ON THE CROSS PERIODOGRAM OF A STATIONARY GAUSSIAN VECTOR PROCESS

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0. Summary. Bartlett [1] obtained an asymptotic relation connecting the periodogram of a linear process and the periodogram of the residuals with uniform spectrum in the case of one dimensional stationary time series, and using this relation he studied the asymptotic properties of the periodogram.

In this paper we obtain some asymptotic relations between the co-periodogram and quadrature periodogram of the residuals in the case of a stationary Gaussian vector process. The covariances of the co-periodogram and quadrature periodogram have also been obtained. Two inequalities connecting the variance and the bandwidth of the cross spectral estimate have been derived.

1. Introduction. Let $x'(t) = (x_1(t), x_2(t))$ $(t = 1, 2, \cdots)$ be a two-dimensional wide sense stationary discrete vector process where the mean value of $x'(t)$ is assumed to be identically zero.

The spectral matrix of $x'(t)$ is

$$F(\lambda) = (f_{ij}(\lambda)) \qquad (i, j = 1, 2) \tag{1.1}$$

where

$$f_{12}(\lambda) = (2\pi)^{-1} \textstyle\sum_{-\infty}^{\infty} e^{it\lambda} R_{12}(t) = c_{12}(\lambda) + iq_{12}(\lambda), \tag{1.2}$$

$$R_{12}(t) = E[x_1(j+t)x_2(j)]. \tag{1.3}$$

$c_{12}(\lambda)$ and $q_{12}(\lambda)$ which are respectively known as co-spectral density and quadrature spectral density are assumed to be absolutely continuous.

Let $x'(t) = (x_1(t), x_2(t))(t = 1, 2, \cdots, N)$ be a realization of size N from a real, stationary, Gaussian two-dimensional vector process considered above.

Let us define the complex cross periodogram as

$$\begin{aligned} f_{12}^{(N)}(\lambda) &= (2\pi N)^{-1} |\textstyle\sum_{t=1}^{N} e^{it\lambda} x_1(t)|\, \overline{|\textstyle\sum_{t=1}^{N} e^{it\lambda} x_2(t)|} \\ &= (2\pi)^{-1} \textstyle\sum_{t=-N}^{N} e^{it\lambda} R_{12}^{(N)}(t) = c_{12}(\lambda) + iq_{12}^{(N)}(\lambda) \end{aligned} \tag{1.4}$$

where $c_{12}^{(N)}(\lambda)$ and $q_{12}^{(N)}(\lambda)$ are respectively known as co-periodogram and quadrature periodogram, and

$$R_{12}^{(N)}(t) = N^{-1} \textstyle\sum_{j=1}^{N-t} x_1(j+t)x_2(j), \qquad 0 \leqq t \leqq N-1. \tag{1.5}$$

2. Cross periodogram analysis. Assume the process to be non-deterministic. Then the Zasuhin's multivariate representation of the linear process of the

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elements of the vector $x'(t)$ is given (Whittle [5]) by

$$x_j(t) = \sum_{l=1}^{2} \sum_{m=-\infty}^{\infty} b_{jlm}\epsilon_{l,t-m} \qquad (j = 1, 2) \tag{2.1}$$

where all ϵ's are uncorrelated and each is distributed normally with mean zero and variance unity.

Consider the quantity

$$J_{1,x}(\lambda) = (2\pi N)^{-\frac{1}{2}} \sum_{t=1}^{N} e^{it\lambda} x_1(t). \tag{2.2}$$

Substitute the expression for $x_1(t)$ from (2.1) in (2.2) and assume that b_{1lj} tends to zero exponentially; then following Bartlett [1], it can be shown that

$$J_{1,x}(\lambda) \sim \sum_{l=1}^{2} h_{1,l}^{*}(\lambda) J_{l,\epsilon}(\lambda) \tag{2.3}$$

and

$$f_{12}^{(N)}(\lambda) = J_{1,x}(\lambda) J_{2,x}^{*}(\lambda) \sim \sum_{l} \sum_{l'} h_{1,l}^{*}(\lambda) h_{2,l'}(\lambda) f_{ll',\epsilon}^{(N)}(\lambda) \tag{2.4}$$

where $h_{1,l}^{*}(\lambda) = \sum_{j} b_{1lj} e^{ij\lambda}$.

As $N \to \infty$, we have

$$E(f_{12}^{(N)}(\lambda)) \sim f_{12}(\lambda) = (2\pi)^{-1} \sum_{l=1}^{2} h_{1,l}^{*}(\lambda) h_{2,l}(\lambda). \tag{2.5}$$

The relation (2.4) can be separated into real and imaginary parts as follows. Let

$$h_{1,l}^{*}(\lambda) h_{2,l'}(\lambda) = H_{ll'}^{R}(\lambda) + iH_{ll'}^{I}(\lambda), \tag{2.6}$$
$$f_{ll',\epsilon}^{(N)}(\lambda) = c_{ll',\epsilon}^{(N)}(\lambda) + iq_{ll',\epsilon}^{(N)}(\lambda)$$

where

$H_{ll'}^{R}(\lambda) = \sum_{j,j'} b_{1lj} b_{2l'j'} \cos\lambda(j - j')$ and $H_{ll'}^{I}(\lambda) = \sum_{j,j'} b_{1lj} b_{2l'j'} \sin\lambda(j - j')$.

