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ON THE DISTRIBUTION OF SOME STATISTICS USEFUL IN THE ANALYSIS OF JOINTLY STATIONARY TIME SERIES[1]

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1. Introduction and summary. Let $\{X(t),\ t = \cdots -1, 0, 1, \cdots\}$ be a P dimensional zero mean stationary Gaussian time series,

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_P(t) \end{pmatrix}$$

we let $R(\tau) = EX(t)X'(t + \tau)$, where $R(\tau) = \{R_{ij}(\tau), i, j = 1, 2, \cdots P\}$, and $F(\omega) = (2\pi)^{-1} \sum_{\tau=-\infty}^{\infty} e^{-i\omega\tau} R(\tau)$. It is assumed that $\sum_{i,j=1}^{P} \sum_{\tau=-\infty}^{\infty} |\tau|\ |R_{ij}(\tau)| < \infty$, and hence $F(\omega)$ exists and the elements possess bounded derivatives. It is further assumed that $F(\omega)$ is strictly positive definite, all ω. Knowledge of $F(\omega)$ serves to specify the process.

$F(\omega)$, and S, the covariance matrix of $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_P \end{pmatrix}$, a Normal $(0, S)$ random vector are known to enjoy many analogous properties. (See [7].) To cite two examples, the hypothesis that $X_i(s)$ is independent of $X_j(t)$ for $i \neq j = 1, 2, \cdots P$, any s, t, is equivalent to the hypothesis that $F(\omega)$ is diagonal, all ω, while the hypothesis that x_i is independent of x_j, for $i \neq j = 1, 2, \cdots P$ is equivalent to the hypothesis that S is diagonal. The conditional expectation of x_1, given $x_2, \cdots x_P$ is

$$E(x_1 \mid x_2, \cdots x_P) = S_{12}S_{22}^{-1} \begin{pmatrix} x_2 \\ \vdots \\ x_P \end{pmatrix}, S = \left(\begin{array}{c|c} s_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right).$$

The corresponding regression problem for stationary Gaussian time series goes as follows. If

$$E\{X_1(t) \mid X_2(s), \cdots X_P(s), s = \cdots -1, 0, 1, \cdots\} = \sum_{j=2}^{P} \sum_{s=-\infty}^{\infty} b_j(t - s)X_j(s)$$

then $B(\omega)$, defined by

$$B(\omega) = (B_2(\omega), \cdots B_P(\omega)), \qquad B_j(\omega) = \sum_{s=-\infty}^{\infty} b_j(s)e^{i\omega s}$$

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satisfies

$$B(\omega) = F_{12}(\omega)F_{22}^{-1}(\omega), \qquad F(\omega) = \left(\begin{array}{c|c} f_{11}(\omega) & F_{12}(\omega) \\ \hline F_{21}(\omega) & F_{22}(\omega) \end{array}\right).$$

It is interesting to ask how well these and similar analogies carry over to sampling theory and hypothesis testing. Goodman [3] gave a heuristic argument to support the conclusion that $\hat{F}_X(\omega_k)$, a suitably formed estimate of the spectral density matrix $F(\omega_k)$ has the complex Wishart distribution. The question is met here by the following results. Firstly if $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots M$ are estimates of the spectral density matrix, each consisting of averages of $(2n + 1)$ periodograms based on a record of length T, with the ω_l equally spaced and $(2n + 1)M \leqq \frac{1}{2}T$, then it is possible to construct, on the same sample space as $X(t)$, M independent complex Wishart matrices $\hat{F}_{\bar{X}}(\omega_l)$, $l = 1, 2, \cdots M$ such that $\{\hat{F}_X(\omega_l), l = 1, 2, \cdots M\}$ converge simultaneously in mean square to $\{\hat{F}_{\bar{X}}(\omega_l), l = 1, 2, \cdots M\}$, as n, M get large. Secondly, it is legitimate to use the natural analogies from multivariate analysis to test hypotheses about time series. One example is presented, as follows. The likelihood ratio test statistic for testing S diagonal is $|\hat{S}|/\prod_{i=1}^{P} \hat{s}_{ii}$ where $\hat{S} = \{\hat{s}_{ij}\}$ is the sample covariance matrix. The analogous statistic ψ for testing $X_i(s)$, $X_j(t)$ independent, i, j $1 =, 2, \cdots P$ from a record of length T is

$$\psi = \prod_{l=1}^{M} [|\hat{F}_X(\omega_l)|/\prod_{i=1}^{P} \hat{f}_{ii}(\omega_l)]$$

where $\hat{F}_X(\omega_l) = \{\hat{f}_{ij}(\omega_l)\}$ are the sample spectral density matrices as above. Letting

$$\bar{\psi} = \prod_{l=1}^{M} [|\hat{F}_{\bar{X}}(\omega_l)|/\prod_{i=1}^{P} \hat{h}_{ii}(\omega_l)]$$

where $\hat{F}_{\bar{X}}(\omega_l) = \{\hat{h}_{ij}(\omega_l)\}$ are the independent complex Wishart matrices referred to above, we show

$$EC_{n,M} |\log \psi - \log \bar{\psi}| \to 0$$

for large n, M, where $C_{n,M}$ are chosen to make the result non-trival. The method of proof applies to any statistic which is a product over l of sufficiently smooth functions of the entries of $\hat{F}_X(\omega_l)$. Applications to estimation and testing in the regression problem will appear elsewhere [8]. The distribution theory of functions of complex Wishart matrices has been well investigated by a number of authors [3] [5] [6], and hence can be easily applied here to statistics like $\bar{\psi}$.

The results above are shown for $P = 2$, it is clear that the proofs extend to any (fixed) finite P. The proofs proceed as follows, via a theorem which has somewhat more general application. For each T, let X be the $2 \times T$ random matrix

