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A POTENTIAL THEORY FOR SUPERMARTINGALES[1]

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0. Summary. A potential theory for supermartingales is presented below. It is much like the classical Newtonian potential theory and is a generalization of the potential theory for transient Markov chains. While dealing with stochastic processes with more general dependence relations the new theory retains what we believe to be the important features of the transient Markov chain theory.

Briefly, a *pure potential* is a non-negative supermartingale $\{Z_n, F_n\}$ which satisfies the condition

$$E[Z_{n+k} \mid F_n] \to 0 \quad \text{a.e. as} \quad k \to \infty$$

for every n. The potential principles of domination, Riesz decomposition, lower envelope, balayage, equilibrium and minimum are proved for these potentials. It is shown how the corresponding results of the transient Markov chain theory can be derived from the new theory. Also, some applications to standard martingale theory are given.

1. Preliminaries. Consider a probability space (Ω, B, P). All stochastic processes and random variables appearing throughout this paper are defined on this space. In addition, all random variables are to be real valued and only discrete parameter processes are considered. The expectation of a random variable X is denoted by $E[X]$. If F is a subfield of B then the conditional expectation of X with respect to F is denoted by $E[X \mid F]$. The conditional probability of a subset A of Ω with respect to F is denoted similarly by $P[A \mid F]$. In the case when $F = B(X_\alpha : \alpha \varepsilon \Lambda)$, i.e. F is the Borel field generated by the random variables X_α, the previous notations are frequently written $E[X \mid X_\alpha, \alpha \varepsilon \Lambda]$ and $P[A \mid X_\alpha, \alpha \varepsilon \Lambda]$. If A and B are sets, then their intersection will be denoted by AB, the complement of A by A^c and the indicator function of A by $I(A)$.

A stochastic process is a collection $\mathbf{X} = \{X_t, F_t, t \varepsilon \Lambda\}$ where X_t is F_t-measurable for each t and $F_s \subseteq F_t \subseteq B$ for all $s \leqq t$. As we will be dealing entirely with discrete parameter processes where the index set Λ consists of the positive integers we will omit specific mention of the set Λ below. Recall that a *martingale* is a stochastic process $\mathbf{X} = \{X_n, F_n\}$ satisfying the conditions

$$M1: E[|X_n|] < \infty \qquad \text{for all} \quad n,$$

$$M2: X_m = E[X_n \mid F_m] \qquad \text{a.e. for all} \quad m < n.$$

The process $\mathbf{X}$ is a *supermartingale* if it satisfies $M1$ and the supermartingale

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inequality

$$SM: X_m \geqq E[X_n \mid F_m] \qquad \text{a.e. for all} \quad n > m.$$

Our study of potential theory will involve repeatedly certain conditional expectations. We will frequently want to derive inequalities between conditional expectations. Due to the definition of conditional expectation via the Radon-Nikodym theorem conditional expectations are random variables defined only as one of a certain set of almost everywhere equal random variables. In order to simplify the notation and illustrate more clearly the problems at hand we make the following:

CONVENTION. A random variable will henceforth mean the equivalence class of random variables which are equal a.e. with respect to the measure P.

Under the convention we can speak of *the* conditional expectation of a random variable with respect to a subfield of B. It is convenient to define the operator $\mathbf{T}$ on processes as follows: Let $\mathbf{X} = \{X_n, F_n\}$ be a stochastic process such that $E[|X_n|] < +\infty$ and define $\mathbf{TX}$ by

$$\mathbf{TX} = \{E[X_{n+1} \mid F_n], F_n\}.$$

The natural extensions of this notation will be used, e.g., $\mathbf{T}^2\mathbf{X} = \mathbf{T}(\mathbf{TX})$, etc. Thus, if X_n is the nth random variable of the process $\mathbf{X}$ then

$$T^k X_n = (T^k X)_n = E[X_{n+k} \mid F_n].$$

The identity operator on processes is denoted $\mathbf{I}$ and $\mathbf{T}^0 = \mathbf{I}$.

Statements made relating two processes mean that the statement holds for the corresponding random variables (class of rv under the convention), e.g., conditions $M1$ and $M2$ in the definition of a martingale could be written

$$M1: E[|\mathbf{X}|] < \infty,$$

$$M2: \mathbf{X} = \mathbf{TX}.$$

Further information on martingale theory can be found in Doob [2], Loève [9], and Meyer [10].

2. Definition of a potential. In this section we introduce the notion of a potential process and a charge process. A simple necessary and sufficient condition for a process to be a potential is given. The reader familiar with the potential theory of transient Markov chains will at once observe similarities.

DEFINITION 1. The stochastic process $\mathbf{Z}$ is a *potential process* and the stochastic process $\mathbf{Y}$ is its *charge process* if the following conditions are satisfied:

$$D1: \mathbf{T}^k\mathbf{Z} \to 0 \qquad \text{as} \quad k \to \infty,$$

$$D2: (\mathbf{I} - \mathbf{T})\mathbf{Z} = \mathbf{Y},$$

$$D3: E[|\mathbf{Z}|] < \infty.$$

A *pure potential process* is a potential process whose charge process is non-negative.

