A COUNTER-EXAMPLE RELATING TO CERTAIN MULTIVARIATE GENERALIZATIONS OF t AND F

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Summary. It is shown that the distributions of certain multivariate analogues of t and F are dependent on the population covariance matrix.

Suppose that $\mathbf{z} = (x_1, \cdots, x_p)'$ has the multivariate normal distribution with mean vector $\boldsymbol{\xi}$ and covariance matrix $\boldsymbol{\Sigma}$. Let $\mathbf{S}$, $\mathbf{S}^*$ be independent Wishart matrices also with covariance matrix $\boldsymbol{\Sigma}$, based on n, n^* degrees of freedom respectively. Then if $\mathbf{T}$ is a $p \times p$ matrix such that

$$\mathbf{T}\mathbf{T}' = \mathbf{S} \tag{1}$$

natural candidates for the multivariate analogues of t and F are

$$\mathbf{t} = \mathbf{T}^{-1}(\mathbf{z} - \boldsymbol{\xi}), \tag{2}$$

$$\mathbf{W} = \mathbf{T}^{-1}\mathbf{S}^*\mathbf{T}'^{-1}. \tag{3}$$

Olkin and Rubin (1964), Theorems 3.2 and 4.2, have shown that if $\mathbf{T}$ is taken to be upper or lower triangular, then $\mathbf{t}$ and $\mathbf{W}$ do in fact have distributions which independent of $\boldsymbol{\Sigma}$. However, if $\mathbf{T}$ is taken to be symmetrical and positive definite, $\mathbf{T} = \mathbf{S}^{\frac{1}{2}}$, then they remark (Section 3) that the distribution of $\mathbf{W}$ is unknown for general $\boldsymbol{\Sigma}$. It seems worthwhile to present the following example, which shows that for $\mathbf{T} = \mathbf{S}^{\frac{1}{2}}$ the distributions of $\mathbf{t}$ and $\mathbf{W}$ depend on $\boldsymbol{\Sigma}$; the contrary is occasionally asserted (see Bennett and Cornish (1964), p. 907).

Let $p = 2$, and assume that

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}. \tag{4}$$

Since $\mathbf{S}^{\frac{1}{2}}$ is positive definite, it may be written in the form

$$\mathbf{S}^{\frac{1}{2}} = \begin{bmatrix} x, & (xy)^{\frac{1}{2}}q \\ (xy)^{\frac{1}{2}}q, & y \end{bmatrix} \quad (0 < x, y < \infty, q^2 < 1). \tag{5}$$

Noting the Jacobian

$$\partial(\mathbf{S})/\partial(x, y, q) = 4(1 - q^2)(xy)^{\frac{1}{2}}(x + y), \tag{6}$$

it is readily found by transforming the Wishart distribution that $\mathbf{S}^{\frac{1}{2}}$ has the distribution

$$\begin{aligned} f(\mathbf{S}^{\frac{1}{2}})\, d\mathbf{S}^{\frac{1}{2}} = \{(nab)^n/\pi\Gamma(n-1)\}&(1 - q^2)^{n-2}(xy)^{n-\frac{1}{2}}(x + y) \\ &\cdot \exp\{-\tfrac{1}{2}n[a^2x^2 + (a^2 + b^2)xyq^2 + b^2y^2]\}\, dx\, dy\, dq, \\ &(0 < x, y < \infty, q^2 < 1). \end{aligned} \tag{7}$$

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(See Olkin and Rubin, p. 266.)

Now

$$\begin{aligned} \mathcal{E}(\mathbf{tt}') &= \mathcal{E}\mathbf{W} \\ &= \mathcal{E}(\mathbf{S}^{-\frac{1}{2}}\mathbf{\Sigma}\,\mathbf{S}^{-\frac{1}{2}}) \qquad (8) \\ &= \begin{bmatrix} \nu_1/a^2 + \nu_3/b^2, & -\nu_4/a^2 - \nu_5/b^2 \\ -\nu_4/a^2 - \nu_5/b^2, & \nu_3/a^2 + \nu_2/b^2 \end{bmatrix}, \end{aligned}$$

where

$$(9)\quad \nu_1 = \mathcal{E}[x^{-2}(1-q^2)^{-2}], \qquad \nu_2 = \mathcal{E}[y^{-2}(1-q^2)^{-2}], \qquad \nu_3 = \mathcal{E}[q^2/xy(1-q^2)^2],$$
$$\nu_4 = \mathcal{E}[q/x^{\frac{3}{2}}y^{\frac{1}{2}}(1-q^2)^2], \qquad \nu_5 = \mathcal{E}[q/x^{\frac{1}{2}}y^{\frac{3}{2}}(1-q^2)^2].$$

