SERIES REPRESENTATIONS OF DISTRIBUTIONS OF QUADRATIC FORMS IN NORMAL VARIABLES II. NON-CENTRAL CASE

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This paper is a direct continuation of Part I. [5] Equation numbers continue from those of the earlier paper.

We recall that

(i) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ is the probability density function of $\sum_{i=1}^n \alpha_i(Z_i + \delta_i)^2$ where $Z_1, Z_2, \cdots, Z_n$ are mutually independent standardized normal variables, and $F_n(\boldsymbol{\alpha}; \delta; y)$ is the corresponding cumulative distribution function.

(ii) $f(n; \lambda; y) = f_n(\mathbf{1}; \boldsymbol{\delta}; y)$ with $\lambda = \sum_{i=1}^n \delta_i^2$ is the non-central χ^2 probability density function with n degrees of freedom and non-centrality parameter λ, and $F(n; \lambda; y)$ the corresponding cumulative distribution function.

(iii) $g(n; y) = f(n; 0; y)$ is the central χ^2 probability density function, and $G(n; y)$ the corresponding cumulative distribution function.

In this paper we seek expansion for $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ in each of the following forms:

$$(101)\quad f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \textstyle\sum_{k=0}^\infty a_k^P(-1)^k(y/2)^{n/2+k-1}/2\Gamma(n/2+k) \qquad (\textit{Power Series}).$$

$$(102)\quad f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \textstyle\sum_{k=0}^\infty a_k^L g(n; y/\beta)[k!\,\Gamma(n/2)/\beta\Gamma(n/2+k)]L_k^{(n/2-1)}(y/2\beta) \qquad (\textit{Laguerre Series}).$$

(See (19) and (20) for definition of $L_k^{(n/2-1)}$.)

$$(103)\quad f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \textstyle\sum_{k=0}^\infty a_k^c \beta^{-1} g(n + 2k; y/\beta) \qquad (\textit{Chi-squared Series}).$$

$$(104)\quad f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \textstyle\sum_{k=0}^\infty a_k^{c'} \beta^{-1} f(n + 2k; \lambda; y/\beta) \qquad (\textit{Non-central Chi-squared Series}).$$

We now restrict ourselves to the positive definite forms with $\alpha_r > 0$. The general methods used are almost identical to those described in the introduction to Part I with $g_n(\boldsymbol{\alpha}; y)$ replaced by $f_n(\boldsymbol{\alpha}; \delta; y)$. Briefly, we first find the Laplace transform of $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ which is

$$(105)\quad \begin{aligned} L_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; s) &= \textstyle\int_0^\infty e^{-sy} f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)\, dy \\ &= \exp\left(-\textstyle\sum_{r=1}^n \delta_r^2 \alpha_r^s/(1 + 2s\alpha_r)\right) \cdot \textstyle\prod_{j=1}^n (1 + 2s\alpha_j)^{-\frac{1}{2}}. \end{aligned}$$

We seek an expansion

$$(106)\quad f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \textstyle\sum_{k=0}^\infty a_k h_k(y),$$

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where

$$\hat{h}_k(y) = \int_0^\infty e^{-sy} h_k(y)\, dy = \xi(s)\eta^k(s), \tag{107}$$

and so if $\theta = \eta(s)$ is equivalent to $s = \zeta(\theta)$, we expand

$$M(\theta) = L_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; \zeta(\theta))/\xi[\zeta(\theta)] = \sum_{k=0}^\infty a_k \theta^k \tag{108}$$

to obtain the coefficients a_k.

1. Power series expansion. Here, $h_k(y) = (-1)^k (y/2)^{n/2+k-1}/2\Gamma(n/2 + k)$, so, as in the central case,

$$\xi(s) = (2s)^{-n/2}, \qquad \eta(s) = -(2s)^{-1}, \qquad \zeta(\theta) = -(2\theta)^{-1}, \tag{109}$$

and

$$\begin{aligned} M(\theta) &= \left(\prod_{j=1}^n \alpha_j^{-\frac{1}{2}}\right) \exp\left(-\tfrac{1}{2}\sum_{r=1}^n \delta_r^2/(1 - \theta/\alpha_r)\right) \cdot \prod_{j=1}^n (1-\theta/\alpha_j)^{-\frac{1}{2}} \\ &= \sum_{k=0}^\infty a_k^P \theta^k, \quad |\theta| < \min_j \alpha_j = \alpha_n . \end{aligned} \tag{110}$$

By Cauchy's inequality

$$|a_k^P| \leqq (\max_{|\theta|=\rho} |M(\theta)|)\rho^{-k} = m(\rho)\rho^{-k}) \quad \text{for any} \quad \rho < \alpha_n. \tag{111}$$

$$m(\rho) \leqq C \exp\left[-\tfrac{1}{2} \sum_{r=1}^n \delta_r^2 \alpha_r/(\alpha_r + \rho)\right] \cdot \prod_{j=1}^n (1 - \rho/\alpha_j)^{-\frac{1}{2}}, \tag{112}$$

where $C = \prod_{j=1}^n \alpha_j^{-\frac{1}{2}}$ for

$$\max_{|\theta|=\rho} [\operatorname{Re}(\theta - \alpha_r)^{-1}] = -(\rho + \alpha_r)^{-1},$$

since $\rho < \alpha_r$. Hence,

$$|a_k^P| \leqq C\rho^{-k} \exp\left[-\tfrac{1}{2} \sum_{r=1}^n \delta_r^2 \alpha_r/(\alpha_r + \rho)\right] \cdot \prod_{j=1}^n (1 - \rho/\alpha_j)^{-\frac{1}{2}} \tag{113}$$

for any ρ with $0 < \rho < \alpha_n$. Now

$$\sum_{k=0}^\infty |a_k^P| \cdot |h_k(y)| \leqq \tfrac{1}{2} m(\rho)(y/2)^{n/2-1} \exp(y/2\rho), \tag{114}$$

for any ρ in $0 < \rho < \alpha_n$.