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1(1), \cdots, X_1(T) \\ X_2(1), \cdots, X_2(T) \end{pmatrix}$$

and let the $2T \times 2T$ covariance matrix Σ be given by

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where $\Sigma_{ij} = EX_i'X_j$. $\{\hat{F}_X(\omega_l)\}$, the sample spectral density matrices described above based on a record of length T, are each of the form $\hat{F}_X(\omega_l) = T^{-1}XQX'$ where Q is a $T \times T$ circulant matrix with largest eigenvalue $= T(2n+1)^{-1} \leqq \frac{1}{2}M < < T$. We define circulant matrices $\bar{\Sigma}_{ij}$ which approximate Σ_{ij}, and a random matrix $\bar{X}$ on the sample space of X,

$$\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix} = \begin{pmatrix} \bar{X}_1(1), \cdots, \bar{X}_1(T) \\ \bar{X}_2(1), \cdots, \bar{X}_2(T) \end{pmatrix}$$

with $E\bar{X}_i'\bar{X}_j = \bar{\Sigma}_{ij}$. The $2T$ eigenvalues of the block circulant matrix

$$\bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} \end{pmatrix}$$

will be the $2T$ eigenvalues of the T matrices $\{F(2\pi j/T), j = 1, 2, \cdots T\}$. The distribution of random matrices of the form $T^{-1}\bar{X}Q\bar{X}'$ where Q is any circulant matrix are relatively simple to investigate due to the fact that all circulant matrices commute, and their eigenvalues may be exhibited as simple functions of the elements. Circulant quadratic forms in random vectors with circulant covariance matrices are well known in the literature, (See [1] and references cited there). Let $\hat{F}_{X,Q} = T^{-1}XQX'$ and $\hat{F}_{\bar{X},Q} = T^{-1}\bar{X}Q\bar{X}'$ where Q is now any $T \times T$ (real or complex) quadratic form with largest absolute eigenvalue $\leqq q$. The main Theorem allows the replacement of X by $\bar{X}$ in the analysis, and is, that under the assumptions on $F(\omega)$ and $R(\tau)$, for any T,

$$E \operatorname{tr} (\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})(\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})^{*\prime} \leqq cq^2/T^2 \tag{1.1}$$

where c is a constant depending only on $F(\omega)$ and $R(\tau)$. A lemma, essentially allowing the replacement of $F(\omega)$ by a suitably chosen step-function, together with the application of (1.1) gives the results concerning the $\{\hat{F}_X(\omega_l)\}$ an λ. Since $\hat{R}(\tau)$, the sample (circularized) autocorrelation function is also of the form $T^{-1}XQX'$ with Q circulant we obtain an easy corollary on the distribution of $\{\hat{R}(\tau)\}$.

2. Circulant matrices. In this section we give some lemmas about circulant matrices.

LEMMA 1. *Let Q (circulant, real or complex) be of the form*

$$Q = \begin{pmatrix} q_0 & q_1 & \cdot & \cdot & \cdot & q_{T-1} \\ q_{T-1} & \cdot & \cdot & \cdot & \cdot & \vdots \\ \vdots & & \cdot & \cdot & \cdot & q_1 \\ q_1 & \cdot & \cdot & \cdot & q_{T-1} & q_0 \end{pmatrix}$$

Let $K(\omega) = (2\pi)^{-1} \sum_{\tau=0}^{T-1} q_\tau e^{-i\omega\tau}$, let W be the $T \times T$ unitary matrix with r, sth element $T^{-\frac{1}{2}}e^{2\pi irs/T}$ *and let* D_K *be the* $T \times T$ *diagonal matrix with* r, rth *element* $K(2\pi r/T)$. *Then*

$$Q = 2\pi W D_K W^*.$$

PROOF.

$$\begin{aligned}[2\pi W D_K W^*]_{r,s} &= 2\pi T^{-1} \sum_{\nu=1}^{T} e^{2\pi i((r-s)/T)\nu} K(2\pi\nu/T) \\ &= T^{-1} \sum_{\nu=1}^{T} \sum_{\tau=0}^{T-1} e^{2\pi i((r-s)/T)\nu} e^{-2\pi i\nu\tau/T} q_\tau \\ &= T^{-1} \sum_{\tau=0}^{T-1} q_\tau \sum_{\nu=1}^{T} e^{2\pi i((r-s-\tau)/T)\nu}.\end{aligned}$$

Since

$$\begin{aligned}T^{-1} \sum_{\nu=1}^{T} e^{-2\pi i\nu\tau/T} q_\tau^{(r-s-\tau)} &= 1, \quad \tau = (r-s) + lT, \quad l = 0, \pm 1, \pm 2, \cdots, \\ &= 0, \quad \text{otherwise},\end{aligned}$$

we have

$$\begin{aligned}[2\pi W D_K W^*]_{r,s} &= q_{r-s}, \quad r - s \geqq 0, \\ &= q_{T-|(r-s)|}, \quad r - s < 0.\end{aligned}$$

REMARKS. Q Hermitian $\Rightarrow K(2\pi\nu/T)$ is real for all integers ν, Q real $\Rightarrow K(2\pi\nu/T) = K^*(2\pi(T-\nu)/T)$.

In the sequel we shall call Q the circulant matrix generated by $K(\omega)$.

LEMMA 2. *Let* $R(\tau)$, $\tau = \cdots -1, 0, 1,$ *be a doubly infinite sequence of real numbers with*

$$\sum_{\tau=-\infty}^{\infty} |\tau|\, |R(\tau)| = \theta < \infty,$$

let

$$f(\omega) = (2\pi)^{-1} \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\omega\tau}$$

and for fixed T, *let* D_f *be the* $T \times T$ *diagonal matrix with* r, rth *entry* $f(2\pi r/T)$. *Let* W *be as in Lemma* 1, *and let*

$$\bar{\Sigma} = \{\bar{\sigma}_{\mu\nu}, \mu, \nu = 1, 2, \cdots, T\} = 2\pi W D_f W^*$$

$$\Sigma = \{\sigma_{\mu\nu}, \mu, \nu = 1, 2, \cdots, T\}, \quad \sigma_{\mu\nu} = R(\mu - \nu).$$

For any matrices of the same dimensions, define

$$\varphi(A - B) = \sum_{\mu,\nu} |a_{\mu\nu} - b_{\mu\nu}|, \qquad A = \{a_{\mu\nu}\}, \qquad B = \{b_{\mu\nu}\}.$$

Then

$$\varphi(\Sigma - \bar{\Sigma}) = \sum_{\mu,\nu=1}^{T} |\sigma_{\mu\nu} - \bar{\sigma}_{\mu\nu}| \leqq 2\theta.$$

