OPTIMAL SEQUENTIAL PROCEDURES WHEN MORE THAN ONE STOP IS REQUIRED[1]

By Gus W. Haggstrom

University of California, Berkeley

1. Introduction. Let $\{Y_m, m = 1, 2, \cdots\}$ be a (possibly finite) sequence of random variables having a known distribution. These random variables can be observed sequentially, perhaps at some cost, by a statistician who must decide when to stop. If he stops after having observed $Y^{(m)} = (Y_1, \cdots, Y_m)$, he is then presented with an optimal stopping problem that depends on $Y^{(m)}$, i.e., he starts taking observations on another sequence of random variables $\{Y_{mk}, k = m + 1, m + 2, \cdots\}$ and his gain if he stops after observing $Y^{(m,n)} = (Y_1, \cdots Y_m, Y_{m,m+1}, \cdots, Y_{mn})$ is $Z_{mn} = f_{mn}(Y^{(m,n)})$, where f_{mn} is a known real-valued function of all the observations up to that stage. The statistician's problem is to choose a procedure to maximize his expected gain.

This formulation provides a model for studying some extensions of optimal stopping problems that were first considered by Mosteller and Gilbert in [5]. The model is specifically intended to include their two-stop problems (see the examples in Section 3) but can be extended to include their r-stop problems.

The formulation above also applies to some statistical situations in which a preliminary sample can be taken before a sequential decision procedure, or perhaps the design, is decided upon for a second stage. As an example, consider the situation of a man who is going into business for at most 40 years. Suppose that at the end of each year he can choose to continue his operation or he can stop, in which case his net gain is the sum of the profits (perhaps negative) for each of the preceding years. It may be plausible to assume that these yearly profits have a joint distribution that depends on a parameter θ, which in turn can be assumed to have a certain prior distribution. Before starting the business, he may be able to gather information about the value of θ by making observations on random variables (perhaps the profits of similar businesses) at some cost per observation. The problem of maximizing the expected net gain falls under the general formulation above. Other examples are given in Section 3.

2. General solution. The following structure will be assumed throughout: (i) a probability space (Ω, F, P) with points ω; (ii) a non-decreasing sequence $\{F_m, m \geqq 1\}$ of sub-fields of F; (iii) for each fixed $m = 1, 2, \cdots$, a stochastic process $\{Z_{mn}, F_{mn}, n > m\}$ such that $F_m \subset F_{mn} \subset F_{m,n+1} \subset F$ for all $n > m \geqq 1$.

In terms of the informal discussion at the beginning of Section 1, F_m and F_{mn}

Received 12 December 1966.

[1] This research was supported in part by Research Grant No. NSF GP 3707 from the Division of Mathematical, Physical and Engineering Sciences of the National Science Foundation, and in part by the Army Research Office, Office of Naval Research and Air Force Office of Scientific Research by Contract No. Nonr-2121 (23), NR 342-043.

are the σ-fields generated by the vectors of observations $Y^{(m)}$ and $Y^{(m,n)}$ respectively. The conditional expectation operators relative to these σ-fields will be denoted below by E_m and E_{mn} respectively.

Although it is implicit in the notation above that all the sequences $\{F_m, m \geqq 1\}$ and $\{Z_{mn}, F_{mn}, n > m\}$ are infinite, the theory below will still apply with only minor notational changes if some or all of these sequences are finite.

DEFINITION. *A compound stopping variable* (csv) is a pair of rv's (s, t) with values in $\{1, 2, \cdots, \infty\}$ such that

(a) $s < t < \infty$ a.e.;

(b) $\{s = m\} \varepsilon F_m$ for all $m \geqq 1$;

(c) $\{s = m, t = n\} \varepsilon F_{mn}$ for all $n > m \geqq 1$.