Applying the method described in the introduction to Part I of this paper [5], we have

$$f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^\infty a_k^P (-1)^k (y/2)^{n/2+k-1}/2\Gamma(n/2 + k). \tag{115}$$

The series (115) converges uniformly in every bounded interval of $y > 0$, (by (114)), so we have

$$F_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^\infty a_k^P (-1)^k (y/2)^{n/2+k}/\Gamma(n/2 + k + 1). \tag{116}$$

The series (116) is uniformly convergent in any bounded y-interval of $y > 0$.

The a_k^P are determined by

$$\sum_{k=0}^\infty a_k^P \theta^k = C \exp\left(-\tfrac{1}{2} \sum_{r=1}^n \delta_r^2/(1 - \theta/\alpha_r)\right) \cdot \prod_{j=1}^n (1 - \theta/\alpha_j)^{-\frac{1}{2}} \tag{117}$$

where the recurrence relation

$$a_0^P = M(0) = \prod_{j=1}^n \alpha_j^{-\frac{1}{2}}, \tag{118}$$
$$a_k^P = k^{-1} \sum_{r=0}^{k-1} b_{k-r}^P a_r^P \qquad (k \geqq 1)$$

with $b_k^P = \frac{1}{2} \sum_{j=1}^n (1 - \theta\delta_j^2)(\alpha_j)^{-k}$ can be obtained.

From (113)

$$\begin{aligned} e_n^P(y) &= |\textstyle\sum_{k=N+1}^\infty a_k h_k(y)| \\ &\leqq \exp[-\tfrac{1}{2} \textstyle\sum_{r=1}^n \delta_r^2\alpha_r/(\alpha_r + \rho)] \cdot \prod_{j=1}^n (\alpha_j - \rho)^{-\frac{1}{2}} \\ &\quad \cdot (2\Gamma(n/2))^{-1}(y/2)^{n/2-1} \cdot (y/2\rho)^{N+1}(N!)^{-1} \exp(y/2\rho), \end{aligned} \tag{119}$$

for any ρ in $0 < \rho < \alpha_n$.

Estimating further, we have

$$\begin{aligned} e_N^P(y) \leqq{} & \exp[-\tfrac{1}{2} \textstyle\sum_{r=1}^n \delta_r^2\alpha_r/(\alpha_r + \alpha_n)] \\ & \cdot (2\Gamma(n/2)N!)^{-1}(y/2)^{n/2-1} \\ & \cdot \min_{0<\rho<\alpha_n} [(y/2\rho)^{N+1} \exp(y/2\rho) \cdot \textstyle\prod_{j=1}^n (\alpha_j - \rho)^{-\frac{1}{2}}]. \end{aligned} \tag{120}$$

The estimate for $E_N^P(y)$ is obtained in the same way. It is

$$\begin{aligned} E_N^P(y) \leqq{} & (C/\Gamma(n/2)) \exp[-\tfrac{1}{2} \textstyle\sum_{r=1}^n \delta_r^2\alpha_r/(\alpha_r + \rho)] \\ & \cdot \textstyle\prod_{j=1}^n (1 - \rho/\alpha_j)^{-\frac{1}{2}} \cdot (y/2)^{n/2}(y/2\rho)^{N+1}((N+1)!)^{-1} \exp(y/2\rho), \end{aligned} \tag{121}$$

where ρ is any number in $0 < \rho < \alpha_n$.

A double series for $F_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ in powers of y was obtained by Shah and Khatri [13], using the method of Pachares [7]. The coefficients in the series were given as expectations of powers of a linear form and a quadratic form and their products. This expansion could be obtained by expanding $M(\theta)$ in a double series, the product of the expansion of $\exp(-\frac{1}{2} \sum_{r=1}^n \delta_r^2\alpha_r/(\alpha_r - \theta))$ in powers of θ, and the expansion of $\prod_{j=1}^n (\alpha_j - \theta)^{-\frac{1}{2}}$ in powers of θ. The single series forms (115) and (116) would seem more suitable for computation. Convergence of these series is more rapid as $\sum_{j=1}^n \delta_j^2$ increases, as can be seen from (120) and (121) since $m(\rho) \to 0$ as $\sum_{j=1}^n \delta_j^2 \to \infty$. However in this case the b_k^P used to determine the a_k^P recursively tend to $-\infty$, and hence, although a_k^P are small they are determined as differences of large numbers. This limits the usefulness of the series (115) and (116), even in the situation in which they are theoretically most useful (i.e. large $\sum \delta_j^2$).