To justify calling $\mathbf{Y}$ the charge of $\mathbf{Z}$ it is necessary to show that $\mathbf{Z}$ can be recovered from $\mathbf{Y}$. This fact is contained in

PROPOSITION 1. *The process* $\mathbf{Z}$ *is a potential and* $\mathbf{Y}$ *its charge iff*

$$P1: \mathbf{Z} = \sum_{k=0}^{\infty} (\mathbf{T}^k\mathbf{Y}),$$
$$P2: \mathbf{T}\sum_{k=0}^{\infty} \mathbf{T}^k\mathbf{Y} = \sum_{k=1}^{\infty} \mathbf{T}^k\mathbf{Y},$$
$$P3: E[|\mathbf{Z}|] < \infty.$$

PROOF. First, let us assume that $\mathbf{Z}$ is a potential process with charge process $\mathbf{Y}$. Then

$$\sum_{i=0}^{k} \mathbf{T}^i\mathbf{Y} = \sum_{i=0}^{k} \mathbf{T}^i[(\mathbf{I} - \mathbf{T})\mathbf{Z}] = \mathbf{Z} - \mathbf{T}^{k+1}\mathbf{Z} \to \mathbf{Z}$$

as $k \to \infty$. Furthermore,

$$\sum_{k=1}^{\infty} \mathbf{T}^k\mathbf{Y} = \mathbf{Z} - \mathbf{Y} = \mathbf{T}\mathbf{Z} = \mathbf{T}\sum_{k=0}^{\infty} \mathbf{T}^k\mathbf{Y}.$$

Now assume that $\mathbf{Z}$ and $\mathbf{Y}$ satisfy conditions $P1$–$P3$. Then

$$(\mathbf{I} - \mathbf{T})\mathbf{Z} = \sum_{k=0}^{\infty} (\mathbf{T}^k\mathbf{Y}) - \mathbf{T}\sum_{k=0}^{\infty} (\mathbf{T}^k\mathbf{Y}) = \mathbf{Y}.$$

Furthermore,

$$\mathbf{T}^i\mathbf{Z} = \mathbf{T}^i\sum_{k=0}^{\infty} (\mathbf{T}^k\mathbf{Y}) = \sum_{k=i}^{\infty} \mathbf{T}^k\mathbf{Y} \to 0. \qquad \square$$

COROLLARY. *If* $\mathbf{Z} = \sum_{k=0}^{\infty} \mathbf{T}^k\mathbf{Y}$, $E[\mathbf{Z}] < \infty$ *and* $\mathbf{Y} \geqq 0$, *then* $\mathbf{Z}$ *is a pure potential with charge* $\mathbf{Y}$.

COROLLARY. *The stochastic process* $\mathbf{Z}$ *is a pure potential* iff $\mathbf{Z}$ *is a nonnegative supermartingale with* $\mathbf{T}^k\mathbf{Z} \to 0$.

We remark that for a pure potential $\mathbf{Z}$ condition $D1$ is equivalent with the generally stronger condition $\lim E[Z_n] = 0$.

3. Potential principles. In this section we derive the basic potential principles which hold for our theory. In order to proceed we must define what it means for a process to dominate another process. To this end we state

DEFINITION 2. A *family* is a sequence $\mathbf{E} = \{E_n\}_{n=1,2,\cdots}$, of subsets E_n of Ω such that for each n, E_n is F_n-measurable.

Note that under our convention a family is in fact a sequence of equivalence classes of sets. Thus $\mathbf{E} = \mathbf{F}$ means that for every n the symmetric difference, $[E_n \cap F_n^c] \cup [E_n^c \cap F_n]$, has measure zero with respect to P.

DEFINITION 3. The process $\mathbf{W}$ is *dominated on the family* $\mathbf{E}$ by the process $\mathbf{X}$ if

$$X_n \geqq W_n \text{ on } E_n$$

for every n. $\mathbf{W}$ is *dominated* by $\mathbf{X}$ if it is dominated by $\mathbf{X}$ on the family $\mathbf{\Omega} \doteq \{E_n\}$ where $E_n = \Omega$ for every n.

A weak domination principle is given in

PROPOSITION 2. *A non-negative supermartingale dominated by a potential is a pure potential.*

PROOF. Let $\mathbf{Z}$ be a potential dominating the non-negative supermartingale $\mathbf{W}$. Then $0 \leqq \mathbf{W} \leqq \mathbf{Z}$ and hence $0 \leqq \mathbf{T}^k\mathbf{W} \leqq \mathbf{T}^k\mathbf{Z}$. However, since $\mathbf{Z}$ is a pure potential

$$\mathbf{T}^k\mathbf{Z} \to 0 \quad \text{as} \quad k \to \infty.$$

It follows from the supermartingale inequality that $\mathbf{W}$ has non-negative charge. $\square$

Using this result we find

THEOREM 1 (Principle of the lower envelope). *The infimum of two pure potentials is a pure potential.*

PROOF. Let $\mathbf{Z}^1$ and $\mathbf{Z}^2$ be pure potentials. Define the process $\mathbf{Z}$ by

$$Z_n = \inf(Z_n^{\ 1}, Z_n^{\ 2}).$$

Observe that $\mathbf{TZ} \leqq \mathbf{TZ}^i \leqq \mathbf{Z}^i$ for $i = 1, 2$. This implies the supermartingale inequality $\mathbf{TZ} \leqq \mathbf{Z}$. From Proposition 2 it follows that $\mathbf{Z}$ is a pure potential. $\square$

Observe that under our convention the infimum appearing above is the infimum of two equivalence classes of random variables, i.e., the essential infimum of the random variables involved. The theorem extends immediately to any collection of pure potentials.