For any csv (s, t), the rv Z_{st}, also denoted by $Z(s, t)$ below, is defined by

$$Z_{st}(\omega) = Z_{mn}(\omega) \quad \text{if} \quad s(\omega) = m, \quad t(\omega) = n, \quad n > m$$
$$= -\infty \quad \text{if} \quad s(\omega) > t(\omega) \quad \text{or} \quad t(\omega) = \infty.$$

The following additional restriction on the rv's Z_{mn} will be assumed throughout:

HYPOTHESIS A. If $U = \sup Z_{mn}^{+}$ and $U_m = E_m U$ for $m = 1, 2, \cdots$, then $E(\sup U_m) < \infty$.

Let T_m denote the class of csv's (s, t) such that $s \geqq m$, and let T_{mn} denote the class for which $s = m$ and $t \geqq n$. A csv (σ, τ) will be said to be *optimal* in T_m (or T_{mn}) if $EZ(\sigma, \tau) = v_m$ (or v_{mn}) where

$$v_m = \sup_{(s,t)\varepsilon T_m} EZ_{st}, \qquad v_{mn} = \sup_{(s,t)\varepsilon T_{mn}} EZ_{st}.$$

As a preliminary to the general case, let us tentatively assume that the statistician always stops for the first time after taking exactly m observations. Then the problem of finding an optimal csv (m, τ) in $T_{m,m+1}$ is clearly equivalent to finding an optimal stopping variable (sv) for the sequence $\{Z_{mn}, F_{mn}, n > m\}$, and the "value of the game" is $v_{m,m+1}$. (For general summaries of optimal stopping theory, see [4] and [6], which are based upon the earlier work of Arrow, Blackwell, and Girshick in [1] and Snell in [7]). The following sequences of rv's play a key role in the solution:

$$X_{mn} = \text{ess sup } E_{mn} Z(s, t), \qquad (s, t) \varepsilon T_{mn}, \tag{1}$$

$$X_m = E_m X_{m,m+1}. \tag{2}$$

[Given a family of rv's $\{Y_t, t \varepsilon T\}$, we define ess sup Y_t, $t \varepsilon T$, as a rv X such that (a) $X \geqq Y_t$ a.e. for each t in T, and (b) if $Z \geqq Y_t$ a.e. for each t in T, then $Z \geqq X$ a.e. Such a rv X always exists and can be taken as the supremum of some countable subset of $\{Y_t, t \varepsilon T\}$; thus, the rv X_{mn} in (1) can be taken to be F_{mn}-measurable.]

From results in optimal stopping theory (see [2] and the references cited there), the X_{mn}-sequence satisfies

$$X_{mn} = \max [Z_{mn}, E_{mn} X_{m,n+1}] \quad \text{a.e.} \tag{3}$$

Moreover, if (m, τ_{mn}) is defined by setting τ_{mn} = the first $k \geqq n$ such that $X_{mk} = Z_{mk}$ (or ∞ if no such k exists), then (m, τ_{mn}) is optimal in T_{mn} if $\tau_{mn} < \infty$ a.e., and in this case

$$(4) \qquad X_{mn} = E_{mn}Z(m, \tau_{mn}) \quad \text{a.e.}$$

The condition that $\tau_{mn} < \infty$ a.e. holds whenever an optimal csv in T_{mn} exists and, in particular, when $\lim_n Z_{mn} = -\infty$ a.e. From (2) and (4), if $\tau_{m,m+1} < \infty$ a.e.,

$$(5) \qquad X_m = E_mZ(m, \tau_{m,m+1}) \quad \text{a.e.}$$

Thus, after the first m observations become known, X_m can be interpreted as the statistician's conditional expected gain if he always stops for the first time at this stage and uses an optimal sv for the second stage. [Even if an optimal csv in $T_{m,m+1}$ does not exist, there is a sequence $\{(m, t_k), k \geqq 1\}$ in $T_{m,m+1}$ such that $E_mZ(m, t_k) \uparrow X_m$ a.e. as $k \to \infty$.]