2. Laguerre series expansion. To obtain a series of the form (102), we proceed just as in the central case. We have (see (48)),

$$\xi(s) = (1 + 2s\beta)^{-n/2}, \qquad \eta(s) = 2s\beta/(1 + 2s\beta), \qquad \zeta(\theta) = \theta/2\beta(1 - \theta), \tag{122}$$

so that, with $\gamma_j = 1 - \alpha_j/\beta$,

$$M(\theta) = \exp\{-\tfrac{1}{2} \textstyle\sum_{k=1}^n (\delta_k^2\alpha_k/\beta) \cdot \theta/(1 - \gamma_k\theta)\} \cdot \prod_{j=1}^n (1 - \gamma_j\theta)^{-\frac{1}{2}} \tag{123}$$

provided

$$|\theta| < \epsilon^{-1}, \tag{124}$$

where

$$\epsilon = \max_j |1 - \alpha_j/\beta|. \tag{125}$$

Thus, we must determine the coefficients a_k^L from

$$\sum_{k=0}^{\infty} a_k^L \theta^k = \exp\{-\tfrac{1}{2}\sum_{k=1}^{n}(\delta_k^2\alpha_k/\beta)\cdot\theta/(1-\gamma_k\theta)\}\cdot\prod_{j=1}^{n}(1-\gamma_j\theta)^{-\frac{1}{2}}. \tag{126}$$

Using Cauchy's inequality and a bound on $m(\rho)$ (similar to one given by (112)), we find:

$$|a_k^L| \leqq (1 - \epsilon\rho)^{-n/2}\rho^{-k}\exp(\lambda/2\epsilon), \tag{127}$$

for any ρ in $0 < \rho < \epsilon^{-1}$. As in the central case, we use (67) of Part I to obtain

$$|L_k^{(n/2-1)}(y/2\beta)| \leqq e^{y/4\beta}(1 - R)^{-n/2}R^{-k}, \tag{128}$$

for any R in $0 < R < 1$. Then, obtaining a bound of the form Ae^{by} (see (5) of Part I), we have

$$\begin{aligned} &\sum_{k=0}^{\infty} |a_k^L|\, g(n; y/\beta)[k!\,\Gamma(n/2)/\beta\Gamma(n/2+k)]\cdot|L_k^{(n/2-1)}(y/2\beta)| \\ (129)\quad &\leqq (1-\epsilon\rho)^{-n/2}(1-R)^{-n/2}e^{\lambda/2\epsilon}e^{y/4\beta}\beta^{-1}g(n; y/\beta)\sum_{k=0}^{\infty}\rho^{-k}R^{-k}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{using (127) and (128)} \\ &\leqq (1-\epsilon\rho)^{-n/2}(1-R)^{-n/2}e^{\lambda/2\epsilon}e^{y/4\beta}\beta^{-1}g(n; y/\beta)\rho R/(\rho R - 1), \end{aligned}$$

provided $\rho R > 1$, $0 < R < 1$, and $0 < \rho < \epsilon^{-1}$, or equivalently

$$\epsilon < \rho^{-1} < R < 1. \tag{130}$$

Thus, using the argument developed in the introduction to Part I, we have

$$f_n(\boldsymbol{\alpha};\boldsymbol{\delta};y) = \sum_{k=0}^{\infty} a_k^L[k!\,\Gamma(n/2)/\Gamma(n/2+k)]\beta^{-1}g(n;y/\beta)L_k^{(n/2-1)}(y/2\beta) \tag{131}$$

and

$$\begin{aligned} (132)\quad F_n(\boldsymbol{\alpha};\boldsymbol{\delta};y) &= G(n;y/\beta) \\ &+ \sum_{k=1}^{\infty}[a_k^L(k-1)!/\Gamma(n/2+k)](y/2\beta)^{n/2}e^{-y/2\beta}L_{k-1}^{(n/2)}(y/2\beta). \end{aligned}$$

These expansions are uniformly convergent for all $y > 0$, provided $\beta > \frac{1}{2}\alpha_1$.

The error bounds are more complicated than in the central case. We have to use (127) in place of (64). Thus, changing the sum in (129) to $k = N + 1$ to ∞, we obtain

$$\begin{aligned} (133)\quad & e_N^L(y) \\ &\leqq (1-\epsilon\rho)^{-n/2}(1-R)^{-n/2}e^{\lambda/2\epsilon}e^{y/4\beta}\beta^{-1}g(n; y/\beta)(\rho R)^{-N}/(\rho R - 1), \end{aligned}$$

where ρ and R satisfy (130). A convenient choice of ρ and R is $\rho = \epsilon^{-\frac{2}{3}}$, $R = \epsilon^{\frac{1}{3}}$. This leads to

$$e_N^L(y) \leqq \beta^{-1}g(n; y/\beta)e^{y/4\beta}e^{\lambda/2\epsilon}(1-\epsilon^{\frac{1}{3}})^{-n-1}\epsilon^{\frac{1}{3}(N+1)}. \tag{134}$$

Similarly, the error term $E_N^L(y)$ may be estimated from (127) and (128) as:

$$(135)\quad E_N^L(y) \leqq 2(1-\epsilon\rho)^{-n/2} \cdot (1-R)^{-n/2-1} e^{\lambda/2\epsilon} e^{y/4\beta} g(n+2; y/\beta)[(\rho R)^{-(N+1)}/(1-1/\rho R)],$$

for any ρ, R with $\epsilon < 1/\rho < R < 1$.