It will be sufficient to show that the matrices in (8) are not independent of a and b. In order to simplify the calculations still further, *we shall take the case* $n = 4$, and consider only the elements $\mathcal{E}(t_1^2)$, $\mathcal{E}(t_2^2)$ on the main diagonal in (8). The density (7) now takes the form:

$$(10)\quad f(\mathbf{S}^{\frac{1}{2}})\,d\mathbf{S}^{\frac{1}{2}} = 2^7\pi^{-1}(ab)^4(1-q^2)^2(xy)^{\frac{3}{2}}(x+y) \cdot \exp\{-2[a^2x^2 + (a^2+b^2)xyq^2 + b^2y^2]\}\,dx\,dy\,dq\,,$$

and we wish to evaluate ν_1, ν_2, ν_3.

For ν_1, let us first consider

$$\begin{aligned} I(q) &= \textstyle\int_0^\infty\int_0^\infty x^{-2}(1-q^2)^{-2}f(\mathbf{S}^{\frac{1}{2}})\,dx\,dy \\ &= 2^7\pi^{-1}(ab)^4 \textstyle\int_0^\infty\int_0^\infty (x^{\frac{1}{2}}y^{\frac{5}{2}} + x^{\frac{1}{2}}y^{7/2})e^{-2a^2x^2-2b^2y^2} \qquad (11) \\ &\quad \cdot \textstyle\sum_{k=0}^\infty k!^{-1}[-2(a^2+b^2)xyq^2]^k\,dx\,dy. \end{aligned}$$

Making use of the formulas

$$\begin{aligned} \Gamma(\omega)\Gamma(\omega+\tfrac{1}{2}) &= \pi^{\frac{1}{2}}\Gamma(2\omega)/2^{2\omega-1}, \\ \textstyle\int_0^\infty \omega^\lambda e^{-a^2\omega^2}\,d\omega &= \Gamma((\lambda+1)/2)/2a^{\lambda+1}, \qquad (12) \\ \textstyle\sum_{k=0}^\infty x^k\Gamma(k+\lambda)/k! &= \Gamma(\lambda)(1-x)^{-\lambda}, \qquad (|x| < 1), \end{aligned}$$

it is found that

$$(13)\quad I(q) = (2a^3/\pi b)^{\frac{1}{2}} \textstyle\sum_{k=0}^\infty k!^{-1}[-(a^2+b^2)q^2/2ab]^k \cdot \{a\Gamma(k+\tfrac{3}{2}) + (a+b)\Gamma(k+\tfrac{5}{2})\}$$

$$(14)\quad = (a^3/2b)^{\frac{1}{2}}\{a\phi^{-\frac{3}{2}}(q) + \tfrac{3}{2}(a+b)\phi^{-\frac{5}{2}}(q)\},$$

where

$$(15)\quad \phi(q) = 1 + (a^2+b^2)q^2/2ab.$$

The series in (13) is convergent only for

$$(16)\quad |q|^2 < 2ab/(a^2+b^2).$$

However, since $I(q)$ and the expression (14) are both analytic functions of the complex variable $q = u + iv$ over the region $u^2 - v^2 > -2ab/(a^2 + b^2)$ in the q plane, it follows by analytic continuation that $I(q)$ is certainly equal to (14) for all real q.

Noting that

$$\int_0^1 \phi^{-\frac{3}{2}}(q)\, dq = (2ab)^{\frac{1}{2}}/(a + b), \tag{17}$$
$$\int_0^1 \phi^{-\frac{5}{2}}(q)\, dq = \tfrac{2}{3}(2ab)^{\frac{1}{2}}(a^2 + 3ab + b^2)/(a + b)^3,$$

we have

$$\begin{aligned} \nu_1 &= 2 \int_0^1 I(q)\, dq \\ &= 2a^2[2 - (b/(a + b))^2], \end{aligned} \tag{18}$$

ν_2 being obtained by interchanging a and b.

Similarly, it is found that

$$\nu_3 = 2(ab/(a + b))^2. \tag{19}$$

Hence

$$\begin{aligned} \mathcal{E}(t_1^2) &= \nu_1/a^2 + \nu_3/b^2 = 4 + 2(a - b)/(a + b), \\ \mathcal{E}(t_2^2) &= \nu_3/a^2 + \nu_2/b^2 = 4 - 2(a - b)/(a + b), \end{aligned} \tag{20}$$

and the matrices in (8) are not independent of a and b.

As a check on the working, we have:

$$\mathcal{E}(t_1^2 + t_2^2) = \mathcal{E}(\mathbf{z} - \boldsymbol{\xi})'\mathbf{S}^{-1}(\mathbf{z} - \boldsymbol{\xi}) = 8, \tag{21}$$

which is independent of $\boldsymbol{\Sigma}$ and in accordance with the known distribution of Hotelling's T^2.

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