Now consider the application of optimal stopping theory to the sequence $\{X_m, F_m, m \geqq 1\}$. It follows easily from Hypothesis A that $E(\sup X_m) < \infty$. For each $m \geqq 1$, define

$$(6) \qquad V_m = \operatorname{ess\,sup} E_mX_s, \qquad s \in S_m,$$

where S_m denotes the class of sv's s relative to $\{F_m, m \geqq 1\}$ such that $s \geqq m$. Then, as above,

$$(7) \qquad V_m = \max [X_m, E_mV_{m+1}] \text{ a.e.}, \qquad EV_m = \sup EX_s, \qquad s \in S_m.$$

Moreover, if an optimal sv in S_m exists for $\{X_m, F_m, m \geqq 1\}$ then σ_m = the first $k \geqq m$ such that $X_k = V_k$ (or ∞ if no such k exists) is an optimal sv in S_m and

$$(8) \qquad V_m = E_mX(\sigma_m) \quad \text{a.e.}$$

If an optimal sv does not exist in S_m, the V_m-sequence can be used to construct an ϵ-good sv by setting s_m = the first $k \geqq m$ such that $X_k \geqq V_k - \epsilon$ (or ∞ if no such k exists); then s_m is a sv and

$$(9) \qquad E_mX(s_m) \geqq V_m - \epsilon \quad \text{a.e.}$$

THEOREM 1. *Let σ = the first $m \geqq 1$ such that $X_m = V_m$ (or ∞ if no such m exists). On $\{\sigma = m\}$, let τ = the first $n > m$ such that $Z_{mn} = X_{mn}$ (or ∞ if no such n exists); on $\{\sigma = \infty\}$, set $\tau = \infty$. If (σ, τ) is finite-valued* a.e., *then* (a) *(σ, τ) is an optimal* csv, i.e., $EZ(\sigma, \tau) = v_1$; *and* (b) $V_1 = E_1Z(\sigma, \tau)$ a.e.

PROOF. It suffices to show that $V_1 \geqq E_1Z_{st}$ a.e. for any csv (s, t) with equality holding for $(s, t) = (\sigma, \tau)$. For any csv (s, t), define the sequence of sv's $\{t_m, m \geqq 1\}$ by $t_m = I_mt + (m + 1)(1 - I_m)$ where I_m is the indicator function of $\{s = m\}$. Then $(m, t_m) \in T_{m,m+1}$ so that by (1), (2), and (4), $I_mX_m \geqq I_mE_mZ(m, t_m)$ a.e., and there is equality here if $(s, t) = (\sigma, \tau)$ because, on $\{\sigma = m\}$, τ_m coincides with the sv $\tau_{m,m+1}$ of (5). Next note that if U and U_m are defined as in Hypothesis A, then $X_m \leqq U_m$ a.e. for each m; also, $E_1U_s = U_1 = E_1U$

a.e. for any sv s relative to $\{F_m, m \geqq 1\}$. Thus,

$$\begin{aligned} E_1 X_s &= E_1 U_s - E_1 \textstyle\sum_{m=1}^{\infty} I_m (U_m - X_m) \\ &\geqq E_1 U - E_1 \textstyle\sum_{m=1}^{\infty} I_m (E_m U - E_m Z(m, t_m)) \\ &= E_1 U - \textstyle\sum_{m=1}^{\infty} E_1 E_m I_m (U - Z(m, t_m)) \\ &= E_1 U - E_1 \textstyle\sum_{m=1}^{\infty} I_m (U - Z(m, t_m)) \\ &= E_1 Z_{st} \quad \text{a.e.,} \end{aligned}$$

with equality holding for $(s, t) = (\sigma, \tau)$. Since $V_1 \geqq E_1 X_s$ a.e. with equality for $s = \sigma$ by (6) and (8), this completes the proof.

Clearly, the pair (σ, τ) defined above cannot be finite-valued a.e. if an optimal csv does not exist. Unfortunately, one can also construct examples in which an optimal csv does exist, but (σ, τ) is still not finite-valued a.e. Sufficient conditions for (σ, τ) to be finite-valued a.e. will be given later.