Choosing $\rho = \epsilon^{-\frac{2}{3}}$ and $R = \epsilon^{\frac{1}{3}}$ once more, we have

$$(136)\qquad E_N^L(y) \leqq 2(1-\epsilon^{\frac{1}{3}})^{-n-1} e^{\lambda/2\epsilon} e^{y/4\beta} g(n+2; y/\beta)\epsilon^{\frac{1}{3}(N+1)}.$$

The discussion in Part I about the best choice for ϵ also holds good here, and choice (79) is recommended.

A Laguerre expansion of $F_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ was given by Shah in [12] using Gurland's method. The expression given differs from (132), in that a double series of terms of the form $a_{k,p}L_p^{(\frac{1}{2}n+k)}$ appears. The coefficients $a_{k,p}$ involve rather complicated sums. Equation (132) would seem a more natural generalization of Gurland's [3] expansion.

The coefficients a_k^L are calculated from equations like (118), with $M(0) = 1$;

$$b_k^L = -\tfrac{1}{2}k \textstyle\sum_{j=1}^n \delta_j^2 \gamma_k^{k-1} + \tfrac{1}{2}\sum_{j=1}^n (1 + k\delta_j^2)\gamma_j^k \qquad (k \geqq 1).$$

3. Expansion in central χ^2 distributions. We desire an expansion for $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ of the form (103). As in (81), (82), we have

$$(137)\quad \xi(s) = (1+2s\beta)^{-n/2}, \qquad \eta(s) = (1+2s\beta)^{-1}, \qquad \zeta(\theta) = (1-\theta)/2\beta\theta,$$

and $\xi[\zeta(\theta)] = \theta^{n/2}$. Thus,

$$(138)\quad \begin{aligned} M(\theta) &= \exp\{-\tfrac{1}{2}\textstyle\sum_{k=1}^n \delta_k^2(1-\theta)/(1-(1-\beta/\alpha_k)\theta)\} \cdot \prod_{j=1}^n (\beta/\alpha_j)^{\frac{1}{2}} \\ &\quad \cdot \{1-(1-\beta/\alpha_j)\theta\}^{-\frac{1}{2}} \\ &= A\exp\{-\tfrac{1}{2}\textstyle\sum_{k=1}^n \delta_k^2(1-\theta)/(1-\gamma_k\theta)\}\cdot\prod_{j=1}^n (1-\gamma_j\theta)^{-\frac{1}{2}}, \end{aligned}$$

where

$$\gamma_j = 1 - \beta/\alpha_j, \qquad A = \textstyle\prod_{j=1}^n (\beta/\alpha_j)^{\frac{1}{2}}, \qquad \epsilon = \max_j |\gamma_j|.$$

By definition,

$$(139)\qquad M(\theta) = \textstyle\sum_{k=0}^\infty a_k^C\theta^k, \quad \text{for} \quad |\theta| < \epsilon^{-1},$$

and the condition $|\theta| < \epsilon^{-1}$ means that s must satisfy (see (87))

$$(140)\qquad \operatorname{Re} s > 2\beta^{-1}(\epsilon - 1).$$

If $m(\rho) = \max_{|\theta|=\rho} |M(\theta)|$, for $\rho < \frac{1}{2}$, then

$$(141)\qquad m(\rho) \leqq A\exp\{\tfrac{1}{2}\textstyle\sum_{k=1}^n \delta_k^2(\rho-1)/(1-\gamma_k\rho)\}\cdot\prod_{j=1}^n (1-\gamma_j\rho)^{-\frac{1}{2}}$$

and, using Cauchy's inequality

$$(142)\qquad |a_k^C| \leqq m(\rho)\rho^{-k},$$

for any ρ in $0 < \rho < \epsilon^{-1}$,

$$(143)\quad \textstyle\sum_{k=0}^\infty |a_k^C|\cdot\beta^{-1}g(n+2k; y/\beta) \leqq m(\rho)\beta^{-1}g(n; y/\beta)[1 + (y/2\beta\rho)\exp(y/2\beta\rho)].$$

Hence, using the general method as described in the introduction to Part I, we conclude that (103) is a valid expansion and the series converges uniformly and absolutely for bounded intervals of $y > 0$. Also if $\epsilon < 1$, so that we may take $\rho^{-1} < 1$ in (143), then the series is uniformly absolutely convergent for all $y > 0$.

Integrating term-by-term, we have

$$F_n(\alpha; \delta; y) = \sum_{k=0}^{\infty} a_k^C G(n + 2k; y/\beta) \tag{144}$$

with the series uniformly convergent for any bounded interval of $y > 0$, and uniformly convergent for all y if $\epsilon < 1$.

The series (144) is due to Ruben [10] who established it by a direct method, giving the expression (140), for the coefficients a_k^C, and also the recursive formula

$$\begin{aligned} a_0^C &= M(0) = Ae^{-\frac{1}{2}\lambda}, \\ a^C &= k^{-1} \sum_{r=0}^{k-1} b_{k-r}^C a_r^C \qquad (k \geqq 1), \end{aligned} \tag{145}$$

where

$$b_k^C = \tfrac{1}{2}k \sum_{j=1}^{n} \delta_j^2 \gamma_j^{k-1} + \tfrac{1}{2} \sum_{j=1}^{n} (1 - k\delta_j^2)\gamma_j^k \qquad (k \geqq 1).$$

Before discussing the errors $e_N^C(y)$ and $E_N^C(y)$ we shall obtain an alternative expression for the coefficients a_k^C which was derived by Ruben in [11], and is of some theoretical and practical interest. This expression depends on the assumption that $0 < \beta \leqq \alpha_n$, which incidentally makes (144) a mixture representation (see Ruben [10]).

