relative efficiency (in the Pitman sense). (Received 25 January 1967.)

8. Nonparametric confidence intervals for a scale parameter. GOTTFRIED E. NOETHER, Boston University.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be random samples from $F(z)$ and $G(z) = F(z/\theta)$, where $F(z)$ is continuous with median 0. If $F(z)$ is symmetric, a confidence interval for the scale parameter θ with confidence coefficient γ is bounded by the $(d + 1)$st smallest and largest among the ratios $|Y_j|/|X_i|$, $i = 1, \cdots, m; j = 1, \cdots, n$, where d is the lower tail critical value of the two-sided Wilcoxon test (Mann-Whitney U) with significance level

$1 - \gamma$. If $F(z)$ is not symmetric, the $(c + 1)$st largest of the ratios Y_j/X_i furnishes an upper confidence limit, while the reciprocal of the $(c + 1)$st largest of the ratios X_i/Y_j furnishes a lower limit, where c is the critical value of the Sukhatme test (*Ann. Math. Statist.* **28** 188–194). The Pitman efficiency of these intervals is equal to the Pitman efficiency of the Ansari-Bradley test (*Ann. Math. Statist.* **31** 1174–1189). For discrete populations, closed intervals have confidence coefficients at least γ, open intervals, at most γ. If $G(z - \eta_2) = F((z - \eta_1)/\theta)$, where η_1 and η_2 are unknown population medians, an exact confidence interval for θ is obtained by applying the first method to the differences $X_1 - X_2$, $X_3 - X_4, \cdots, Y_1 - Y_2, Y_3 - Y_4, \cdots$. For large samples, the earlier methods can be applied to the centered observations $X_i - \tilde{X}$ and $Y_j - \tilde{Y}$, where $\tilde{X}$ and $\tilde{Y}$ are sample medians. (Received 6 January 1967.)

9. On the asymptotic normality of one sample Chernoff-Savage test statistics. MADAN L. PURI and PRANAB K. SEN, Courant Institute of Mathematical Sciences, New York University and University of North Carolina. (By title)

Asymptotic normality of a class of one sample Chernoff-Savage type of rank order statistics has been established by Govindarajulu [(1960)], Central limit theorems and asymptotic efficiency for one sample nonparametric procedures, Technical report no. 11, Department of Statistics, University of Minnesota]. His proof is very lengthy and cumbersome. The object of this note is to provide a greatly shortened and simplified proof of the same theorem. (Received 16 December 1966.)

10. A new class of conditions for the existence of partially balanced arrays, including BIBD's and orthogonal arrays (preliminary report). J. N. SRIVASTAVA, Colorado State University.

A partially balanced array (PBA) T (see e.g. Chakravarti, *Ann. Math. Statist.* **32** 1181) with parameters (m, N, s, t) and (for $s = 2$) an index set $(\mu_0^{t'}, \cdots, \mu_t^{t})$ is an $(m \times N)$ matrix with elements 0 or 1 such that in every $(t \times N)$ submatrix, every vector of weight i occurs μ_i^t times. In this body of work, given *any* array T with 2 symbols and considering the columns of T as assemblies from a 2^m factorial, we first develop a new polynomial ring P, so that T corresponds to a polynomial $p_T \varepsilon P$. A useful symbolism and a related calculus are developed along with the properties of P; these condense various properties of (general) arrays into properties of polynomials. Necessary and sufficient conditions C for the existence of a (PBA) T are derived from this theory; C being a large set even for relatively small $(m - t)$. For $m \leqq t + 2$, C is shown equivalent to the existence of a solution of a linear integer programing problem in $(m + 1)$ variables and all solutions for $t = 4$, are tabulated for many interesting cases. Using these, necessary conditions for T (for $m > t + 2$) are obtained in the form of necessary systems of linear diophantine equations. One such set is $A\mathbf{x} = \mathbf{d} = B\mathbf{y}$, where all symbols involved have integer elements $\mathbf{x} > \mathbf{0}$, $\mathbf{y} > \mathbf{0}$, A and $\mathbf{d}$ are known functions of m, t and μ_j^i $(i \leqq t + 2)$, and the values of elements of B arise out of condition C with $m = t + 2$. (The special case of $A\mathbf{x} = \mathbf{d}$, for $m = 8$, $t = i = 4$, is independently obtained by D. V. Chopra). Some generalizations for $s \geqq 2$ are also considered, obtaining as a by-product new type of conditions for the existence of BIBD's with $\lambda = 1$. (Received 17 January 1967.)

11. On an inverse Gaussian process II. M. T. WASAN, Queen's University, Kingston.

A simple method of derivation of characteristic function of an inverse Gaussian process is given. The orthogonal functions with respect to inverse Gaussian measure are discussed.

A differential equation and a canonical representation of an inverse Gaussian process are obtained and their consequences are investigated. Furthermore, a continuity property of the process is also discussed. It is proved that inverse Gaussian density is complete and a Pólya type. Bayes estimate of a function of a parameter is obtained. Comparison of maximum likelihood estimate and unbiased estimate for a parameter of mixed density is made. A sequential estimation procedure for a parameter is also discussed. (Received 16 January 1967.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. A class of infinitely divisible random variables. CHARLES GOLDIE, University of Cambridge. (Introduced by J. H. Kingman.)

The product of a non-negative random variable and an exponentially-distributed random variable is infinitely divisible.