THEOREM 2. *Given $\epsilon > 0$, let s = the first $m \geqq 1$ such that $X_m \geqq V_m - \epsilon/2$ (or ∞ if no such m exists). On $\{s = m\}$, let t = the first $n > m$ such that $Z_{mn} \geqq X_{mn} - \epsilon/2$ (or ∞ if no such n exists); on $\{s = \infty\}$, set $t = \infty$. Then* (a) *(s,t) is a* csv; (b) *$E_1 Z_{st} \geqq V_1 - \epsilon$ a.e.; and* (c) *$EZ_{st} \geqq EV_1 - \epsilon$.*

PROOF. (a) The finiteness of (s, t) follows from the corresponding result in optimal stopping theory (see Theorem 3.6 of [7]).

(b) By (9), $E_1 X_s \geqq V_1 - \epsilon/2$ a.e. Similarly, if τ_m = the first $n > m$ such that $Z_{mn} \geqq X_{mn} - \epsilon/2$, then τ_m is a sv for $\{Z_{mn}, F_{mn}, n > m\}$ and $E_{m+1} Z(m, \tau_m) \geqq X_{m,m+1} - \epsilon/2$ a.e. By (2), this implies that $E_m Z(m, \tau_m) \geqq X_m - \epsilon/2$ a.e. Now define $t_m = I_m t + (m + 1)(1 - I_m)$ for each $m \geqq 1$ where I_m denotes the indicator function of $\{s = m\}$; then t_m coincides with τ_m on $\{s = m\}$, and by the same type of proof as in Theorem 1 it follows that $E_1 Z_{st} \geqq E_1 X_s - \epsilon/2 \geqq V_1 - \epsilon$ a.e.

(c) This follows by taking expectations in (b).

THEOREM 3. *Let $\{V_m, m \geqq 1\}$ be defined as in* (6). *Then*

(a) $V_m = \text{ess sup } E_m Z_{st}$, $(s, t) \ \varepsilon \ T_m$;

(b) $EV_m = v_m$;

(c) *if (σ, τ) is any optimal* csv *in T_m, then $V_m = E_m Z(\sigma, \tau)$* a.e.

PROOF. (a) It was shown in the proof of Theorem 1 that $V_1 \geqq E_1 Z_{st}$ a.e. for any csv (s, t). This combines with Theorem 2(b) to prove (a) for the case $m = 1$, and the proof for arbitrary m follows immediately.

(b) Since $E_1 Z_{st} \leqq V_1$ a.e. for any csv (s, t), $v_1 = \sup EZ_{st} \leqq EV_1$. The opposite inequality follows from Theorem 2(c).

(c) By (a), $V_m \geqq E_m Z(\sigma, \tau)$ a.e. Contrary to the assertion, suppose there is an $\epsilon > 0$ such that $E_m Z(\sigma, \tau) \leqq V_m - \epsilon$ on a set A in F_m for which $P(A) > 0$. By a proof analogous to that given in Theorem 2(b), there is a csv (s, t) in T_m such that $E_m Z(s, t) \geqq V_m - \epsilon/2$ a.e. Now consider the csv (s^*, t^*) which coincides with (s, t) on A and with (σ, τ) on A^c. Then $(s^*, t^*) \ \varepsilon \ T_m$ and it is easily seen that $EZ(s^*, t^*) > EZ(\sigma, \tau)$, thus yielding a contradiction.

If optimal csv's exist in each of the classes T_m (as in the "truncated case" below or under the conditions of Theorem 4 below), then by (7) and Theorem 3(c), $V_m = \max [X_m, E_m Z(\sigma_m, \tau_m)]$ a.e. for each $m \geqq 1$ where (σ_m, τ_m) is any optimal csv in T_{m+1}. Thus, stopping at the first m such that $X_m = V_m$ (as in Theorem 1) is equivalent to stopping at the first m such that $X_m \geqq E_m Z(\sigma_m, \tau_m)$.