Just as the supremum of two superharmonic functions is not in general superharmonic, it is not generally the case that the supremum of two supermartingales is a supermartingale. However, in analogy with the classical situation we have

PROPOSITION 3. *The supremum of an increasing sequence of non-negative supermartingales which are uniformly bounded by a process with finite expectation is a supermartingale.*

PROOF. Let $\{\mathbf{X}^k\}_{k=1,2,\cdots}$, be such a sequence of supermartingales. The supremum of the sequence is the process $\mathbf{X}$ defined by $X_n = \sup_k (X_n^{\ k})$. From the assumption on boundedness it follows that $E[|X_n|] < \infty$. To verify the supermartingale inequality for $\mathbf{X}$ observe that

$$T \sup_k (X_n^{\ k}) = T \lim_k X_n^{\ k} \leqq \lim_k TX_n^{\ k} \leqq \lim_k X_n^{\ k} = \sup_k (X_n^{\ k}). \quad \square$$

Corresponding to the classical Riesz decomposition of a non-negative superharmonic function into a potential part and a harmonic part, we have (see also [1], [10])

THEOREM 2. (Riesz decomposition). *Every non-negative supermartingale can be uniquely represented as the sum of a pure potential and a non-negative martingale.*

PROOF. Let $\mathbf{W}$ be a non-negative supermartingale. Consider the processes $\mathbf{X}$ and $\mathbf{Z}$ defined by

$$X_n = \lim_{k\to\infty} T^k W_n,$$
$$Z_n = W_n - \lim_{k\to\infty} T^k W_n = \sum_{k=0}^{\infty} T^k(W_n - TW_n).$$

Observe that $W_n = X_n + Z_n$.

By virtue of monotone convergence the non-negative process $\mathbf{X}$ is seen to satisfy the martingale inequality

$$TX_n = T \lim_{k\to\infty} T^k W_n = \lim_{k\to\infty} T^{k+1} W_n = X_n .$$

Noting that $\mathbf{Z}$ is a pure potential with charge process $\{W_n - TW_n\}$ we obtain the desired representation.

Suppose now that there were two distinct decompositions of $\mathbf{W}$, say

$$\mathbf{W} = \mathbf{X} + \mathbf{Z} = \mathbf{X}' + \mathbf{Z}',$$

where $\mathbf{X}$ and $\mathbf{X}'$ are non-negative martingales and $\mathbf{Z}$ and $\mathbf{Z}'$ are pure potentials. Applying the operator $\mathbf{T}^k$ and allowing $k \to \infty$ we see

$$\begin{array}{ccc} \mathbf{T}^k\mathbf{X} + \mathbf{T}^k\mathbf{Z} & = & \mathbf{T}^k\mathbf{X}' + \mathbf{T}^k\mathbf{Z}' \\ \downarrow \quad\quad & & \quad\quad \downarrow \\ \mathbf{X} + 0 & & \mathbf{X}' + 0 \end{array}$$

thus proving uniqueness. □

Appealing to the fundamental martingale convergence theorem $\lim W_n = W_\infty$ exists and is uniformly integrable. Hence $X_n = E[W_\infty \mid F_n]$. If the random variable W_∞ is thought of as being, "at the boundary" then we have a representation of martingales in terms of their "boundary values."

THEOREM 3. (Minimum principle). *A supermartingale dominating the negative of a pure potential is non-negative.*

PROOF. Let $\mathbf{S}$ be a supermartingale and $\mathbf{Z}$ a pure potential such that $\mathbf{S} \geqq -\mathbf{Z}$. Then $\mathbf{S} \geqq \mathbf{T}^k\mathbf{S} \geqq \mathbf{T}^k(-\mathbf{Z}) \to 0$ as $k \to \infty$. □

COROLLARY. *The supremum of all martingales dominated by a pure potential is zero.*

In the analogy with the classical terminology the martingale part of the Riesz decomposition would be called the greatest martingale minorant of the supermartingale.

The next concept to be considered is that of the balayage potential. The balayage operator, defined below, will lead to the principle of domination and the existence of equilibrium potentials.

DEFINITION 4. Let $\mathbf{E}$ be a family and $\mathbf{X}$ a non-negative process. The *balayage operator* $\mathbf{B}^E$ takes the process $\mathbf{X}$ into the process $\mathbf{B}^E\mathbf{X} = \{B^E X_n , F_n\}$ where

$$B^E X_n = E[X_n I(E_n) + X_{n+1} I(E_n^{\,c} E_{n+1}) + X_{n+2} I(E_n^{\,c} E_{n+1}^c E_{n+2}) + \cdots \mid F_n].$$

PROPOSITION 4. *Let* $\mathbf{X}$ *be a non-negative supermartingale and* $\mathbf{E}$ *a family. Then* $\mathbf{B}^E\mathbf{X}$ *is a non-negative supermartingale and* $\mathbf{X} \geqq \mathbf{B}^E\mathbf{X}$.

