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THE CENTRAL LIMIT PROBLEM FOR GENERALIZED RANDOM FIELDS

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1. Introduction. For probability measures in the real line two main results concerning sums of independent random variables are the Lèvy-Khintchine representation of an infinitely divisible probability measure, and the criteria for weak convergence of probability measures. During the last two decades or so these theorems have been extended to a variety of topological Abelian groups, including vector spaces.

We mention in particular the work of Parthasarathy, Rao, and Varadhan [5] in locally compact Abelian groups, and the earlier extension by Takano [6] of the classical theory to finite-dimensional space.

On the whole, the theory for non-locally compact Abelian groups seems incomplete, due chiefly to the lack of an adequately well-behaved Fourier transform. It is therefore natural to seek to extend the theory to non-locally-compact Abelian groups in which it can be reduced to the locally compact theory. For example, one might apply the locally compact theory to the projections of one's probability measures into the locally compact quotient groups. Among the topological vector spaces, the natural domain of this technique is of course the class of duals $\mathfrak{X}$ of nuclear spaces (and duals of strict inductive limits of nuclear spaces).

In this paper we introduce the notions of bounded variances for a double sequence of measures in $\mathfrak{X}$, weak convergence of measures, and convolution. These notions coincide with the usual definitions in the special case of finite-dimensional spaces, and seem natural in terms of applications. Taking the aforementioned approach yields the following results, in terms of our generalized notions. The class of weak limits of sums of random variables with bounded variances coincides with the class of infinitely divisible measures having covariances. If μ is any infinitely divisible measure, then the log of its Fourier transform has value at each φ in the original strict inductive limit space $\mathfrak{X}'$ given by

$$A(\varphi) - \tfrac{1}{2}\|\varphi\|^2 + \int [e^{ix(\varphi)} - 1 - ix(\varphi)]\, d\nu(x)$$

where $A \varepsilon \mathfrak{X}$, $\|\varphi\|$ is a Hilbert norm on $\mathfrak{X}'$, and ν is a σ-finite measure on $\mathfrak{X}$ which integrates the function $e^{ix(\varphi)} - 1 - ix(\varphi)$ and has finite mass outside each neighborhood of the null element.

2. The Lèvy continuity theorem. Henceforth $\mathfrak{X}$ will denote the dual space of a strict inductive limit $\mathfrak{X}'$ of nuclear spaces $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$; $\mathfrak{X}$ is given the strong topology. By a *measure* in $\mathfrak{X}$ we mean a σ-finite non-negative regular Borel measure ν with values in $[0, \infty]$, such that for some closed set $\mathfrak{N}$ with $\nu(\mathfrak{N}) = 0$,

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$\nu(\mathcal{C}) < \infty$ for every closed cylinder set $\mathcal{C}$ missing $\mathfrak{N}$, and the measure $\nu(\cdot \cap \mathcal{C})$ is continuous on cylinder sets, in the sense of Gel'fand [1]. (The definition of this continuity is given below). *Regular* is meant in the sense that for any Borel set $\mathcal{B}$, $\nu(\mathcal{B}) = \inf \nu(\mathcal{Z})$, the infimum being over all countable unions $\mathcal{Z} \supseteq \mathcal{B}$ of closed cylinder sets. Thus if $\mathfrak{X}'$ arises in the usual way from the infinitely differentiable compactly supported functions on an open region in finite-dimensional space, then the probability measures are precisely the *random generalized fields* in the region, as defined by Gel'fand.

For each n-dimensional subspace $\mathcal{S}$ of $\mathfrak{X}'(1 \leqq n \leqq \infty)$ let $\mathcal{S}^\circ$ be its annihilating subspace of $\mathfrak{X}$. If ν is a function on the Borel class in $\mathfrak{X}$, $\nu_{\mathcal{S}}$ will denote its projection into the (n-dimensional) quotient space $\mathfrak{X}/\mathcal{S}^\circ$. The term *measure of cylinder sets* in $\mathfrak{X}$ will refer to a function ν the Borel class such that for each $\mathcal{S}$, $\nu_{\mathcal{S}}$ is a measure in $\mathfrak{X}/\mathcal{S}^\circ$. A sequence μ_α of finite measures in $\mathfrak{X}$ will be said to *converge weakly* to $\mu(\mu_\alpha \to \mu)$ if for every one-dimensional subspace $\mathcal{S}$ the sequence $\mu_{\alpha\mathcal{S}}$ converges weakly to $\mu_{\mathcal{S}}$ in the usual sense for locally compact spaces (i.e., in the sense that $\int f\, d\mu_{\alpha\mathcal{S}} \to \int f\, d\mu_{\mathcal{S}}$ for every bounded continuous real-valued function f on $\mathfrak{X}/\mathcal{S}^\circ$). The arrow $\to$ with measures will denote weak convergence, whether they be measures in $\mathfrak{X}$ or some associated finite-dimensional space. If $\mathcal{S}$ is spanned by a single φ in $\mathfrak{X}'$ the Fourier transforms of a finite measure μ and its projection $\mu_{\mathcal{S}}$ agree on $\mathcal{S}$. (For all finite-dimensional $\mathcal{S}$ we identify $\mathfrak{X}/\mathcal{S}^\circ$ with the dual space of $\mathcal{S}$). Hence the classical one-dimensional Lèvy continuity theorem, applied to the sequence $\mu_{\alpha\mathcal{S}}$, yields the "if" part of