PROOF. By a calculation similar to that of Lemma 1,

$$\bar{\sigma}_{\mu\nu} = \sum_{l=-\infty}^{\infty} R(\mu - \nu + lT)$$

$$\begin{aligned}\sum_{\mu,\nu=1}^{T} |\sigma_{\mu\nu} - \bar{\sigma}_{\mu\nu}| &= \sum_{\tau=-(T-1)}^{T-1} (T - |\tau|) \left| \sum_{l=-\infty, l\neq 0}^{\infty} R(\tau + lT) \right| \\ &\leqq \sum_{\tau=-(T-1)}^{T-1} (T - |\tau|) \sum_{l=-\infty, l\neq 0}^{\infty} |R(\tau + lT)|\end{aligned}$$

$$\leqq T\{\textstyle\sum_{l=-\infty,l\neq 0,-1,+1}^{\infty}\sum_{\tau=-(T-1)}^{T-1}|R(\tau+lT)|$$
$$+\textstyle\sum_{\tau=0}^{T-1}|R(\tau+T)|+\sum_{\tau=-(T-1)}^{0}|R(\tau-T)|\}$$
$$+\textstyle\sum_{\tau=-(T-1)}^{-1}(T-|\tau|)|R(\tau+T)|$$
$$+\textstyle\sum_{\tau=1}^{T-1}(T-|\tau|)R(\tau-T)$$
$$\leqq 2T\textstyle\sum_{|\tau|\geqq T}|R(\tau)|+\sum_{\tau=-(T-1)}^{T-1}|\tau||R(\tau)|$$
$$\leqq 2\textstyle\sum_{-\infty}^{\infty}|\tau||R(\tau)|=2\theta.$$

3. The joint distribution of circulant forms in jointly circulant normal random vectors. For each $T \geqq 2$ define the $T \times T$ circulant matrices $\bar{\Sigma}_{ij}$ as

$$\bar{\Sigma}_{ij} = 2\pi W D_{ij} W^*, \qquad i, j = 1, 2, \tag{3.1}$$

where W is the unitary matrix defined in Lemma 1 and D_{ij} is the $T \times T$ diagonal matrix with r, rth entry $f_{ij}(2\pi r/T)$. Since $f_{ii}(2\pi r/T) = f_{ii}(2\pi(T-r)/T)$ and $f_{ij}(2\pi r/T) = f_{ij}(2\pi(T - r/T) = f_{ji}^*(2\pi r/T)$, the $2T \times 2T$ matrix $\bar{\Sigma}$ given by

$$\bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} \end{pmatrix} \tag{3.2}$$

is real symmetric. It is readily verified that upper and lower bounds for the eigenvalues of $\bar{\Sigma}$ are given by $\Lambda = \max \Lambda(\omega)$ and $\lambda = \min_\omega \lambda(\omega)$ where $\lambda(\omega)$ and $\Lambda(\omega)$ are the smallest and largest eigenvalues of $F(\omega)$.

LEMMA 3. *Let $(\bar{X}_1, \bar{X}_2)$ be a $2T$ dimensional zero mean Gaussian random vector with covariance matrix $\bar{\Sigma}$ defined by* (3.2). *Let $K(\omega)$ be a real function of ω defined on* $[0, 2\pi]$ *and let Q be the $T \times T$ Hermitian circulant matrix generated by $K(\omega)$. Let $\bar{X}$ be the $2 \times T$ matrix $\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$. Then the random matrix*

$$\hat{F}_{\bar{X}} = T^{-1}\bar{X}Q\bar{X}'$$

is distributed as the random matrix

$$(2\pi/T)\textstyle\sum_{r=1}^{T} z_r z_r^{*\prime} K(2\pi r/T) \tag{3.3}$$

where, for each r, z_r is a 2 dimensional complex normal [3] *random (column) vector with complex covariance*

$$E z_r z_r^{*\prime} = F(2\pi r/T), \qquad r = 1, 2, \cdots, T,$$

with $z_r \equiv z_{T-r}^*$ and z_r and z_s independent for $s \neq r$ or $T - r$.

PROOF.

$$\bar{X}Q\bar{X}' = 2\pi\bar{X}WD_K W^*\bar{X}'.$$

Let Z be the $2 \times T$ dimensional matrix $\bar{X}W$ with rth column z_r, $r = 1, 2, \cdots, T$ and jth row denoted by Z_j, $j = 1, 2$. It is readily verified that $z_r = z_{T-r}^*$. Furthermore,

$$E Z_i^{*\prime} Z_j = E W^* \bar{X}_i' \bar{X}_j W = D_{ij}, \qquad i, j = 1, 2,$$

from which the assertion follows.

Let $T^{-1}Q_{T,n}(\omega_0)$ be the quadratic form corresponding to the average of $2n+1$ periodograms centered at $\omega_0 = 2\pi j_0/T$, and spaced by $2\pi/T$; that is,

$$T^{-1}X_iQ_{T,n}(\omega_0)X_j' = [2\pi(2n+1)T]^{-1} \sum_{j=-n}^{n} \sum_{r=1}^{T} X_i(r)e^{(2\pi i r/T)(j_0+j)} \cdot \sum_{s=1}^{T} X_j(s)e^{(-2\pi i s/T)(j_0+j)},$$

where $X_k(\nu)$ is the νth entry of X_k. $Q_{T,n}(\omega_0)$ is the Hermitian circulant quadratic form generated by $K_{T,n}(\omega-\omega_0)$ where

$$\begin{aligned} K_{T,n}(\omega) &= (2\pi)^{-1}[T/(2n+1)], \qquad |\omega| \leqq 2\pi n/T, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

For each T, let

$$\begin{aligned} \hat{F}_{\bar{X},T,n}(\omega_0) &= T^{-1}\bar{X}Q_{T,n}(\omega_0)\bar{X}' \\ &= (2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} z_r z_r^{*\prime}. \end{aligned} \tag{3.4}$$

LEMMA 4. *Let $\hat{F}(\omega_0)$ be a random variable distributed as $W_c(\underline{F}(\omega_0), 2, 2n+1)$, where $W_c(\underline{F}(\omega_0), 2, 2n+1)$ is the complex Wishart distribution* [3], [6], *with $2n+1$ degrees of freedom, with $\underline{F}(\omega_0)$ defined as*

$$\underline{F}(\omega_0) = \underline{F}_{T,n}(\omega_0) = (2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} F(2\pi r/T),\ \omega_0 = 2\pi j_0/T.$$

Let λ, Λ defined above satisfy $0 < \lambda \leqq \Lambda < \infty$, and suppose

$$\sum_{i,j=1}^{2} \sum_{\tau=-\infty}^{\infty} |\tau|\,|R_{ij}(\tau)| = \theta < \infty.$$

Then it is possible to construct an $\hat{F}(\omega_0)$ on the same sample space as $\hat{F}_{\bar{X},T,n}(\omega_0)$ so that

$$\begin{aligned} &E\ \mathrm{tr}(\hat{F}_{\bar{X},T,n}(\omega_0) - \hat{F}(\omega_0))(\hat{F}_{\bar{X},T,n}(\omega_0) - \hat{F}(\omega_0))^{*\prime} \\ &\qquad \leqq 6(\Lambda/\lambda)\theta^2(2n+1)T^{-2}. \end{aligned} \tag{3.5}$$