PROOF. We prove the inequality first. Consider

$$\begin{aligned} X_n &= E[X_n \mid F_n] \\ &= E[X_n I(E_n) + X_n I(E_n^{\,c}) \mid F_n]. \end{aligned}$$

Since $\mathbf{X}$ is a supermartingale and $I(E_n^{\,c})$ is F_n-measurable

$$X_n I(E_n^{\,c}) \geqq E[X_{n+1} I(E_n^{\,c}) \mid F_n].$$

Therefore,

$$(*) \qquad X_n \geqq E[X_n I(E_n) + X_{n+1} I(E_n^c) \mid F_n].$$

Iterating this equation we obtain

$$X_n \geqq E[X_n I(E_n) + X_{n+1} I(E_n^c E_{n+1}) + X_{n+2} I(E_n^c E_{n+1}^c) \mid F_n].$$

Consider the nondecreasing sequence $\{S_k^n\}_{k=1,2,\cdots}$, of F_n-measurable random variables defined by

$$S_{k+1}^n = E[X_n I(E_n) + \cdots + X_{n+k} I(E_n^c \cdots E_{n+k-1}^c E_{n+k}) \mid F_n].$$

Using inequality $(*)$ above, it may be shown that $X_n \geqq S_k^n$ for every k. From the definition of $\mathbf{B}^E\mathbf{X}$ it follows that $S_k^n \to B^E X_n$. Therefore, $X_n \geqq B^E X_n$.

Since $\mathbf{X}$ dominates $\mathbf{B}^E\mathbf{X}$ the expectations $E[B^E X_n]$ are finite. It remains to show that $\mathbf{B}^E\mathbf{X}$ satisfies the supermartingale inequality. Indeed,

$$\begin{aligned} TB^E X_n &= E[B^E X_{n+1} \mid F_n] \\ &= E[B^E X_{n+1} I(E_n) + B^E X_{n+1} I(E_n^c) \mid F_n] \\ &\leqq E[X_{n+1} I(E_n) + B^E X_{n+1} I(E_n^c) \mid F_n] \\ &\leqq E[X_n I(E_n) + B^E X_{n+1} I(E_n^c) \mid F_n] \\ &= B^E X_n . \end{aligned}$$

∎

Corollary. *If* $\mathbf{Z}$ *is a pure potential and* $\mathbf{E}$ *is a family, then* $\mathbf{B}^E\mathbf{Z}$ *is a pure potential and* $\mathbf{Z} \geqq \mathbf{B}^E\mathbf{Z}$.

In order to consider the remaining potential principles it is necessary to have the notion of the support of a potential. This is given in

Definition 5. Let $\mathbf{Z}$ be a potential with charge $\mathbf{Y}$. The *support* of $\mathbf{Z}$ is the family $\mathbf{E}$ defined by

$$E_n = \{Y_n \neq 0\}.$$

Proposition 5. *Let* $\mathbf{E}$ *be a family containing the support of the pure potential* $\mathbf{Z}$. *Then* $\mathbf{Z} = \mathbf{B}^E\mathbf{Z}$.

Proof. Since $\mathbf{Z}$ has support in $\mathbf{E}$

$$Z_n - E[Z_{n+1} \mid F_n] = (I - T)Z_n = 0 \quad \text{on} \quad E_n^c$$

for every n. Hence

$$Z_n I(E_n^c) = I(E_n^c) E[Z_{n+1} \mid F_n] = E[Z_{n+1} I(E_n^c) \mid F_n].$$

In this case inequality $(*)$ of Proposition 4 becomes equality

$$(**) \qquad Z_n = E[Z_n I(E_n) + Z_{n+1} I(E_n^c) \mid F_n].$$

Using $(**)$ we find for every k

$$Z_n = S_{k+1}^n + E[Z_{n+k+1} I(E_n^c \cdots E_{n+k}^c) \mid F_n]$$

where the S_k^n are as defined in Proposition 4. As before, $S_k^n \to B^E Z_n$ as $k \to \infty$.

However, since $\mathbf{Z}$ is a pure potential

$$E[Z_{n+k+1}I(E_n{}^cE_{n+1}^c \cdots E_{n+k}^c) \mid F_n] \to 0 \quad \text{as} \quad k \to \infty$$

and hence

$$S_k{}^n \to Z_n \quad \text{as} \quad k \to \infty . \quad \square$$

These results lead to our next potential principle.

THEOREM 4 (Principle of Domination). *Let* $\mathbf{Z}$ *and* $\mathbf{W}$ *be pure potentials. If* $\mathbf{Z}$ *dominates* $\mathbf{W}$ *on a family* $\mathbf{E}$ *which contains the support of* $\mathbf{W}$ *then* $\mathbf{Z}$ *dominates* $\mathbf{W}$.