Now suppose that the collection $\{Z_{mn}, m \geqq 1, n > m\}$ is finite, say of the form $\{Z_{mn}, 1 \leqq m \leqq M, m < n \leqq N(m)\}$ where $M, N(1), \cdots, N(M)$ are positive integers. The above theory clearly applies to this case with only minor notational changes. As in the corresponding optimal stopping theory for the truncated case (see [6]), the sequences $\{X_{mn}\}$ and $\{V_m\}$ can be determined (at least in theory) by backward induction using the following relations:

(10) $$X_{m,N(m)} = Z_{m,N(m)} \quad \text{for} \quad 1 \leqq m \leqq M;$$

(11) $$X_{mn} = \max [Z_{mn}, E_{mn} X_{m,n+1}] \quad \text{for} \quad m < n < N(m), \quad 1 \leqq m \leqq M;$$

(12) $$V_M = X_M;$$

(13) $$V_m = \max [X_m, E_m V_{m+1}] \quad \text{for} \quad 1 \leqq m < M.$$

Examples illustrating the calculations necessary to carry out a solution using these relations will be given in Section 3.

The csv (σ, τ) of Theorem 1 is clearly finite-valued, and therefore optimal, in the truncated case above. For sufficient conditions in the non-truncated case, the following theorem appears useful, especially for possible statistical applications in which $Z_{mn} = -(r_{mn} + c_m + d_n)$ where $c_m \uparrow \infty$ as $m \to \infty$, $d_n \uparrow \infty$ as $n \to \infty$, and $r_{mn} \geqq K$ for all m and n.

THEOREM 4. *Suppose that*

(a) $\lim_n Z_{mn} = -\infty$ a.e. *for each* $m \geqq 1$, *and*

(b) $\lim_m \sup_n Z_{mn} = -\infty$ a.e.

Then an optimal csv (σ, τ) *exists and can be defined as in Theorem* 1.

PROOF. It suffices to show that the pair (σ, τ) of Theorem 1 satisfies: (i) $\sigma < \infty$ a.e., and (ii) $\tau < \infty$ a.e. on $\{\sigma = m\}$ for each $m \geqq 1$. Condition (ii) follows from (a) and Snell's condition for the existence of an optimal sv (see Corollary 3.1 of [6]). Similarly, (i) will hold if we can show that $\lim_m X_m = -\infty$ a.e. Set

$$Y_m = \sup_{k \geqq m} \sup_n Z_{kn}.$$

Then $Y_m \downarrow -\infty$ a.e. as $m \to \infty$ by (b), and, since $X_m \leqq E_m Y_m$ a.e. by (1) and (2), it suffices to show that $\lim_m E_m Y_m = -\infty$ a.e. From Hypothesis A we have that $E(\sup Z_{mn}^+) < \infty$ so that $EY_m^+ < \infty$ for each $m \geqq 1$. Thus, since $E_n Y_m \geqq E_n Y_n$ whenever $n > m$, it follows from Theorem 2.4 of [7] that for any $m \geqq 1$

(14) $$E_\infty Y_m \geqq \lim_n E_n Y_m \geqq \limsup_{n\to\infty} E_n Y_n \text{ a.e.}$$

where E_∞ denotes the conditional expectation operator relative to the σ-field generated by $\bigcup_{n=1}^{\infty} F_n$. The result now follows by taking the limit in (14) as $m \to \infty$.

3. Examples. a. *An investment problem.* Let $Y_1, Y_2, \cdots, Y_N$ be independent, each uniformly distributed on [0, 1]. These rv's can be interpreted as prices of a commodity that a statistician observes sequentially. He must make two stops, buying at the first stop and selling at the second; thus, if his stops are at stages m and $n(m < n \leqq N)$, his gain is $Z_{mn} = Y_n - Y_m$.