PROOF. From Lemma 3 and (3.4)

$$\begin{aligned} \hat{F}_{\bar{X},T,n}(\omega_0) &= (2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} z_r z_r^{*\prime} \\ &\sim (2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} F^{\frac{1}{2}}(2\pi r/T)\xi_r\xi_r^{*\prime}F^{\frac{1}{2}}(2\pi r/T) \end{aligned}$$

where the $\{\xi_r = F^{-\frac{1}{2}}(2\pi r/T)z_r\}$ are independent complex normal random vectors with complex covariance $E\,\xi_r\xi_r^{*\prime} = I_{2\times 2}$. (Here, as in the sequel, for A Hermitian or symmetric, $A^{\frac{1}{2}}$ is the Hermitian or symmetric square root). Hence

$$\hat{F}_{\bar{X},T,n}(\omega_0) \sim \hat{F}(\omega_0) + (2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} G_r$$

where

$$\hat{F}(\omega_0) = (2n+1)^{-1}\underline{F}^{\frac{1}{2}}(\omega_0)\Big(\sum_{r=j_0-n}^{j_0+n} \xi_r\xi_r^{*\prime}\Big)\underline{F}^{\frac{1}{2}}(\omega_0) \sim W_c(\underline{F}(\omega_0), 2, 2n+1) \tag{3.6}$$

and

$$G_r = F^{\frac{1}{2}}(2\pi r/T)\xi_r\xi_r^{*\prime}F^{\frac{1}{2}}(2\pi r/T) - \underline{F}^{\frac{1}{2}}(\omega_0)\xi_r\xi_r^{*\prime}\underline{F}^{\frac{1}{2}}(\omega_0).$$

Using the facts that

$$E \text{ tr } G_r G_s^{*\prime} = \text{tr } (F(2\pi r/T) - \underline{F}(\omega_0))(F(2\pi r/T) - \underline{F}(\omega_0)),\ r \neq s,$$

and

$$(2n+1)^{-2} \text{ tr } \textstyle\sum_{r=j_0-n}^{j_0+n} \sum_{s=j_0-n}^{j_0+n} \cdot (F(2\pi r/T) - \underline{F}(\omega_0))(F(2\pi s/T) - \underline{F}(\omega_0))^{*\prime} = 0,$$

$$(2n+1)^{-2} \text{ tr } \textstyle\sum_{r=j_0-n}^{j_0+n} (F(2\pi r/T) - \underline{F}(\omega_0))(F(2\pi r/T) - \underline{F}(\omega_0))^{*\prime} \geqq 0$$

gives the inequality

$$E \text{ tr } ((2n+1)^{-1} \textstyle\sum_{r=j_0-n}^{j_0+n} G_r)((2n+1)^{-1} \sum_{r=j_0-n}^{j_0+n} G_r)^{*\prime} \leqq (2n+1)^{-2} \sum_{r=j_0-n}^{j_0+n} E \text{ tr } (G_r G_r^{*\prime}). \tag{3.7}$$

Observe now that

$$\begin{aligned}\varphi(F(2\pi r/T) - \underline{F}(\omega_0)) &\leqq \max_{2\pi(j_0-n)/T \leqq \omega \leqq 2\pi(j_0+n)/T} \textstyle\sum_{i,j=1}^{2} |f_{ij}(2\pi r/T) - f_{ij}(\omega)| \\ &\leqq \theta(2n+1)T^{-1},\end{aligned}$$

since, for $|\omega - (2\pi r/T)| \leqq 2\pi(2n+1)T^{-1}$,

$$\begin{aligned}|f_{ij}(2\pi r/T) - f_{ij}(\omega_0)| &\leqq (2\pi)^{-1} \textstyle\sum_{\tau=-\infty}^{\infty} |R_{ij}(\tau)|\ |1 - e^{((2\pi i r/T)-\omega)\tau}| \\ &\leqq (2n+1)T^{-1} \textstyle\sum_{\tau=-\infty}^{\infty} |\tau|\ |R_{ij}(\tau)|.\end{aligned}$$

Lemma A.4 of the appendix then yields the inequality

$$\begin{aligned}\text{tr}(G_r G_r^{*\prime}) &\leqq [\text{max eigenvalue } \xi_r \xi_r^{*\prime}]^2 (\Lambda/\lambda)((2n+1)/T)^2\theta^2 \\ &\leqq (\|\xi_r\|^2)^2(\Lambda/\lambda)((2n+1)/T)^2\theta^2.\end{aligned} \tag{3.8}$$

Observing that $E\|\xi_r\|^4 = 6$, and putting together (3.7) and (3.8) gives the result.

4. Asymptotic behavior of quadratic forms in stationary gaussian processes. We first prove a lemma which puts a bound on the mean square difference between a general quadratic form in a normal vector Y and the same quadratic form in $\bar{Y}$, a normal random vector approximating Y.

Lemma 5. *Let $Y \sim \mathfrak{N}(0, S)$, let A be any (real or complex) quadratic form with q^2 the largest eigenvalue of $AA^{*\prime}$. Let $\bar{S}$ be a symmetric positive definite matrix of the same dimensions as S, let $\varphi(S - \bar{S}) \leqq \theta$, and let Λ and λ be common upper and lower bounds for the eigenvalues of S and $\bar{S}$, $0 < \lambda < \Lambda < \infty$. Let $\bar{Y} = YS^{-\frac{1}{2}}\bar{S}^{\frac{1}{2}}$. Then*

$$E|YAY' - \bar{Y}A\bar{Y}'|^2 \leqq (1 + 2(\Lambda/\lambda))q^2\theta^2. \tag{4.1}$$

Proof.

$$|YAY' - \bar{Y}A\bar{Y}'|^2 = |Y(A - S^{-\frac{1}{2}}\bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}}S^{-\frac{1}{2}})Y'|^2.$$

Letting $H = A - S^{-\frac{1}{2}}\bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}}S^{-\frac{1}{2}}$, we have, using the relations between 4th order mixed moments of Gaussian random variables

$$(4.2)\qquad E|YHY'|^2 = (\operatorname{tr} HS)^2 + 2 \operatorname{tr} HSH'S,$$

$$(4.3)\qquad (\operatorname{tr}\mathrm{HS})^2 = (\operatorname{tr} A(S - \bar{S}))^2 \leqq [\text{largest absolute entry of } A \times \varphi(S - \bar{S})]^2 \leqq q^2\theta^2,$$

$$(4.4)\qquad \operatorname{tr} HSH'S = \operatorname{tr}(A - S^{-\frac{1}{2}}\bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}}S^{-\frac{1}{2}})S(A - S^{-\frac{1}{2}}\bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}}S^{-\frac{1}{2}})'S = \operatorname{tr}(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})'.$$

Using Lemma A.4 of the appendix on the right-hand side of (4.4) yields the inequality

$$(4.5)\qquad \operatorname{tr} HSH'S \leqq (\Lambda/\lambda)q^2\theta^2.$$

Combining (4.2), (4.3) and (4.5) gives the result.