PROOF. $\mathbf{Z} \geqq \mathbf{B}^E\mathbf{Z} \geqq \mathbf{B}^E\mathbf{W} = \mathbf{W}$. $\square$

COROLLARY. *Pure potentials equal on a family containing their supports are equal.*

In order to prove our next potential principle we will need some additional facts concerning supermartingales. The two following lemmas are restatements of results of Snell [11]. (Recall the previous remarks regarding the infimum and the essential infimum.)

LEMMA 1. *The infimum of a collection of non-negative supermartingales is a non-negative supermartingale.*

LEMMA 2. *If* $\mathbf{X}$ *is a non-negative process which is dominated by a supermartingale then the infimum,* $\mathbf{Y}$, *of all supermartingales dominating* $\mathbf{X}$ *satisfies the relation*

$$\mathbf{Y} = \max (\mathbf{X}, \mathbf{TY}).$$

The next lemma gives another characterization of the infimum $\mathbf{Y}$ of Lemma 2. It provides a recursive procedure for calculating the infimum and is an extension of a result of Dynkin [6].

LEMMA 3. *Let* $\mathbf{X}$ *be a non-negative process which is dominated by a supermartingale. If* $\mathbf{Y}$ *is the infimum of all supermartingales dominating* $\mathbf{X}$ *then* $\mathbf{Y} = \lim \mathbf{Q}^k\mathbf{X}$ *where* $\mathbf{Q}$ *is the operator defined by* $\mathbf{Q}^1\mathbf{X} = \max (\mathbf{X}, \mathbf{TX})$ *and* $\mathbf{Q}^{k+1} = \mathbf{Q}(\mathbf{Q}^k)$.

PROOF. $\{\mathbf{Q}^k\mathbf{X}\}$ is a monotone non-decreasing sequence of processes which is dominated by a supermartingale. Thus $\lim \mathbf{Q}^k\mathbf{X}$ exists and dominates $\mathbf{X}$. Furthermore,

$$\mathbf{T} \lim \mathbf{Q}^k\mathbf{X} \leqq \lim \mathbf{TQ}^k\mathbf{X} \leqq \lim \mathbf{Q}^k\mathbf{X}$$

so that $\lim \mathbf{Q}^k\mathbf{X}$ is a supermartingale. The process $\mathbf{Y}$ equal to the infimum of all supermartingales dominating $\mathbf{X}$ exists by virtue of Lemma 1 and satisfies the inequality $\mathbf{Y} \leqq \lim \mathbf{Q}^k\mathbf{X}$. The reverse inequality follows from Lemma 2. $\square$

We are now ready to derive our next potential principle.

THEOREM 5 (Principle of Balayage). *Let* $\mathbf{Z}$ *be a pure potential and* $\mathbf{E}$ *a family. Then there is a unique pure potential* $\mathbf{W}$ *with support in* $\mathbf{E}$ *such that*

$$\mathbf{W} \leqq \mathbf{Z}, \qquad \mathbf{W} = \mathbf{Z} \quad \text{on} \quad \mathbf{E}.$$

Also, $\mathbf{W} = \mathbf{B}^E\mathbf{Z}$ = *infimum of all pure potentials which dominate* $\mathbf{Z}$ *on* $\mathbf{E}$ = *infimum of all non-negative supermartingales which dominate* $\mathbf{Z}$ *on* $\mathbf{E}$. *The potential* $\mathbf{W}$ *is called the* balayage potential *of* $\mathbf{Z}$ *on* $\mathbf{E}$.

PROOF. Let $\mathbf{Z}$ be a pure potential and $\mathbf{E}$ a family. The process $\mathbf{X}$ defined by $X_n = I(E_n)Z_n$ is non-negative and is dominated by a supermartingale. By Lemma 1 the process $\mathbf{W}$ defined as the infimum of all supermartingales dominating $\mathbf{X}$ exists and is a supermartingale. By the Principle of the lower envelope $\mathbf{W}$ is a pure potential.

Clearly, $\mathbf{Z}$ dominates $\mathbf{W}$ and $\mathbf{Z} = \mathbf{W}$ on $\mathbf{E}$.

We will now prove that the support of $\mathbf{W}$ is contained in $\mathbf{E}$. By Lemma 2, for every n

$$\begin{aligned} W_n &= \max\{I(E_n)Z_n\,,\, E[W_{n+1}\mid F_n]\} \\ &= I(E_n)Z_n + I(E_n^c)E[W_{n+1}\mid F_n] \\ &= E[I(E_n)Z_n \mid F_n] + E[I(E_n^c)W_{n+1}\mid F_n]. \end{aligned}$$

Iterating this equation to eliminate W_{n+1} we obtain

$$W_n = E[I(E_n)Z_n + I(E_n^c E_{n+1})Z_{n+1}\mid F_n] + E[I(E_n^c E_{n+1}^c)W_{n+2}\mid F_n].$$

After k such iterations we obtain

$$W_n = E[I(E_n)Z_n + \cdots + I(E_n^c \cdots E_{n+k-1}^c E_{n+k})Z_{n+k}\mid F_n] + E[I(E_n^c \cdots E_{n+k}^c)W_{n+k+1}\mid F_n].$$

As k increases the first term on the right increases to $B^E Z_n$ while the second term, being dominated by $T^{k+1}W_n$, decreases to 0. Thus $\mathbf{W} = \mathbf{B}^E\mathbf{Z}$. It follows that the support of $\mathbf{W}$ is contained in $\mathbf{E}$.