Rewrite $M(\theta)$ from (138), in the form

$$M(\theta) = Ae^{-\frac{1}{2}\lambda} \exp\{\textstyle\sum_{k=1}^{n} \frac{1}{2}\delta_k^2(1 - \gamma_k)\theta/(1 - \gamma_k\theta)\} \cdot \prod_{j=1}^{n} (1 - \gamma_j\theta)^{-\frac{1}{2}}. \tag{146}$$

Assuming $0 < \beta \leqq \alpha_n$, we have

$$1 - \gamma_k = 1 - \beta/\alpha_k \geqq 0, \qquad k = 1, 2, \cdots, n. \tag{147}$$

Thus, the expression $A^{-1}e^{\frac{1}{2}\lambda}M(\theta)$ is the moment generating function of the density $f_n(\gamma/2; \delta'; 2y)$, as may be seen by comparing it with (105). Here,

$$\delta_k' = \delta_k(\gamma_k^{-1} - 1)^{\frac{1}{2}}, \qquad k = 1, 2, \cdots, n. \tag{148}$$

Thus,

$$\begin{aligned} M(\theta) &= \sum_{k=0}^{\infty} a_k^C \theta^k \\ &= Ae^{-\frac{1}{2}\lambda} L_n(\gamma/2; \delta'; -\theta) \\ &= Ae^{-\frac{1}{2}\lambda} \sum_{k=0}^{\infty} E(\hat{Q}^k)\theta^k/k!, \end{aligned} \tag{149}$$

where $\hat{Q}$ is the quadratic form defined by

$$\begin{aligned} \hat{Q}(X) &= \tfrac{1}{2} \sum_{j=1}^{n} \gamma_j(X_j + \delta_j')^2 \\ &= \tfrac{1}{2} \sum_{j=1}^{n} (\gamma_j^{\frac{1}{2}}X_j + \delta_j(1 - \gamma_j)^{\frac{1}{2}})^2, \end{aligned} \tag{150}$$

and the X_j are independent $N(0, 1)$ variables.

Equating coefficients in (149) yields

$$a_k^C = Ae^{-\frac{1}{2}\lambda}E(\hat{Q}^k)/k!. \tag{151}$$

This can be used to obtain bounds for the a_k^C in much the same way as was done in the central case. These are:

$$0 \leqq a_k^C \leqq Ae^{-\frac{1}{2}\lambda}\epsilon^k L_k^{(n/2-1)}(-\tfrac{1}{2}\lambda'), \tag{152}$$

provided $0 < \beta \leqq \alpha_n$, with $\lambda' = \sum_{j=1}^n \gamma_j^{-1}\delta_j^2(1 - \gamma_j)$.

This inequality may be combined with the following, derived from Cauchy's inequality

$$L_k^{(n/2-1)}(-\tfrac{1}{2}\lambda') \leqq R^{-k}(1 - R)^{-n/2} \exp\{+\tfrac{1}{2}\lambda' R/(1 - R)\}, \tag{153}$$

for any R in $0 < R < 1$. Recall that

$$\sum_{k=0}^\infty L_k^{(a)}(x)t^k = (1 - t)^{-a-1} \exp(-xt/(1 - t)), \qquad (|t| < 1.)$$

We now obtain bounds for $e_N^C(y)$ and $E_N^C(y)$ which differ from Ruben's [10] bounds. The inequality used ([10], equation (4.14)), was

$$|a_k^C| \leqq A[e^{-\frac{1}{2}\lambda}\Gamma(\tfrac{1}{2}n + k)/\Gamma(n/2)]\cdot\mu^k/k!, \tag{154}$$

where

$$\mu = \tfrac{1}{2} \sum_{j=1}^n \delta_j^2\beta/\alpha_j + \epsilon. \tag{155}$$

The use of (142) is preferable to (154) where great accuracy is required since, for large k, the estimate (142) decreases like r^k for any $r > \epsilon$ while (154) decreases like μ^k. If $\delta \neq 0$, then $\mu > \epsilon$, and we can always take $r = \rho^{-1} < \mu$. For small values of k, however, the estimate (154) is usually better than that in (142), since the factor multiplying μ^k is less than the factor multiplying ρ^{-k}.

We have, using (142), (compare (143)),

$$\begin{aligned} e_N^C(y) \leqq (A/\beta)g(n; y/\beta) \exp(y/2\beta\rho) \exp\{\tfrac{1}{2} \textstyle\sum_{k=1}^n \delta_k^2 \\ \cdot(\rho - 1)/(1 - \gamma_k\rho)\}\textstyle\prod_{j=1}^n (1 - \gamma_j\rho)^{-\frac{1}{2}}(N!)^{-1}(y/2\beta\rho)^{N+1}, \end{aligned} \tag{156}$$

for any ρ with $0 < \rho < \epsilon^{-1}$.