THEOREM 1. *Let $X(t)$, $t = \cdots -1, 0, 1, \cdots$ be a two dimensional stationary zero mean Gaussian stochastic process possessing a spectral density matrix $F(\omega)$. Suppose:*

(1) $0 < \lambda \leqq$ *eigenvalues of* $F(\omega) \leqq \Lambda < \infty$;

(2) $\sum_{i,j=1}^{2} \sum_{\tau=-\infty}^{\infty} |\tau|\,|R_{ij}(\tau)| = \theta < \infty$.

For each T, let X be defined as in Section 1. Let $\bar{X}_1$ and $\bar{X}_2$ be T-dimensional zero mean Gaussian (row) vectors with $E\bar{X}_i'\bar{X}_j = \bar{\Sigma}_{ij}$, $i, j = 1, 2$, defined in (3.1), let $\bar{X} = \begin{pmatrix}\bar{X}_1\\ \bar{X}_2\end{pmatrix}$ and let Q be any $T \times T$ quadratic form with largest absolute eigenvalue $\leqq q$.

Let

$$\hat{F}_{X,Q} = T^{-1}XQX'$$

and

$$\hat{F}_{\bar{X},Q} = T^{-1}\bar{X}Q\bar{X}'.$$

Then for each T, it is possible to construct an $\bar{X}$ independent of Q on the sample space of X such that

$$(4.6)\qquad E\operatorname{tr}(\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})(\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})^{*\prime} \leqq 4(1 + 2(\Lambda/\lambda))(q^2/T^2)(2\theta)^2.$$

PROOF. Let

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix},$$

where the Σ_{ij} are as in Section 1. It follows, by application of an argument directly analogous to [4], p. 64, that the eigenvalues of Σ are bounded above and below by Λ and λ.

Let the $2T$ dimensional vector $(\bar{X}_1, \bar{X}_2)$ be given by

$$(\bar{X}_1, \bar{X}_2) = (X_1, X_2)\Sigma^{-\frac{1}{2}}\bar{\Sigma}^{\frac{1}{2}},$$

then $E\bar{X}_i'\bar{X}_j = \bar{\Sigma}_{ij}$. Applying Lemma 2 to each of the blocks $\Sigma_{ij} - \bar{\Sigma}_{ij}$ of $\Sigma - \bar{\Sigma}$ gives

$$\varphi(\Sigma - \bar{\Sigma}) \leqq 2 \sum_{i,j=1}^{2} \sum_{\tau=-\infty}^{\infty} |\tau|\, |R_{ij}(\tau)| = 2\theta.$$

Now

$$E \operatorname{tr} (\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})(\hat{F}_{X,Q} - \hat{F}_{\bar{X},Q})^{*\prime}$$
$$= T^{-2}E \sum_{i,j=1}^{2} |(X_1, X_2)Q_{ij}(X_1, X_2)' - (\bar{X}_1, \bar{X}_2)Q_{ij}(\bar{X}_1, \bar{X}_2)'|^2$$

where Q_{ij} is the $2T \times 2T$ matrix with Q in the i, jth block (of dimension $T \times T$) and zeroes elsewhere; with the largest eigenvalue of $Q_{ij}Q_{ij}^{*\prime} \leqq q^2$. Applying Lemma 5 with $Y = (X_1, X_2)$, $S = \Sigma$ and $\bar{S} = \bar{\Sigma}$ gives the result.

An application of this theorem will allow us to show that M sample spectral density matrices, calculated at M appropriately spaced frequencies converge jointly in mean square to M independent complex Wishart matrices, even though M be large. Let $\hat{F}_{X,T,n}(\omega_l)$, $= T^{-1}XQ_{T,n}(\omega_l)X'$, $l = 1, 2, \cdots M$ be the sample spectral density matrices formed from the average of $2n + 1$ periodograms centered at $\omega_l = (2\pi j_l T^{-1})$, where the j_l are chosen so that the M sets of integers $\{j_l + j, j = 0, \pm 1, \pm 2, \cdots \pm n\}$, $l = 1, 2, \cdots M$, are disjoint, and $0 < 2\pi(j_l - n)/T < 2\pi(j_l + n)/T \leqq \pi$. We have necessarily $(2n + 1)M \leqq \frac{1}{2}T$. It will be convenient to define $\bar{\bar{X}}$, a $2 \times T$ random matrix by $\bar{Z} = \bar{\bar{X}}W$, where the rth column $\bar{z}_r$ of $\bar{Z}$ is given by

$$\begin{aligned} &\bar{z}_r = \underline{F}^{\frac{1}{2}}(\omega_l)F^{-\frac{1}{2}}(2\pi r/T)z_r, && j_l - n \leqq r \leqq j_l + n, \quad \omega_l = (2\pi j_l T^{-1}) \\ (4.7)\quad &\bar{z}_s = \bar{z}_r^{*}, && s = T - r \\ &\bar{z}_r = z_r, && \text{otherwise} \end{aligned}$$

with $\underline{F}(\omega_l) = (2n + 1)^{-1} \sum_{r=j_l-n}^{j_l+n} F(2\pi r/T)$, and z_r defined from $\bar{X}$ as in the proof of Lemma 3. Then $\hat{F}_{\bar{\bar{X}},T,n}(\omega_l) = T^{-1}\bar{\bar{X}}Q_{T,n}(\omega_l)\bar{\bar{X}}'$ for each $l = 1, 2, \cdots M$ are constructed exactly as the $\hat{F}(\omega_0)$ of (3.6) and, by Lemmas 3 and 4 are a set of independent $W_c(\underline{F}(\omega_l), 2, 2n + 1)$ random matrices.