Proposition 2 yields the third representation of $\mathbf{W}$ as the infimum of pure potentials dominating $\mathbf{Z}$ on $\mathbf{E}$.

Uniqueness is a consequence of the corollary to the principle of domination. □

The last potential principle we shall discuss concerns the existence of an equilibrium potential. The equilibrium potential is a balayage potential where certain restrictions are placed on the families $\mathbf{E}$ considered. The appropriate condition to insure the existence of an equilibrium potential is given in

DEFINITION 6. An *equilibrium family* is a family $\mathbf{E}$ satisfying the condition

$$P[E_k \cup E_{k+1} \cup E_{k+2} \cup \cdots \mid F_n] \to 0 \quad \text{as} \quad k \to \infty \quad \text{for every} \quad n.$$

The expression $P[E_n \cup E_{n+1} \cup \cdots \mid F_r]$ is a generalized hitting probability and is seen to be equal to $\mathbf{B}^E\mathbf{1}$. Thus $\mathbf{E}$ is an equilibrium family iff $\mathbf{T}^k\mathbf{B}^E\mathbf{1} \to 0$ as $k \to \infty$. From monotone convergence it follows that: $\mathbf{E}$ is an equilibrium family iff

$$P[E_k \cup E_{k+1} \cup E_{k+2} \cup \cdots] \to 0 \quad \text{as} \quad k \to \infty.$$

PROPOSITION 6. *Let* $\mathbf{X}$ *be a non-negative supermartingale. If* $\mathbf{E}$ *is an equilibrium family then* $\mathbf{B}^E\mathbf{X}$, *the balayage of* $\mathbf{X}$ *on* $\mathbf{E}$, *is a pure potential.*

PROOF. Let $\mathbf{X}$ and $\mathbf{E}$ be as above. Then $\mathbf{X}$ can be represented as a sum $\mathbf{X} = \mathbf{Z} + \mathbf{M}$ where $\mathbf{Z}$ is a pure potential and $\mathbf{M}$ is a non-negative martingale.

Applying $\mathbf{B}^E$ we have $\mathbf{B}^E\mathbf{X} = \mathbf{B}^E\mathbf{Z} + \mathbf{B}^E\mathbf{M}$. By the corollary to Proposition 4, $\mathbf{B}^E\mathbf{Z}$ is a potential. The discussion following Theorem 2 yields the representation $M_n = E[X_\infty \mid F_n]$ where $X_\infty = \lim_n X_n$. Therefore,

$$\begin{aligned} B^E M_n &= E[I(E_n)M_n + I(E_n{}^c E_{n+1})M_{n+1} + \cdots \mid F_n] \\ &= E[I(E_n)X_\infty + I(E_n{}^c E_{n+1})X_\infty + \cdots \mid F_n] \end{aligned}$$

which implies

$$E[B^E M_n] = \int_{E_n \cup E_{n+1} \cup \cdots} X_\infty \, dP.$$

However, $\mathbf{E}$ an equilibrium family implies

$$P(E_n \cup E_{n+1} \cup \cdots) \to 0.$$

Hence $E[B^E M_n] \to 0$. Since $\mathbf{B}^E\mathbf{Z}$ is a potential $E[B^E Z_n] \to 0$. Therefore,

$$(***) \qquad E[B^E X_n] \to 0.$$

Hence, since $\mathbf{X}$ is a non-negative supermartingale it follows from Proposition 4 that $\mathbf{B}^E\mathbf{X}$ is also a non-negative supermartingale. This fact combined with condition (***) above implies that $\mathbf{B}^E\mathbf{X}$ is indeed a pure potential (see the remark following the corollaries to Proposition 1). □

It will be noted that the condition given in the preceding proposition is sufficient but not necessary for $\mathbf{B}^E\mathbf{X}$ to be a pure potential. We have already shown that if $\mathbf{X}$ is a pure potential then so is $\mathbf{B}^E\mathbf{X}$.

Definition 7. Let $\mathbf{E}$ be a family. The *equilibrium potential of* $\mathbf{E}$ is a pure potential taking the value one on $\mathbf{E}$ with support in $\mathbf{E}$.

If the equilibrium potential of a given family exists, it is unique by the principle of domination. To see that the equilibrium potential does not always exist, consider the family $\mathbf{E}$ where $E_n = \Omega$ for every n. By the definition of an equilibrium potential, each of the random variables of the potential would have to be equal to one. However, such a sequence of random variables could not have conditional expectations converging to zero as required of potentials.

A necessary and sufficient condition for the existence of an equilibrium potential is given in

Theorem 6 (Principle of the Equilibrium Potential). *The family* $\mathbf{E}$ *supports an equilibrium potential* iff $\mathbf{E}$ *is an equilibrium family.*

Proof. From the equation

$$B^E 1_n = P[E_n \cup E_{n+1} \cup \cdots \mid F_n]$$

it follows that $B^E 1_n$ is a potential iff $\mathbf{E}$ is an equilibrium family.