Or, using (154), we have

$$\begin{aligned} e_N^C(y) &\leqq Ae^{-\frac{1}{2}\lambda} \textstyle\sum_{k=N+1}^\infty [\Gamma(\tfrac{1}{2}n + k)\mu^k/\Gamma(n/2)k!](2\beta)^{-1}e^{-y/2\beta} \\ &\qquad \cdot(y/2\beta)^{n/2+k-1}\Gamma(n/2 + k))^{-1} \\ &\leqq (A/\beta)e^{-\frac{1}{2}\lambda}g(n; y/\beta) \exp(y\mu/2\beta)\cdot(y\mu/2\beta)^{N+1}((N + 1)!)^{-1} \end{aligned} \tag{157}$$

where μ is given by (155).

As bounds for $E_N^C(y)$, we find

$$E_N^C(y) \leqq Ae^{-\frac{1}{2}\lambda}(y/\beta)g(n; y/\beta) \exp(y\mu/2\beta)\cdot(y\mu/2\beta)^{N+1}((N + 1)!)^{-1} \tag{158}$$

for $\mu > 1$, and

$$\begin{aligned} E_N^C(y) \leqq Ae^{-\frac{1}{2}\lambda}[\Gamma(\tfrac{1}{2}n + N + 1)/\Gamma(n/2)(N + 1)!]\mu^{N+1}(1 - \mu)^{-n/2-N} \\ \cdot G(n + 2N + 2; (1 - \mu)y/\beta), \end{aligned} \tag{159}$$

for $0 < \mu < 1$.

4. Expansion in non-central χ^2 distributions. An expansion of $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ of the form given in (104) will now be sought.

We first need the Laplace transform of each term on the right of (104). We have

$$(160)\quad \int_0^\infty e^{-sy}\beta^{-1}f(n+2k;\lambda;y/\beta)\,dy = \exp\{-s\beta\lambda/(1+2s\beta)\}(1+2s\beta)^{-n/2-k}$$

(using (105), with $\alpha_j = \beta$ for $j = 1, \cdots, n+2k$).

Hence, in the notation of (6) of Part I,

$$(161)\quad \begin{aligned} \xi(s) &= \exp\{-s\beta\lambda/(1+2s\beta)\}(1+2s\beta)^{-n/2};\\ \eta(s) &= (1+2s\beta)^{-1} = \theta;\\ \zeta(\theta) &= (1-\theta)/2\beta\theta = s; \end{aligned}$$

and $\xi[\zeta(\theta)] = \theta^{n/2}\exp(-(1-\theta)\lambda/2)$. Hence,

$$(162)\quad M(\theta) = A\exp[-\tfrac{1}{2}\theta(1-\theta)\textstyle\sum_{k=1}^n \delta_k^2\gamma_k/(1-\gamma_k\theta)]\prod_{j=1}^n(1-\gamma_j\theta)^{-\frac{1}{2}},$$

where

$$(163)\quad \gamma_k = 1 - \beta/\alpha_k\,; \qquad A = \textstyle\prod_{j=1}^n(\beta/\alpha_j)^{\frac{1}{2}}.$$

The proof that series (104) is uniformly convergent and of exponential order as $y \to \infty$ requires a bound on $f(n+2k;\lambda;y)$. To obtain a suitable bound, we use the special case (non-central χ^2) of (131) with $\beta = 1$, $\alpha_j = 1$ for all j, and $\boldsymbol{\delta} = (\lambda^{\frac{1}{2}}, 0, 0, \cdots, 0)$ to obtain

$$(164)\quad f(n;\lambda;x) = \textstyle\sum_{k=0}^\infty a_k^{L}[k!\,\Gamma(n/2)/\Gamma(n/2+k)]g(n;x)L_k^{(n/2-1)}(x/2).$$

In this case, (126) reduces to

$$(165)\quad \textstyle\sum_{k=0}^\infty a_k^{L}\theta^k = \exp(-\tfrac{1}{2}\lambda\theta),$$

so that

$$(166)\quad a_k^{L} = (-\tfrac{1}{2}\lambda)^k/k!,$$

and

$$(167)\quad f(n;\lambda;x) = \textstyle\sum_{k=0}^\infty(-\tfrac{1}{2}\lambda)^k(\Gamma(n/2+k))^{-1}\tfrac{1}{2}e^{-x/2}(x/2)^{n/2-1}L_k^{(n/2-1)}(x/2).$$

From (128) with $\beta = 1$, $R = \frac{1}{2}$, $|L_k^{(n/2-1)}(x/2)| \leqq e^{x/4}2^{n/2+k}$. Then, from (167)

$$(168)\quad \begin{aligned} f(n;\lambda;x) &\leqq 2^{n/2-1}e^{-x/2}(x/2)^{n/2-1}e^{x/4}\textstyle\sum_{k=1}^\infty\lambda^k/(k-1)!\,\Gamma(n/2+1)\\ &= x^{n/2-1}e^{-x/4}(\Gamma(n/2))^{-1}(1+(2/n)\lambda e^{\lambda}). \end{aligned}$$