We have the following

Corollary 1. *Under the conditions of the theorem,* $\{\hat{F}_{X,T,n}(\omega_l), l = 1, 2, \cdots M\}$, *where* ω_l *are chosen as above, jointly converge in mean square to* M *independent* $W_c(\underline{F}(\omega_l), 2, 2n + 1)$ *matrices, as* $n, M, T \to \infty$, *provided only that* $\log_2 M \leqq n$. *More precisely, let* $\{\hat{F}_{\bar{\bar{X}},T,n}(\omega_l)\}$ *be the* M *independent complex Wishart matrices defined above. Then*

$$(4.8)\quad E \sum_{l=1}^{M} \operatorname{tr} (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))^{*\prime}$$
$$\leqq (12(\Lambda/\lambda)\theta^2)(2n + 1)MT^{-2} + 32(1 + 2(\Lambda/\lambda)\theta^2(\log_2 M/(2n + 1)^2) \to 0.$$

Proof. Using Lemma **A.1, the le**ft hand side of (4.8) is less than

$$(4.9)\quad (2 \sum_{l=1}^{M} E \operatorname{tr} (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))^{*\prime})$$
$$+ (2 \sum_{l=1}^{M} E \operatorname{tr} (\hat{F}_{\bar{X},T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))(\hat{F}_{\bar{X},T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))^{*\prime}).$$

By Lemma 4, the second term in (4.9) is bounded by $12(\Lambda/\lambda)\theta^2(2n+1)MT^{-2}$. To bound the first term, let $Q = Q_{T,n}(s_1, s_2, \cdots s_M) = \sum_{l=1}^M s_l Q_{T,n}(\omega_l)$. Then

$$(4.10) \quad \sum_{l=1}^M s_l(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l)) = T^{-1}(XQX' - \bar{X}Q\bar{X}').$$

Since, by the choice of ω_l, $\{(2n+1)T^{-1}Q_{T,n}(\omega_l),\ l = 1, 2, \cdots M\}$ are a family of orthogonal projections, the largest eigenvalue of QQ^* is equal to $\max_l |s_l|^2(T/(2n+1))^2$. Applying Theorem 1 to (4.10) then gives

$$(4.11) \quad \sum_{l=1}^M \sum_{k=1}^M s_l s_k E \operatorname{tr} (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_k) - \hat{F}_{\bar{X},T,n}(\omega_k)) \leqq 4(1 + (2\Lambda/\lambda)(2\theta)^2(2n+1)^{-2} \max_l |s_l|^2.$$

Now let A be the (non-negative definite) $M \times M$ matrix with lkth element a_{lk} given by $a_{lk} = E \operatorname{tr} (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_k) - \hat{F}_{\bar{X},T,n}(\omega_k))^{*\prime}$. Using Lemma A.5, we see that (4.11) implies that

$$\operatorname{tr} A = \sum_{l=1}^M E \operatorname{tr} (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))^{*\prime} \leqq 4(1 + (2\Lambda/\lambda))(2\theta)^2 \log_2 M(2n+1)^{-2}.$$

Let $\{\hat{R}_X(\tau) = T^{-1}XU_\tau X',\ \tau = 0, 1, 2, \cdots L < T\}$ be the sample circularized autocorrelation matrices, where U_τ is the symmetric circulant matrix with $\frac{1}{2}$ down the τth and $T - \tau$th diagonal and zeroes elsewhere on and above diagonal (with $U_0 = I$). As is well known (and obvious from Lemma 3), $\{\hat{R}_{\bar{X}}(\tau) = T^{-1}\bar{X}U_\tau\bar{X}',\ \tau = 0, 1, 2, \cdots L\}$ are jointly distributed as the L random matrices

$$T^{-1} \sum_{r=1}^T \cos(2\pi r\tau/T) z_r z_r^{*\prime}, \qquad \tau = 1, 2, \cdots L,$$

where $z_r = z_{T-r}^*$, z_r and z_s are independent, for $s \neq r$ or $T - r$, complex normal vectors with $Ez_r z_r^{*\prime} = F(2\pi r/T)$. We have

Corollary 2.

$$E \operatorname{tr} (\hat{R}_X(\tau) - \hat{R}_{\bar{X}}(\tau))(\hat{R}_X(\tau) - \hat{R}_{\bar{X}}(\tau))' \leqq 4(1 + 2(\Lambda/\lambda))(2\theta)^2T^{-2}, \qquad \tau = 0, 1, 2, \cdots L.$$

5. Applications to hypothesis testing for stationary time series. Suppose we observed a $2 \times T$ random matrix distributed as $\bar{\bar{X}}$. Recall that $\bar{\bar{X}}$ is an approximation to the random matrix X, which is a record of length T of the process $X(t)$ where the approximation proceeded in two steps, first by circularizing X to get $\bar{X}$, and then by replacing $F(\omega)$ by $\underline{F}(\omega) = \underline{F}(\omega_l)$ for $2\pi(j_l - n)T^{-1} \leqq \omega \leqq 2\pi(j_l + n)T^{-1}$. (Here we let the union of the intervals $\{[(2\pi j_l - n)T^{-1}, (2\pi j_l + n)T^{-1}]\}$ cover the points $\{2\pi jT^{-1},\ j = 1, 2, \cdots [\frac{1}{2}T]\}$.) The hypothesis that the 2 rows of $\bar{\bar{X}}$ are independent is equivalent to the hypothesis that $\underline{F}(\omega)$ is diagonal, all ω, which represents an approximation to the hypothesis that the two time series $X_1(t)$ and $X_2(s)$ are independent, i.e. $|f_{12}(\omega)|^2 = 0$, all ω. Considering $\bar{\bar{X}}$; the likelihood ratio statistic for the hypothesis $\underline{F}(\omega) =$ diagonal, all ω, may be readily gotten by examining the well known results [2] in the analogous situation of testing for the diagonality of the covariance matrix of a (real) normal random vector. It is

$$\bar{\bar{\psi}} = [\prod_{l=1}^{M} (1 - \hat{U}(\omega_l))]^{1/M}$$

where

$$\hat{U}(\omega_l) = |\hat{h}_{12}(\omega_l)|^2/(\hat{h}_{11}(\omega_l)\hat{h}_{22}(\omega_l)), \qquad \{\hat{h}_{ij}(\omega_l)\}_{i,j=1,2} = \hat{F}_{\bar{\bar{X}},T,n}(\omega_l).$$

This suggests using ψ to test the hypothesis of independence of $X_1(t)$ and $X_2(s)$, where

$$\psi = [\prod_{l=1}^{M} (1 - \hat{W}(\omega_l))]^{1/M},$$

$$\hat{W}(\omega_l) = |\hat{f}_{12}(\omega_l)|^2/(\hat{f}_{11}(\omega_l)\hat{f}_{22}(\omega_l)), \qquad \{f_{ij}(\omega_l)\}_{i,j=1,2} = \hat{F}_{X,T,n}(\omega_l).$$

Under the null hypothesis, $\hat{U}(\omega_k)$, $k = 1, 2, \cdots M$ are distributed as M independent $\beta_{1,2n}$ random variables, hence Var $-2nM^{\frac{1}{2}} \log \bar{\bar{\psi}} = 1$, and under the alternative, for large M, n, var $-$ $(nM)^{\frac{1}{2}} \log \bar{\bar{\psi}} \rightarrow$ constant. (See [3] for the density of $\hat{U}(\omega_k)$.)