Observe that $\mathbf{B}^E\mathbf{1}$ takes the value one on $\mathbf{E}$.

To see that $\mathbf{B}^E\mathbf{1}$ has the support in $\mathbf{E}$ consider its charge process $(\mathbf{I} - \mathbf{T})\mathbf{B}^E\mathbf{1}$.

$$\begin{aligned} (I - T)B^E 1_n &= P[E_n \cup E_{n+1} \cup \cdots \mid F_n] - P[E_{n+1} \cup E_{n+2} \cup \cdots \mid F_n] \\ &= P[E_n \cap E_{n+1}^c \cap E_{n+2}^c \cap \cdots \mid F_n] \end{aligned}$$

which is zero off E_n.

Thus $\mathbf{B}^E\mathbf{1}$ is the equilibrium potential of $\mathbf{E}$. ▯

The charge of the equilibrium potential can be naturally interpreted as a generalized escape probability. Indeed, if the fields F_n arise from a sequence $\{U_n\}$ of random variables and $E_n = \{\omega\colon U_n(\omega) \varepsilon A, \omega \varepsilon \Omega\}$ where A is a Borel set of real numbers, then

$$\begin{aligned}(I - T)B^E 1_n &= P[E_n \cap E_{n+1}^c \cap E_{n+2}^c \cap \cdots \mid F_n] \\ &= P[U_n \varepsilon A \quad \text{and} \quad U_k \not\varepsilon A, k > n \mid F_n].\end{aligned}$$

4. Markov potentials. In this section we consider the relation of the potential theory of transient Markov chains to the martingale potential theory described above.

Consider a transient, stationary, denumerable Markov chain $\{X_n\}_{n=1,2,\cdots}$, with transition matrix P and state space S. The potential theory of such a process deals with real valued functions with domain the state space of the chain. The analogue of the classical superharmonic (harmonic) functions are the superregular (regular) functions, i.e. functions $h\colon S \to R^1$ such that $h \geqq Ph$, $(h = Ph)$. (Here we are using matrix notation. The inequality $h \geqq Ph$ means $h(i) \geqq \sum_{j\varepsilon S} P(i, j)h(j)$ for every i in S.) A function $g\colon S \to R^1$ is a potential function with charge function f if $g = \lim_{k\to\infty} [(I + P + \cdots + P^k)f]$ is finite. From these notions a potential theory has been constructed which includes all the potential principles we have given for martingales. For details in this case and in the case of other types of Markov processes, see, for example, Doob [3], Hunt [7], and Kemeny, Snell and Knapp [8].

The potential theory of martingales which we have presented is a natural generalization of the Markov theory. Consider a superregular function h for the chain $\{X_n\}_{n=1,2,\cdots}$, with transition matrix P. The random variables X_n which make up the Markov chain are defined on a probability space (Ω, P). Now $\{h \circ X_n\}$, the chain evaluated by h, is also a stochastic process on (Ω, P). (Note: $(h \circ X_n)(\omega) = h(X_n(\omega))$ for $\omega \varepsilon \Omega$ and $n = 1, 2, \cdots$.) What is more, $\{h \circ X_n\}_{n=1,2,\cdots}$, is a supermartingale. Indeed,

$$\begin{aligned}h \circ X_n &\geqq (Ph) \circ X_n \\ &= E[h \circ X_{n+1} \mid X_n] \\ &= E[h \circ X_{n+1} \mid X_1, \cdots, X_n] \\ &= E[h \circ X_{n+1} \mid h \circ X_1, \cdots, h \circ X_n].\end{aligned}$$

Thus the Markov chain potential theory can be translated into a potential theory of martingales of the form $\{h \circ X_n\}_{n=1,2,\cdots}$. The theory we have given thus generalizes the Markov chain theory to the case of arbitrary martingales.

The general results in the previous section can be used to prove the corresponding results for the Markov chain theory (although it is perhaps simpler to prove them directly). The procedure is to: (1) consider the desired statement in the Markov chain theory; (2) translate it into a statement about martingales of the form $\{h \circ X_n\}$; (3) apply the general theorems of the preceding section; (4) prove that the result is still of the form $\{g \circ X_n\}$; (5) translate the result back to the Markov chain theory. Only in step (4) can difficulty arise.

5. Applications. In this section we use the potential theory of Section 3 to derive some standard inequalities of martingale theory.

The principle of balayage provides a simple proof of the well known downcrossing lemma and a related inequality of Dubins [5]. The proof of the downcrossing lemma is suggested in Doob [4]. Recall that a process $\mathbf{X}$ performs a downcrossing of the interval $[a, b]$ along the path $\{X_n(\omega)\}$ if there are integers m and n such that $m < n$ and $X_m(\omega) \geqq b > a \geqq X_n(\omega)$. The number of downcrossings of $[a, b]$ by the path $\{X_n(\omega)\}$ will be denoted by

$$\#[\text{down}: \ \mathbf{X}, a, b](\omega).$$

It is computed as follows: the first downcrossing begins at the first time j such that $X_j(\omega) \geqq b$ and continues until the first time $k > j$ such that $X_k(\omega) \leqq a$; the second downcrossing begins at the first time $m > k$ such that $X_m(\omega) \geqq b$ and continues until the first time $n > m$ such that $X_n(\omega) \leqq a$; etc. If time j does not exist then $\#$ [down: $\mathbf{X}, a, b] = 0$. Upcrossings are defined similarly.