2. A rank test for skewness. MILAN K. GUPTA, Presidency College, Calcutta. (Introduced by Atindra Mohan Gurr.)

A rank test is proposed for testing whether two populations are identical against the alternative that they differ in skewness. Random samples of sizes r_1 and r_2 are drawn from the two populations. The $r = r_1 + r_2$ observations are combined and arranged in increasing order. This is next divided into three groups, there being U, $V - U$ and $r - V$ observations in the lower, middle and upper group respectively. The observations in the lower group are ranked outward to the left beginning from the Uth observation of the combined sample. Let S_1 be the sum of the new ranks of the observations from the first sample and lying in the lower group. Similarly rank the observations in the upper group outward to the right beginning from the $(V + 1)$st observation of the combined sample. Let S_2 be the sum of the new ranks of the observations from the first sample and lying in the upper group. Then $S = S_1 + S_2$ is the proposed test statistic. Too small or too large a value of S will reject the null hypothesis that the two populations are identical. It is shown that S is asymptotically normally distributed under the null hypothesis. (Received 2 December 1966.)

3. Characterization of independence in bivariate families with regression dependence. KUMAR JOGDEO and G. P. PATIL, Courant Institute of Mathematical Science, New York University and Pennsylvania State University.

Lehmann (*Ann. Math. Statist.* **37** (1966) 1137–1153) showed that in the family of bivariate distributions with *quadrant positive* (*or negative*) *dependence* uncorrelatedness implies independence. He further introduced a subfamily having regression dependence, i.e. $P[Y \leqq y \mid X = x]$ is nonincreasing (or nondecreasing) in x for each y. The present authors show that if this subfamily is parametrized suitably then the independence is characterized simply by the independence of *any* two events of the type $[X \leqq a]$, $[Y \leqq b]$ whose probabilities are bounded away from 0 and 1. In particular it follows that if (X, Y) has a bivariate normal distribution then the independence of above events is enough for that of X and Y. The same holds true if dependence is given by a model $Y = \alpha + \beta X + Z$, where X and Z are independent. To show that one cannot do away with parametrization completely, an example is given where $P[Y \leqq y \mid X = x]$ is nondecreasing, there exist two independent events of the above type and still X and Y are not independent. (Received 19 December 1966.)

4. On Bahadur's study of sample quantiles. J. KIEFER, Cornell University.

Let $X_1, X_2, \cdots$ be independent with common df F satisfying $F(\xi) = p$ (with $0 < p < 1$), $F'(\xi) > 0$, F'' bounded near ξ. Let $Y_{p,n}$ be the sample p-quantile and let S_n be the sample df, both based on $(X_1, \cdots, X_n)$. Let $R_n(p) = Y_{p,n} - \xi + [S_n(\xi) - p]/F'(\xi)$. Bahadur (*Ann. Math. Statist.* **37** (1966), 577–580) initiated the study of $R_n(p)$, showed it is $O(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{4}})$ wp 1 as $n \to \infty$, and raised the question of finding the exact order. The present paper demonstrates that, for either choice of sign, $\limsup_{n\to\infty} \pm F'(\xi)R_n(p)/[p^{\frac{1}{4}}(1-p)^{\frac{1}{4}}n^{-\frac{3}{4}}(\log \log n)^{\frac{3}{4}}2^{\frac{5}{4}}3^{-\frac{3}{4}}] = 1$ wp 1. Moreover, for any positive T, the process $\Gamma_n = \{W_{p,n}(t), -T \leqq t \leqq T\}$, defined by $W_{p,n}(t) = n^{\frac{3}{4}}R_n(p + n^{-\frac{1}{2}}t)$, has supremum and infimum over $[-T, T]$ which have the same behavior as that just exhibited for $n^{\frac{3}{4}}|R_n(p)|$. (The supremum and infimum over $[-n^{\frac{1}{2}}p, n^{\frac{1}{2}}(1-p)]$ are more complicated in behavior.) The process Γ_n tends in law (as $n \to \infty$) to a limiting process, whose law is computed. (Received 2 December 1966.)

5. A non-parametric test for the bivariate two sample location problem, IV: small sample power in the non-normal case and the effect of non-normality to T^2 (preliminary report). K. V. MARDIA, University of Newcastle upon Tyne. (Introduced by R. L. Plackett.)

In this paper, we deal with the empirical small sample powers of the tests U^2 (*Ann. Math. Statist.* **36** (1966) 1075, abstract), and Hotelling's T^2 when the underlying populations are (i) contingency-type rectangular (Plackett, J., *Amer. Statist. Assoc.* **59** (1965) 516–522) and (ii) Pareto-type 1 (Mardia, *Ann. Math. Statist.* **33** (1962) 1008–1015). The empirical bivariate distributions are generated with the help of Rosenblatt's transformation (*Ann. Math. Statist.* **23** (1952) 470–472) applied to pseudo-random numbers. Sampling trials indicate that U^2 does better (locally or uniformly) than T^2 for the translation type of alternatives when the following quantity $A = [\gamma_{20} + \gamma_{02} + 2\gamma_{22} + 4\rho^2(\gamma_{22} - \gamma_{13} - \gamma_{31})]/(1 - \rho^2)^2$ is large, where $\gamma_{20} = (\mu_{40}/\mu_{20}^2 - 3)$, $\gamma_{02} = (\mu_{40}/\mu_{02}^2 - 3)$, $\gamma_{22} = (\mu_{22}/\mu_{02}\mu_{20}) - (1 + 2\rho^2)$, $\gamma_{31} = (\mu_{31}/\mu_{11}\mu_{20}) - 3$, $\gamma_{13} = (\mu_{13}/\mu_{11}\mu_{02}) - 3$ and $(\mu_{rs}) = (r, s)$th central moment of population. The quantity A appears in the approximation to the non-normal T^2 by its permutation distribution and is zero for the normal theory T^2. Even after correcting the 5 per cent level of T^2 for non-normality, the foregoing conclusion regarding the powers of U^2 and T^2 remains true. Further, the size and power of T^2 are not seriously invalidated for these cases. (Received 14 December 1966.)