Then from (2.4) and (2.6) we have

$$c_{12}^{(N)}(\lambda) = \sum_{l} \sum_{l'} [H_{ll'}^{R}(\lambda) c_{ll',\epsilon}^{(N)}(\lambda) - H_{ll'}^{I}(\lambda) q_{ll',\epsilon}^{(N)}(\lambda)], \tag{2.7}$$

$$q_{12}^{(N)}(\lambda) = \sum_{l} \sum_{l'} [H_{ll'}^{R}(\lambda) q_{ll',\epsilon}^{(N)}(\lambda) + H_{ll'}^{I}(\lambda) c_{ll',\epsilon}^{(N)}(\lambda)]. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$E(c_{12}^{(N)}(\lambda)) \sim c_{12}(\lambda) = (2\pi)^{-1} \sum_{l=1}^{2} H_{ll}^{R}(\lambda), \tag{2.9}$$

$$E(q_{12}^{(N)}(\lambda)) \sim q_{12}(\lambda) = (2\pi)^{-1} \sum_{l=1}^{2} H_{ll}^{I}(\lambda). \tag{2.10}$$

The two equations (2.7) and (2.8) give asymptotic relations between the co-periodogram and quadrature periodogram of the process and co-periodogram and quadrature periodogram of the residual series.

THEOREM 1. *Let $x_i(t)$ $(i = 1, 2)$ have the representation* (2.1). *Let $g_a(w) = f_{11}^{(N)}(w) + 2ac_{12}^{(N)}(w) + a^2 f_{22}^{(N)}(w)$ be the periodogram of the series $x_1(t) + ax_2(t)$ and $K_a(w) = f_{11}^{(N)}(w) + 2aiq_{12}^{(N)}(w) - a^2 f_{22}^{(N)}(w)$ be the cross periodogram of the series $x_1(t) - ax_2(t)$ and $x_1(t) + ax_2(t)$ for all a. Then*

$$
(2.11)\quad
\begin{aligned}
&\text{(i)} & \operatorname{Cov}(g_a(w_1), g_a(w_2)) &= O(N^{-2}), & K_4 &= E(\epsilon_i^4) - 3 = 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0;\\
&\text{(ii)} & \operatorname{Cov}(K_a(w_1), K_a(w_2)) &= O(N^{-2}), & K_4 &= 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0;\\
&\text{(iii)} & \operatorname{Cov}(g_a(w_1), K_a(w_2)) &= O(N^{-2}), & K_4 &= 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0.
\end{aligned}
$$

PROOF. Define

$$J(w_1) = (2\pi N)^{-\frac{1}{2}} \textstyle\sum_{t=1}^{N} (x_1(t) + ax_2(t))e^{itw_1}$$

so that

$$
(2.12)\quad
\begin{aligned}
g_a(w_1) &= J(w_1)J^*(w_1)\\
&= (2\pi N)^{-1} \textstyle\sum_{t,s}^{N} \sum_{l,l'}^{2} \sum_{m,m'=-\infty}^{\infty} B_{lm}B_{l'm'}\epsilon_{l,t-m}\epsilon_{l',s-m'}e^{-iw_1(t-s)}
\end{aligned}
$$

where $B_{lm} = (b_{1lm} + ab_{2lm})$. Applying the results of Bartlett [1], p. 278, to the periodogram $g_a(w_1)$ of $x_1(t) + ax_2(t)$, it can be shown that result (i) follows.

Similarly one can proceed to show the results (ii) and (iii) of (2.11).

COROLLARY. *Let $x_i(t)$ $(i = 1, 2)$ have the representation* (2.1). *Then*

$$
(2.13)\quad
\begin{aligned}
&\text{(i)} & \operatorname{Cov}(c_{12}^{(N)}(w_1), c_{12}^{(N)}(w_2)) &= O(N^{-2}), & K_4 &= 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0;\\
&\text{(ii)} & \operatorname{Cov}(q_{12}^{(N)}(w_1), q_{12}^{(N)}(w_2)) &= O(N^{-2}), & K_4 &= 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0;\\
&\text{(iii)} & \operatorname{Cov}(c_{12}^{(N)}(w_1), q_{12}^{(N)}(w_2)) &= O(N^{-2}), & K_4 &= 0,\\
& & &= O(N^{-1}), & K_4 &\neq 0.
\end{aligned}
$$