(Since $\sum_{u=1}^\infty\lambda^u/(u-1)! = \lambda e^{\lambda}$.) Thus,

$$(169)\quad \begin{aligned} &\textstyle\sum_{k=0}^\infty|\beta^{-1}a_k^{C'}f(n+2k;\lambda;y/\beta)|\\ &\qquad\leqq \beta^{-1}m(\rho)(1+2\lambda e^{\lambda})(y/\beta)^{n/2-1}(\Gamma(n/2))^{-1}e^{-y/4\beta}[1+2(y/\beta\rho)e^{y/\beta\rho}]. \end{aligned}$$

Hence the series (167) is uniformly convergent in any bounded y-interval and, if $\rho > 4$ for all $y > 0$. Thus, we have,

$$f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^{C'} \beta^{-1} f(n; \lambda; y/\beta) \tag{170}$$

for any $\beta > 0$. The series is uniformly convergent in any bounded y-interval. Also

$$F_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^{C'} F(n; \lambda; y/\beta) \tag{171}$$

and the series is uniformly convergent over any bounded y-interval as well.

The $a_k^{C'}$ are defined by

$$\begin{aligned} \sum_{k=0}^{\infty} a_k^{C'} \theta^k \\ = A \exp[-\tfrac{1}{2}\theta(1-\theta) \textstyle\sum_{k=1}^{n} \delta_k^2 \gamma_k/(1-\gamma_k\theta)] \cdot \prod_{j=1}^{n} (1-\gamma_j\theta)^{\frac{1}{2}}, \end{aligned} \tag{172}$$

where

$$A = \prod_{j=1}^{n} (\beta/\alpha_j)^{\frac{1}{2}}, \quad \text{and} \quad \gamma_j = 1 - \beta/\alpha_j . \tag{173}$$

The following recurrence formulae hold:

$$\begin{aligned} a_0^{C'} &= M(0) = A, \\ a_k^{C'} &= k^{-1} \textstyle\sum_{r=0}^{k-1} b_{k-r}^{C'} a_r^{C'}, \qquad k \geqq 1, \end{aligned} \tag{174}$$

with

$$\begin{aligned} b_1^{C'} &= \tfrac{1}{2} \textstyle\sum_{j=1}^{n} (1-\delta_j^2)\gamma_j , \\ b_n^{C'} &= \tfrac{1}{2}k \textstyle\sum_{j=1}^{n} \delta_j^2 \gamma_j^{k-1} + \tfrac{1}{2} \sum_{j=1}^{r} (1 - k\delta_j^2)\gamma_j^k, \qquad (k \leqq 2). \end{aligned}$$

Error bounds may be obtained in the usual way, using (168), and the well-known inequality,

$$F(n; \lambda; y/\beta) \leqq G(n; y/\beta), \tag{175}$$

with equality only if $\lambda = 0$. Now,

$$m(\rho) = \max_{|\theta|=\rho} |M(\theta)| \leqq A \exp(\tfrac{1}{2}\lambda\rho(\rho-1)\epsilon/(1-\epsilon\rho)) \cdot (1-\epsilon\rho)^{-n/2}. \tag{176}$$

In particular, if $\epsilon < 1$

$$m(\epsilon^{-\frac{1}{2}}) \leqq A \exp(\tfrac{1}{2}\lambda)(1-\epsilon^{\frac{1}{2}})^{-n/2}, \tag{177}$$

and

$$a_k^{C'} \leqq A e^{\frac{1}{2}\lambda}(1-\epsilon^{\frac{1}{2}})^{-n/2} \epsilon^{\frac{1}{2}k}, \qquad \text{if} \quad 0 < \epsilon < 1. \tag{178}$$

Since the expansions (170) and (171) are of more theoretical than practical interest we leave the required estimates of $E_N^{C'}(y)$ and $e_N^{C'}(y)$ to the interested reader. Expressions (170) and (171) were given by Ruben [10] who also gave the recursion formulae (174).

5. Some comparisons. It is to be expected that the error bounds obtained for the non-central case will not be, in general, as sharp as those for the central case. In order to assess differences between the two kinds of bounds, we put $\delta_r = 0$ in the non-central formulae and compare with the corresponding central formulae.

(1) First, considering the *Laguerre series* expansions, from (134), with $\lambda = 0$,

$$(179) \qquad e_N'^L(y) \leqq \beta^{-1} g(n; y/\beta) e^{y/4\beta} (1 - \epsilon^{\frac{1}{3}})^{-n-1} \epsilon^{\frac{1}{3}(N+1)} = b_N^L(0)$$

while from (74) of Part I

$$(180) \qquad e_N^L(y) \leqq \beta^{-1} g(n; y/\beta) e^{y/4\beta} (1 - \epsilon^{\frac{1}{2}})^{-\frac{1}{2}n-1} \epsilon^{\frac{1}{2}(N+1)} = b_N^L.$$

The ratio of the bounds $b_N^L/b_N^L(0)$ is

$$(181) \qquad \epsilon^{\frac{1}{6}(N+1)} (1 - \epsilon^{\frac{1}{3}})^{n+1} / (1 - \epsilon^{\frac{1}{2}})^{\frac{1}{2}n+1}.$$

Since $0 < \epsilon < 1$, $(1 - \epsilon^{\frac{1}{3}}) < (1 - \epsilon^{\frac{1}{2}})$ and so $(1 - \epsilon^{\frac{1}{3}})^{n+1} < (1 - \epsilon^{\frac{1}{2}})^{\frac{1}{2}n+1}$ for $n \geqq 1$. It follows that the ratio (181) is less than 1 and so the (central) bound (180) is (uniformly in y) less than the special case of the non-central bound (179). In fact, as N and/or n increase, the ratio of the bounds $b_N^L/b_N^L(0)$ decreases to 0 geometrically.