Dubins' inequalities are stated in

PROPOSITION 7. *Let* $\mathbf{X}$ *be a non-negative supermartingale and* $a < b$ *real numbers. Then for any position integer* k,

(i) $P[\#[\textit{down}: \ \mathbf{X}, a, b] \geqq k] \leqq E[\min(X_1, b)](a/b)^{k-1}/b,$

(ii) $P[\#[\textit{up}: \ \mathbf{X}, a, b] \geqq k] \leqq E[\min(X_1, a)](a/b)^{k-1}/b.$

PROOF. Define families $\mathbf{E}$ and $\mathbf{F}$ by

$$E_n = \{\omega \mid X_n(\omega) \in [0, a]\}$$

$$F_n = \{\omega \mid X_n(\omega) \in [b, +\infty]\}.$$

Then $\mathbf{X} \geqq \mathbf{B}^F\mathbf{X} \geqq b\mathbf{B}^F\mathbf{1}$, and $a\mathbf{B}^E\mathbf{1} \geqq \mathbf{B}^E\mathbf{X} \geqq b\mathbf{B}^E(\mathbf{B}^F\mathbf{1})$. Hence,

$$(a/b)\mathbf{B}^E\mathbf{1} \geqq \mathbf{B}^E(\mathbf{B}^F\mathbf{1}).$$

Balayage these processes on $\mathbf{F}$ and use the fact that $\mathbf{1} \geqq \mathbf{B}^E\mathbf{1}$ to obtain

$$\begin{aligned}(a/b)\mathbf{B}^F(\mathbf{B}^E\mathbf{1}) &\geqq \mathbf{B}^F(\mathbf{B}^E(\mathbf{B}^F\mathbf{1})) \\ &\geqq \mathbf{B}^F(\mathbf{B}^E(\mathbf{B}^F(\mathbf{B}^E\mathbf{1}))).\end{aligned}$$

This equation can be iterated, giving us

$$(a/b)^{k-1}\mathbf{B}^F(\mathbf{B}^E\mathbf{1}) \geqq \mathbf{B}^F(\mathbf{B}^E(\cdots(\mathbf{B}^F(\mathbf{B}^E\mathbf{1}))\cdots)),$$

where the term on the right has k factors of $\mathbf{B}^E$ and $\mathbf{B}^F$.

Now observe that

$$\begin{aligned}B^E 1_1 &= P[E_1 \cup E_2 \cup \cdots \mid F_1] \\ &= P[\mathbf{X} \quad \text{passes below} \quad a \mid F_1].\end{aligned}$$

Then

$$\begin{aligned} B^F(B^E1)_1 &= P[F_1(E_2 \cup E_3 \cup \cdots) \cup F_2(E_3 \cup E_4 \cup \cdots) \cup \cdots \mid F_1] \\ &= P[\mathbf{X} \text{ passes above } b \text{ and then below } a \mid F_1] \\ &= P[\#[\text{down}: \ \mathbf{X}, a, b] \geqq 1 \mid F_1]. \end{aligned}$$

Similarly

$$\begin{aligned} P[\#[\text{down}: \ \mathbf{X}, a, b] \geqq k] &= B^F(B^E(\cdots(B^F(B^F1))\cdots))_1 \qquad (k \text{ pairs } B^FB^E) \\ &\leqq (a/b)^{k-1}B^F(B^E1)_1 \\ &\leqq (a/b)^{k-1} \min(1, X_1/b) \end{aligned}$$

since

$$X_1 \geqq bB^F1_1 \geqq bB^F(B^E1)_1,$$

and

$$1 \geqq B^F1_1 \geqq B^F(B^E1)_1.$$

Taking the expectation of the above inequality yields (i). Inequality (ii) can be proved similarly. □

The standard downcrossing inequality now follows easily.

PROPOSITION 8. *Let* $\mathbf{X}$ *be a non-negative supermartingale. Then*

(i) $E[\#[\mathit{down}: \ \mathbf{X}, a, b] \mid F_1] \leqq \min(b, X_1)/(b-a)$,

(ii) $E[\#[\mathit{up}: \ \mathbf{X}, a, b] \mid F_1] \leqq \min(a, X_1)/(b-a)$.

PROOF. Using the notation and intermediate results of the proof of the preceding proposition we have

$$\begin{aligned} E[\#[\text{down}: \ \mathbf{X}, a, b] \mid F_1] &\leqq \textstyle\sum_{k=1}^{\infty} P[\#[\text{down}: \ \mathbf{X}, a, b] \geqq k \mid F_1] \\ &\leqq \textstyle\sum_{k=1}^{\infty} (a/b)^{k-1} \min(1, X_1/b) \\ &= \min(b, X_1)/(b-a). \end{aligned}$$

Inequality (ii) is obtained similarly. □

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