Let F_m and F_{mn} be the σ-fields generated by $Y^{(m)}$ and $Y^{(n)}$ respectively where $Y^{(k)} = (Y_1, \cdots, Y_k)$. Then choosing a procedure to maximize the expected gain amounts to choosing an optimal csv. By the independence of the Y_i's, equations (10)–(13) become

$$X_{mn} = \max(Y_n - Y_m, \alpha_n - Y_m) = \max(Y_n, \alpha_n) - Y_m$$

where $\alpha_N = 0$, and $\alpha_{n-1} = E\max(Y_n, \alpha_n) = (1 + \alpha_n^2)/2$ for $n \leqq N$;

$$V_m = \max(X_m, \beta_m) = \max(\alpha_m - Y_m, \beta_m) \quad \text{for} \quad m < N$$

where $\beta_{N-1} = \alpha_{N-1} - 1 = -\frac{1}{2}$, and for $m < N$

$$\beta_{m-1} = E\max(\alpha_m - Y_m, \beta_m) = \beta_m + \tfrac{1}{2}(\alpha_m - \beta_m)^2.$$

It follows from these relations and Theorem 1 that an optimal procedure is to stop at the first m such that $\alpha_m - Y_m \geqq \beta_m$ (or $Y_m \leqq \alpha_m - \beta_m$) and thereafter at the first n such that $Y_n \geqq \alpha_n$. The expected gain using this procedure is $\beta_0 = \beta_1 + \frac{1}{2}(\alpha_1 - \beta_1)^2$. A short table of the α's and β's is in Table 1.

TABLE 1

k	α_{N-K}	β_{N-K}	k	α_{N-K}	β_{N-K}
1	.5000	−.5000	7	.8203	.5287
2	.6250	.0000	8	.8364	.5712
3	.6953	.1953	9	.8498	.6064
4	.7417	.3203	10	.8611	.6360
5	.7751	.4091	11	.8707	.6613
6	.8004	.4761	12	.8791	.6833

b. *Dowry problem with two choices.* This problem was originally posed by Mosteller and Gilbert in [5]; their solution used a heuristic argument that can now be made precise by appealing to the results in Section 2. Let $(w_1, w_2, \cdots, w_N)$ denote a random permutation of the integers $1, 2, \cdots, N$ where all $N!$ permutations are equally likely. Let Y_i denote the relative rank of w_i among $(w_1, w_2, \cdots, w_i)$; i.e., $Y_i = 1+$ (number of terms $w_1, \cdots, w_{i-1}$ less than w_i). Then $Y_1, \cdots, Y_N$ are independent, and $P(Y_i = j) = 1/i$ for $j = 1, \cdots, i$. A statistician observes the relative ranks $Y_1, \cdots, Y_N$ sequentially and is permitted two stops. If his stops are after stages m and n, he wins one unit if either $w_m = 1$ or $w_n = 1$. Finding a stopping procedure to maximize his probability of winning amounts to finding an optimal csv given that F_m and F_{mn} for $m < n \leqq N$ are the σ-fields generated by $Y^{(m)}$ and $Y^{(n)}$ respectively, and

$$Z_{mn} = P(w_m = 1 \mid Y^{(n)}) + P(w_n = 1 \mid Y^{(n)}) = Q_{mn} + R_n, \quad \text{say.}$$

Here, $R_n = n/N$ or 0 according as $Y_n = 1$ or >1. By the independence of the Y_i's, equations (10) and (11) become

$$X_{mn} = Q_{mn} + \max(R_n, \alpha_n)$$

where $\alpha_N = 0$, and for $n \leqq N$

$$(15) \quad \begin{aligned} \alpha_{n-1} = E\max(R_n, \alpha_n) &= 1/N + (n-1)\alpha_n/n \quad \text{if} \quad n/N \geqq \alpha_n \\ &= \alpha_n \quad \text{if} \quad n/N < \alpha_n . \end{aligned}$$

Solving this difference equation for the α_n's gives $\alpha_n = nL_n/N$ for $s - 1 \leqq n < N$ and $\alpha_n = \alpha_{s-1}$ for $n < s - 1$, where $L_n = \sum_{i=n}^{N-1} 1/i$ and s is the largest integer for which $L_s \leqq 1$. By Theorem 1, on $\{\sigma = m\}$, τ is the first $n > m$ such that $R_n \geqq \alpha_n$. Note that $R_n \geqq \alpha_n$ if and only if $n \geqq s$ and $Y_n = 1$. Since $X_m = E(X_{m,m+1} \mid Y^{(m)}) = R_m + \alpha_m$ for $m < N$, equations (12) and (13) become

$$V_m = \max(R_m + \alpha_m, \beta_m) \quad \text{for} \quad m < N$$

where $\beta_{N-1} = \alpha_{N-1}$, and for $1 \leqq m < N$

$$\begin{aligned} \beta_{m-1} &= E\max(R_m + \alpha_m, \beta_m) \\ &= (m/N + \alpha_m)/m + \beta_m(m-1)/m \quad \text{if} \quad m/N + \alpha_m \geqq \beta_m \\ &= \beta_m \quad \text{otherwise.} \end{aligned}$$