We have the following

COROLLARY 3. *As n, M, $T \rightarrow \infty$, in such a way that* $(\log_2 M)/n \rightarrow 0$

$$E|(nM)^{\frac{1}{2}} (\log \bar{\bar{\psi}} - \log \psi)| \rightarrow 0.$$

If $|f_{12}(\omega)| \equiv 0$, *then*

$$E|nM^{\frac{1}{2}} (\log \bar{\bar{\psi}} - \log \psi)| \rightarrow 0.$$

PROOF. Using the fact that $|\log (1 - \mu) - \log (1 - \nu)| \leqq |\mu - \nu||(1 - \mu)^{-1} + (1 - \nu)^{-1}|$ for $0 \leqq \mu, \nu < 1$, and rewriting $\hat{U}(\omega_l) - \hat{W}(\omega_l)$ in terms of the entries of $\hat{F}_{X,T,n}(\omega_l)$, $\hat{F}_{\bar{\bar{X}},T,n}(\omega_l)$ and $(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))$, we may obtain

$$\begin{aligned} &E(nM)^{\frac{1}{2}} |\log \bar{\bar{\psi}} - \log \psi| \\ &\leqq (n/M)^{\frac{1}{2}} \textstyle\sum_{l=1}^{M} E\, |\log (1 - \hat{U}(\omega_l)) - \log (1 - \hat{W}(\omega_l)| \\ &\leqq (n/M)^{\frac{1}{2}} \textstyle\sum_{l=1}^{M} E[g(\omega_l)(\sum_{i,j=1}^{2} |\hat{h}_{ij}(\omega_l) - \hat{f}_{ij}(\omega_l)|)] \qquad (5.1)\\ &\leqq \text{const } (n/M)^{\frac{1}{2}} \textstyle\sum_{l=1}^{M} [Eg^2(\omega_l)]^{\frac{1}{2}} \\ &\qquad \cdot [E \text{ tr } (\hat{F}_{X,T,n}(\omega_l) - F_{\bar{\bar{X}},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))^{*\prime}]^{\frac{1}{2}} \end{aligned}$$

where $Eg^2(\omega_l)$ is bounded by a constant depending only on $F(\omega)$. Observe that an inequality of the type (5.1) applies to a fairly general class of ψ's formed from products of the functions of $\hat{F}_{X,T,n}(\omega_l)$. This results in

$$\begin{aligned} E(nM)^{\frac{1}{2}} |\log \bar{\bar{\psi}} - \log \psi| \leqq \text{const } n^{\frac{1}{2}} [&\textstyle\sum_{l=1}^{M} E \text{ tr } (\hat{F}_{X,T,n}(\omega_l) \\ &- \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{\bar{X}},T,n}(\omega_l))^{*\prime}]^{\frac{1}{2}} \end{aligned}$$

which, by Corollary 1 is less than const $n^{\frac{1}{2}} (\log_2 M/(2n + 1)^2 + M(2n + 1)/T^2)^{\frac{1}{2}}$ $\rightarrow 0$ as $M, n \rightarrow \infty$ in such a way that $\log_2 M/n \rightarrow 0$.

When the null hypothesis is true, $|\hat{h}_{12}(\omega_l)| + |\hat{f}_{12}(\omega_l)|$ may be factored out of $g(\omega_l)$ to get an expression of the form

$$\begin{aligned} &EnM^{\frac{1}{2}} |\log \bar{\bar{\psi}} - \log \psi| \\ &\quad \leqq \text{constant } nM^{-\frac{1}{2}} \textstyle\sum_{l=1}^{M} (Eh^4(\omega_l))^{\frac{1}{4}}(E|\hat{h}_{12}(\omega_l)|^4 + E|\hat{f}_{12}(\omega_l)|^4)^{\frac{1}{4}} \\ &\quad \cdot [E \text{ tr } (\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))(\hat{F}_{X,T,n}(\omega_l) - \hat{F}_{\bar{X},T,n}(\omega_l))^{*\prime}]^{\frac{1}{2}} \end{aligned}$$

where $Eh^4(\omega_l)$ is bounded by a constant depending only on $F(\omega)$. Using the facts that $\hat{h}_{12}(\omega_l)$ and $\hat{f}_{12}(\omega_l)$ are quadratic forms in normal random variables, and for such quadratic forms t, $Et^4 \leqq c(Et^2)^2$, where c is a universal constant, and under the null hypothesis, $E|\hat{f}_{12}(\omega_l)|^2$ and $E|\hat{h}_{12}(\omega_l)|^2$ are bounded by a constant $\times$ $(2n+1)^{-1}$, the result is

$$\begin{aligned} &EnM^{-1} |\log \bar{\bar{\psi}} - \log \psi| \\ &\quad \leqq \text{constant} \cdot n(2n+1)^{-\frac{1}{2}}(\log_2 M/(2n+1)^2 + M(2n+1)/T^2)^{\frac{1}{2}} \to 0. \end{aligned}$$

APPENDIX

In Lemmas A.1–A.3, A and B are strictly positive definite matrices, U, U_i and V are any square matrices, and λ_Z, Λ_Z are the smallest and largest eigenvalues of the matrix Z. The fact that $\lambda_A \text{ tr } B \leqq \text{tr } AB \leqq \Lambda_A \text{ tr } B$ is repeatedly used.