PROOF. From Theorem 1, we have

$$
(2.14)\quad
\begin{aligned}
&4a^2 \operatorname{Cov}(c_{12}^{(N)}(w_1), c_{12}^{(N)}(w_2))\\
&\qquad = \operatorname{Cov}(g_a(w_1), g_a(w_2) - \operatorname{Cov}(g_a(w_1), f_{11}^{(N)}(w_2))\\
&\qquad\quad - a^2 \operatorname{Cov}(g_a(w_1), f_{22}^{(N)}(w_2)) - \operatorname{Cov}(f_{11}^{(N)}(w_1), g_a(w_2))\\
&\qquad\quad + \operatorname{Cov}(f_{11}^{(N)}(w_1), f_{11}^{(N)}(w_2)) + a^2 \operatorname{Cov}(f_{11}^{(N)}(w_1), f_{22}^{(N)}(w_2))\\
&\qquad\quad - a^2 \operatorname{Cov}(f_{22}^{(N)}(w_1), g_a(w_2)) + a^2 \operatorname{Cov}(f_{22}^{(N)}(w_1), f_{11}^{(N)}(w_2))\\
&\qquad\quad + a^4 \operatorname{Cov}(f_{22}^{(N)}(w_1), f_{22}^{(N)}(w_2)).
\end{aligned}
$$

From Theorem 1 and Bartlett [1] it follows that each term of (2.14) is $O(N^{-2})$ if $K_4 = 0$ and $O(N^{-1})$ if $K_4 \neq 0$. Hence the result (i) of the corollary.

Similarly one can proceed to show the results (ii) and (iii).

The results (i), (ii) and (iii) of the above corollary can also be obtained even without assuming the representation $x_j(t)$ given in (2.1).

From (2.9), (2.13) and (2.10), it follows that co-periodogram and quadrature periodogram do not provide consistent estimates of co-spectral density and quadrature spectral density. Hence, to ensure consistency, we consider estimates of the form (Rosenblatt [4])

$$c_{12}^*(\lambda) = (2\pi)^{-1} \sum_{t=-N}^{N} k(B_N t) R_{12}^{(N)}(t) \cos t\lambda, \tag{2.15}$$
$$q_{12}^*(\lambda) = (2\pi)^{-1} \sum_{t=-N}^{N} k(B_N t) R_{12}^{(N)}(t) \sin t\lambda,$$

where the function $k(x)$ is assumed to be continuous and square integrable (Parzen [2], [3]). B_N is a sequence of constants such that $B_N \to 0$ as $N \to \infty$. We can choose $B_N = M^{-1}$, where M is known as the truncation point. It has to be noted that $q_{12}^*(\lambda)$ is zero at $\lambda = 0$.

Parzen [3] has shown that for algebraic type of kernels $k(x)$, the bandwidth of the estimates is given by

$$\beta(c_{12}^*(\lambda)) = \beta(q_{12}^*(\lambda)) = [2\pi/M \int k(x)\, dx] \tag{2.16}$$

which is inversely proportional to M.

We now derive two inequalities.

THEOREM 2. *Let algebraic type of kernels $k(x)$ be chosen to estimate the co-spectral density and quadrature spectral density. Then*

$$(c_{12}^2(\lambda))^{-1} \operatorname{var}(c_{12}^*(\lambda)) \cdot \beta(c_{12}^*(\lambda)) \geqq N^{-1} G(k), \tag{2.17}$$

$$(q_{12}^2(\lambda))^{-1} \operatorname{var}(q_{12}^*(\lambda)) \cdot \beta(q_{12}^*(\lambda)) \geqq N^{-1} G(k) \tag{2.18}$$

where

$$G(k) = [4\pi \int k^2(x)(1 + d_\lambda)/\int k(x)\, dx]$$

and

$$d_\lambda = 1 \quad \text{if } \lambda = 0 \text{ or } \pm\pi$$
$$= 0 \quad \text{otherwise.}$$

PROOF. From Rosenblatt [4], we have the variance of the estimate (2.15)

$$\operatorname{var}(c_{12}^*(\lambda)) = (M/N)[f_{11}(\lambda) f_{22}(\lambda) + c_{12}^2(\lambda) - q_{12}^2(\lambda)] \int k^2(x)\, dx \cdot (1 + d_\lambda). \tag{2.19}$$

Using the coherency inequality $f_{11}(\lambda) f_{22}(\lambda) \geqq c_{12}^2(\lambda) + q_{12}^2(\lambda)$, we can write (2.19) as

$$(c_{12}^2(\lambda))^{-1} \operatorname{var}(c_{12}^*(\lambda)) \geqq (2M/N) \int k^2(x)\, dx (1 + d_\lambda). \tag{2.20}$$

The result (2.17) then follows immediately from (2.20) and (2.16). Similarly one can proceed to show the result (2.18).

For designing the cross spectral estimate, one can consider the equality sign of (2.17) and (2.18) and proceed as in the case of single time series (Parzen [3]).

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