By Theorem 1, σ is the first $m \geqq r$ such that $Y_m = 1$, where r is the first integer for which $r/N + \alpha_r \geqq \beta_r$. (The sequence $m/N + \alpha_m$ is increasing with m by (15), whereas β_m decreases with m.) The pair (r, s), called "starting numbers" in [5], completely characterize the optimal procedure. To find r, Mosteller and Gilbert compute the probability of winning for all such procedures that use a pair of starting numbers (q, s) where $q < s$; then r is the minimizing value of q. Table 3 in [5] gives these pairs (r, s) [denoted by (r^*, s^*) there] for many values of N.

c. *The burglar problem.* Let $\theta_1, \theta_2, \cdots$ be independent, positive rv's having a common distribution with finite second moment. Here θ_m represents the mean yield of burglaries executed successfully in the mth city on a burglar's list. By traveling from city to city, the burglar can observe the rv's $\theta_1, \theta_2, \cdots$ sequentially at a constant cost c per observation. If he stops after observing θ_m, he must restrict his burgling to the mth city in which case the successive yields from burglaries in that city are rv's $Y_{m1}, Y_{m2}, \cdots$. These rv's are assumed to be conditionally independent given θ_m, each having conditional cdf $pI + qH(\theta_m)$ where $0 < p < 1$, $q = 1 - p$, I is the cdf of a degenerate distribution at 0, and $H(\theta)$ is the cdf for a negative exponential distribution with mean θ. We shall say that the burglar "gets caught" while observing Y_{mj} if $Y_{mj} = 0$. Thus, there is a constant probability p of getting caught on each burglary. If he stops before getting caught, he gets to keep all the yields from the jobs performed successfully. If he gets caught, he loses all his earlier gains.

To formulate this as a compound stopping problem, let F_m and F_{mn} be the σ-fields generated by $(\theta_1, \cdots, \theta_m)$ and $(\theta_1, \cdots, \theta_m, Y_{m1}, \cdots, Y_{m,n-m})$ respectively, and set

$$Z_{mn} = \sum_{i=1}^{n-m} Y_{mi} - cm \quad \text{if} \quad \prod_{i=1}^{n-m} Y_{mi} \neq 0,$$
$$= -cm \qquad \text{otherwise.}$$

To assure that Hypothesis A is satisfied, we now suppose that $Y_{mk} = \theta_m Y_k'$ where $Y_1', Y_2', \cdots$ are mutually independent (and independent of the θ_i's), each having cdf $pI + qH(1)$. Then if N is the first integer n such that $Y_n' = 0$, $Z_{mn} \leqq \theta_m S_N - cm$ where $S_n = Y_1' + \cdots + Y_n'$; hence, $U = \sup Z_{mn}^+ \leqq Q$ where $Q = \sup(\theta_m S_N - cm)^+$. However, $EQ < \infty$ by Corollary 2 of [3], because $E\theta_i^2 S_N^2 = (E\theta_i^2)(ES_N^2) < \infty$. Therefore,

$$U_m = E_m U \leqq \sup_{i \leqq m} (\theta_i S_N - ci)^+ + E \sup_{i > m} (\theta_i S_N - ci)^+ \leqq Q + EQ$$