LEMMA A.1

$$\text{tr } (\textstyle\sum_{j=1}^N U_j)(\sum_{j=1}^N U_j)^{*\prime} \leqq N \text{ tr } (\sum_{j=1}^N U_j U_j^{*\prime}).$$

PROOF.

$$\begin{aligned} \text{tr } (\textstyle\sum_{j=1}^N U_j)(\sum_{j=1}^N U_j)^{*\prime} &= \text{tr } \textstyle\sum_{j=1}^N \sum_{k=1}^N U_j U_k^{*\prime} \\ &\leqq \text{tr } \textstyle\sum_{j=1}^N \sum_{k=1}^N \frac{1}{2}(U_j U_j^{*\prime} + U_k U_k^{*\prime}) \\ &= N \text{ tr } (\textstyle\sum_{j=1}^N U_j U_j^{*\prime}). \end{aligned}$$

LEMMA A.2

$$\text{tr } (A-B)^2 \leqq (\lambda_A + \lambda_B)^{-2} \text{ tr } (A^2 - B^2)^2.$$

PROOF.

$$\begin{aligned} (\lambda_A + \lambda_B) \text{ tr } (A-B)^2 &\leqq \text{tr } (A-B)^2(A+B) \\ &= \text{tr } (A-B)(A+B)(A-B), \end{aligned}$$

$$\begin{aligned} &(\lambda_A + \lambda_B) \text{ tr } (A-B)(A+B)(A-B) \\ &\qquad \leqq \text{tr } (A-B)(A+B)(A-B)(A+B) \end{aligned}$$

giving

$$(\lambda_A + \lambda_B)^2 \text{ tr } (A-B)^2 \leqq \text{tr } (A-B)(A+B)(A-B)(A+B).$$

$$\begin{aligned}
\operatorname{tr}&(A - B)(A + B)(A - B)(A + B) \\
&= \operatorname{tr}(A^2 - BA + AB - B^2)(A^2 - BA + AB - B^2) \\
&= \operatorname{tr}[A^4 - A^2BA + A^3B - A^2B^2 - BA^3 + BABA - BAAB + BAB^2 \\
&\qquad + ABA^2 - ABBA + ABAB - AB^3 - B^2A^2 + B^3A - B^2AB + B^4] \\
&= \operatorname{tr}[A^4 - 4B^2A^2 + 2BABA + B^4].
\end{aligned}$$

Now,

$$\operatorname{tr} BABA \leqq \operatorname{tr} A^2B^2,$$

hence

$$\operatorname{tr}[A^4 - 4B^2A^2 + 2BABA + B^4] \leqq \operatorname{tr}[A^4 - 2\operatorname{tr} A^2B^2 + B^4] = \operatorname{tr}[A^2 - B^2]^2.$$

Combining these last inequalities gives the lemma. It is easy to see that equality is obtained for $A = aI$, $B = bI$.

LEMMA A.3

$$\operatorname{tr}(U - V)(U - V)^{*\prime} \leqq [\varphi(U - V)]^2.$$

PROOF.

$$\operatorname{tr}(U - V)(U - V)^{*\prime} = \sum_{\mu,\nu}(u_{\mu\nu} - v_{\mu\nu})^2 \leqq [\sum_{\mu,\nu}|u_{\mu\nu} - v_{\mu\nu}|]^2 = [\varphi(U - V)]^2.$$

LEMMA A.4. *Let A be any (real or complex) quadratic form with q^2 the largest eigenvalue of $AA^{*\prime}$. Let S, $\bar{S}$ be strictly positive (real or complex) matrices of the same dimension as A and let $0 < \lambda \leqq \Lambda < \infty$ be common lower and upper bounds for the eigenvalues of S and $\bar{S}$, and suppose $\varphi(S - \bar{S})$, defined in Lemma 2 satisfies $\varphi(S - \bar{S}) \leqq \theta$. Then*

$$\operatorname{tr}(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})^{*\prime} \leqq (\Lambda/\lambda)q^2\theta^2.$$

PROOF.

$$\begin{aligned}
\operatorname{tr}(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}}AS^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}}A\bar{S}^{\frac{1}{2}})^{*\prime} &= \operatorname{tr}[(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})AS^{\frac{1}{2}} + \bar{S}^{\frac{1}{2}}A(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})] \\
&\quad \cdot[(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})AS^{\frac{1}{2}} + \bar{S}^{\frac{1}{2}}A(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})]^{*\prime},
\end{aligned}$$

which by Lemma A.1 is less than

$$\begin{aligned}
2\operatorname{tr}&(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})^{*\prime}(ASA^{*\prime} + A\bar{S}A^{*\prime}) \\
&\leqq 4q^2\Lambda \operatorname{tr}(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})^{*\prime}.
\end{aligned}$$

Lemmas A.2 and A.3 give

$$\begin{aligned}
\operatorname{tr}&(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})(S^{\frac{1}{2}} - \bar{S}^{\frac{1}{2}})^{*\prime} \\
&\leqq (4\lambda)^{-1}\operatorname{tr}(S - \bar{S})(S - \bar{S})^{*\prime} \leqq (4\lambda)^{-1}[\varphi(S - \bar{S})]^2 = (4\lambda)^{-1}\theta^2,
\end{aligned}$$

which gives the result.

LEMMA A.5 *Let A be a non-negative definite $M \times M$ matrix, and suppose it is known that, for any $s = (s_1, s_2, \cdots s_M)$, $sAs' \leqq \max_l |s_l|^2 c^2$. Then, if $M = 2^k$ for some k,* $\text{tr } A \leqq c^2$. *In general,* $\text{tr } A \leqq (\log_2 M)c^2$.

PROOF. We use the fact, that if $y^1, y^2, \cdots y^M$ are any orthonormal set of M dimensional (row) vectors, then $\text{tr } A = \sum_{k=1}^M y^k A y^{k'}$. If $M = 2^k$, for some integer k, then there exists a set of M orthonormal vectors, each of the form $y^k = M^{-\frac{1}{2}}(y_1^k, y_2^k, \cdots y_M^k)$ where y_l^k, $l = 1, 2, \cdots M$ is ± 1. In this case, the hypothesis gives $y^k A y^{k'} \leqq c^2 M^{-1}$ and $\text{tr } A \leqq c^2$. In general, if $2^k < M < 2^{k+1}$, write $M = \sum_{\nu=0}^k \theta_\nu 2^\nu$ where $\theta_k = 1$ and $\theta_\nu = 0$ or 1, $\nu = 0, 1, 2, \cdots, k - 1$. Let l be the number of non-zero θ_ν's. An $M \times M$ orthogonal matrix can be constructed with l non-zero blocks down the diagonal, the mth block of dimension $2^{\nu_m} \times 2^{\nu_m}$, $m = 1, 2, \cdots l$, where ν_m corresponds to the mth non-zero θ_ν. In the mth block place an orthogonal matrix with rows of the form $(2^{\nu_m})^{-\frac{1}{2}}(u_1, u_2, \cdots u_{2^{\nu_m}})$ where $u_r = \pm 1$. Now let $y^1, y^2, \cdots y^M$ be the rows of this matrix. If part of y^k is contained in the mth block, $y^k A y^{k'} \leqq c^2 2^{-\nu_m}$, and there are 2^{ν_m} such y^k's. Hence

$$\text{tr } A \leqq \sum_{m=1}^l 2^{\nu_m} \cdot c^2 2^{-\nu_m} = lc^2 \leqq kc^2 \leqq (\log_2 M)c^2.$$

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