Similarly, comparing the bounds (136) (with $\lambda = 0$) and (76) of Part I for $E_N^L(y)$, we find that the latter is the smaller if

$$(182) \qquad n(N + 1)^{-1} \epsilon^{(N+4)/6} (1 - \epsilon^{\frac{1}{3}})^{n+1} / (1 - \epsilon^{\frac{1}{2}})^{\frac{1}{2}n+2} < 1.$$

This is certainly the case if $N + 1 \geqq n \geqq 3$. The ratio (182) tends to 0 geometrically when $N \to \infty$ and as $n\delta^{n/2-1}$ (where δ is less than $1 - \epsilon^{\frac{1}{2}}$) with $n \to \infty$.

In the case of expansions in terms of *central* χ^2 *distributions*, however, the situation is different. Formula (158), with $\mu = \epsilon$ and $\lambda = 0$, gives the *same* bound for $E_N^C(y)$ as does the (central) formula (96) of Part I. This is, of course, a satisfactory result.

Turning now to expansions in *power series*, formula (121), with $\delta_r = 0$, gives

$$(183) \quad E_N'^P(y) \leqq [\Gamma(\tfrac{1}{2}n)(N + 1)!]^{-1} (\tfrac{1}{2}y)^{\frac{1}{2}n} \cdot \min_{0<\rho<\alpha_n} [(y/2\rho)^{N+1} \exp (y/2\rho) \cdot \textstyle\prod_{j=1}^n (\alpha_j - \rho)^{-\frac{1}{2}}] = B_N^P(0)$$

while, from the (central) formula (51) of Part I,

$$(184) \quad E_N^P(y) \leqq [(\tfrac{1}{2}n + N + 1)\Gamma(\tfrac{1}{2}n)(N + 1)!]^{-1} (\tfrac{1}{2}y)^{\frac{1}{2}n} (y/2\alpha_n)^{N+1} \textstyle\prod_{j=1}^n \alpha_j^{-\frac{1}{2}} = B_N^P.$$

The right-hand side of (183) cannot be less than

$$[\Gamma(\tfrac{1}{2}n)(N + 1)!]^{-1} (\tfrac{1}{2}y)^{\frac{1}{2}n} (y/2\alpha_n)^{N+1} \exp (y/2\alpha_n) \textstyle\prod_{j=1}^n \alpha_j^{-\frac{1}{2}}$$

which is the bound in (184) multiplied by $(\frac{1}{2}n + N + 1)e^{(y/2\alpha_n)}$.

Therefore the central case bound is here better than the corresponding non-central case bound, especially for large y, due to the exponential term.

REFERENCES

[1] GRAD, A. and SOLOMON, H. (1955). Distribution of quadratic forms and some applications. *Ann. Math. Statist.* **26** 464–477.

[2] GURLAND, J. (1955). Distribution of definite and of indefinite quadratic forms. *Ann. Math. Statist.* **26** 122–127. Correction: *Ann. Math. Statist.* **33** (1962) 813.

[3] GURLAND, J. (1956). Quadratic forms in normally distributed random variables. *Sankhyā.* **17** 37–50.

[4] HOTELLING, H. (1948). Some new methods for distribution of quadratic forms (abstract). *Ann. Math. Statist.* **19** 119.

[5] KOTZ, S., JOHNSON N. L. and BOYD, D. W. (1967) Series representations of quadratic forms in normal variables I. Central Case. *Ann. Math. Statist.* **38** 823–837.

[6] PACHARES, J. (1952). The distribution of the difference of two independent chi-squares (abstract). *Ann. Math. Statist.* **23** 639.

[7] PACHARES, J. (1955). Note on the distribution of a definite quadratic form. *Ann. Math. Statist.* **26** 128–131.

[8] ROBBINS, H. (1948). The distribution of a definite quadratic form. *Ann. Math. Statist.* **19** 266–270.

[9] ROBBINS, H. and PITMAN, E. J. G. (1949). Application of the method of mixtures to quadratic forms in normal variables. *Ann. Math. Statist.* **20** 552–560.

[10] RUBEN, H. (1962). Probability content of regions under spherical normal distributions, IV: The distribution of homogeneous and non-homogeneous quadratic functions of normal variables. *Ann. Math. Statist.* **33** 542–570.

[11] RUBEN, H. (1963). A new result on the distribution of quadratic forms. *Ann. Math. Statist.* **34** 1582–1584.

[12] SHAH, B. K. (1963). Distribution of definite and of indefinite quadratic forms from a non-central normal distribution. *Ann. Math. Statist.* **34** 186–190.

[13] SHAH, B. K. and KHATRI, C. G. (1961). Distribution of a definite quadratic form for non-central normal variates. *Ann. Math. Statist.* **32** 883–887. Correction: *Ann. Math. Statist.* **34** (1963) 673.

14] SZEGO, G. (1939). *Orthogonal Polynomials.* Amer. Math. Soc. Colloquium Publ. **23.**