so that $E(\sup U_m) \leqq 2EQ < \infty$

For fixed m, the sequence $\{Z_{mn}, F_{mn}, n > m\}$ satisfies the conditions of the "monotone case" considered in [3]. To see this, set $B_n = \{E(Z_{m,n+1} \mid F_{mn}) \leqq Z_{mn}\}$ for $n > m$; then it follows that $B_n = \{Z_{mn} = -cm \text{ or } Z_{mn} \geqq b_m - cm\}$ where $b_m = E_m Y_{m1}/p = q\theta_m/p$, so that $B_{m+1} \subset B_{m+2} \subset \cdots$. By Theorem 1 of [3], an optimal csv $\tau_{m,m+1}$ in $T_{m,m+1}$ is given by stopping the first time that B_n occurs or, equivalently, the first time that $Y_{m,n-m} = 0$ or $\sum_{i=1}^{n-m} Y_i \geqq b_m$. By (5), $X_m = E_m Z(m, \tau_{m,m+1})$ a.e., because, as is easily seen, any two optimal csv's in $T_{m,m+1}$ yield the same X_m.

Computing the X_m's entails solving the following random walk problem. Let $Y_1, Y_2, \cdots$ be independent, each having cdf $pI + qH(\theta)$, and let N be the first k such that $Y_k = 0$ or $u + \sum_1^k Y_i \geqq b$ where $b > 0, u < b$. We want to find $f(0)$ if $f(u) = ES(u)$ where $S(u) = u + \sum_1^N Y_i$ if $Y_N > 0$, $S(u) = 0$ if $Y_N = 0$. It is easily seen that f satisfies the functional equation

$$(16) \qquad f(u) = q\left[\int_0^{b-u} f(u+y)h_\theta(y)\,dy + \int_{b-u}^\infty (u+y)h_\theta(y)\,dy\right]$$

where $h_\theta(y) = (1/\theta)e^{-y/\theta}$. After a change of variable on the right in (16) to $y + u$, one can differentiate both sides to obtain $f'(u) = pf(u)/\theta$, so that $f(u) = Ce^{pu/\theta}$. Since $f(b-) = q(b+\theta)$ by (16), it follows that $f(u) = q(b+\theta)e^{p(u-\theta)/\theta}$ so that, if $b = q\theta/p$, $f(0) = q\theta/pe^q$.

Applying this result to the computation of the X_m-sequence gives us that $X_m = r\theta_m - cm$ for $m \geqq 1$ where $r = q/pe^q$. Let G denote the common cdf of $r\theta_1, r\theta_2, \cdots$, and let α be defined by $\int (t-\alpha)^+ dG(t) = c$. Then (see [3]), an optimal sv for $\{X_m, F_m, m \geqq 1\}$ is given by $\sigma =$ the first m such that $r\theta_m \geqq \alpha$, and $EX_\sigma = \alpha$. Thus, an optimal csv is given by (σ, τ) where τ is defined on $\{\sigma = m\}$ as the first n such that $Y_{m,n-m} = 0$ or $\sum_{i=1}^{n-m} Y_{m,i} \geqq q\theta_m/p$. Moreover, $EZ(\sigma, \tau) = \alpha$. The computation of α becomes particularly simple if the common distribution of the θ_i's is negative exponential with mean λ and $r\lambda \geqq c$; in this case, $\alpha = r\lambda \ln(r\lambda/c)$.

REFERENCES

[1] Arrow, K. J., Blackwell, D. and Girshick, M. A. (1949). Bayes and minimax solutions of sequential decision problems. *Econometrics* **17** 213–244.

[2] CHERNOFF, HERMAN. A note on risk and maximal regular generalized submartingales in stopping problems. *Ann. Math. Statist.* **38** 606–607.

[3] CHOW, Y. S. and ROBBINS, HERBERT (1961). A martingale system theorem and applications. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 93–104. Univ. of California Press.

[4] CHOW, Y. S. and ROBBINS, HERBERT. On values associated with a stochastic sequence. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 427–440. Univ of California Press.

[5] GILBERT, JOHN and MOSTELLER, FREDERICK (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* **61** 35–73.

[6] HAGGSTROM, GUS W. (1966). Optimal stopping and experimental design. *Ann. Math. Statist.* **37** 7–29.

[7] SNELL, J. L. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.* **73** 293–312.