THEOREM 2.1. (Lèvy continuity theorem). *If $\mu_\alpha \to \mu$ then for each φ in $\mathfrak{X}'$ $\mu_\alpha{}^\wedge(\varphi) \to \mu^\wedge(\varphi)$. Conversely if, for some function g on $\mathfrak{X}'$ continuous at $\varphi \equiv 0$,*

$$\mu_\alpha{}^\wedge(\varphi) \to g(\varphi) \qquad (\varphi \text{ in } \mathfrak{X}'),$$

then there exists a unique finite measure μ such that $\mu_\alpha \to \mu$ and $\mu^\wedge = g$.

PROOF OF THE CONVERSE. Given any finite-dimensional subspace $\mathcal{S}$ of $\mathfrak{X}'$,

$$\hat{\mu_{\alpha\mathcal{S}}}(\varphi) = \mu_\alpha{}^\wedge(\varphi) \to_\alpha g(\varphi) \qquad (\varphi \text{ in } \mathfrak{X}').$$

By the finite-dimensional continuity theorem there exists a unique finite measure $\mu^{\mathcal{S}}$ in $\mathfrak{X}/\mathcal{S}^\circ$ such that $g(\varphi) = \mu^{\mathcal{S}\wedge}(\varphi)$ for φ in $\mathcal{S}$. If $\mathfrak{I} \supseteq \mathcal{S}$ is another finite-dimensional subspace and $P(X + \mathcal{S}^\circ) = X + \mathfrak{I}^\circ$ then the consistency condition $\mu^{\mathcal{S}} = \mu^{\mathfrak{I}} \circ P^{-1}$ is seen from uniqueness of Fourier transforms: For all φ in $\mathcal{S}$

$$\begin{aligned}\mu^{\mathcal{S}\wedge}(\varphi) &= \int_{\mathfrak{X}/\mathcal{S}^0} e^{ix(\varphi)}\, d\mu^{\mathcal{S}}(x) = g(\varphi) = \int_{\mathfrak{X}/\mathfrak{I}^0} e^{iy(\varphi)}\, d\mu^{\mathfrak{I}}(y)\\ &= \int_{\mathfrak{X}/\mathcal{S}^0} e^{ix(\varphi)}\, d(\mu^{\mathfrak{I}} \circ P^{-1})(x) = (\mu^{\mathfrak{I}} \circ P^{-1})^\wedge(\varphi).\end{aligned}$$

If $\mathcal{C}$ is any cylinder set, say $\mathcal{C} = \{(X(\varphi_1), \cdots, X(\varphi_N)) \,\varepsilon\, C\}$, let $\mathcal{S}$ be the span of $\varphi_1, \cdots, \varphi_N$, define $Q(X) = X + \mathcal{S}^\circ$, and define $\mu(\mathcal{C}) = \mu^{\mathcal{S}}(Q(\mathcal{C}))$. $\mu(\mathcal{C})$ is well-defined, since if $\mathcal{C} = \{(X(\theta_1), \cdots, X(\theta_M)) \,\varepsilon\, B\}$ is another representation of $\mathcal{C}$, $\mathfrak{I}$ the span of $\theta_1, \cdots, \theta_M$, and $R(X) = X + \mathfrak{I}^\circ$, then $\mu^{\mathcal{S}}(Q(\mathcal{C}))$ and $\mu^{\mathfrak{I}}(R(\mathcal{C}))$ both coincide with $\mu^{\mathfrak{U}}(S(\mathcal{C}))$, where $\mathfrak{U}$ is the span of $\varphi_1, \cdots, \varphi_N$, $\theta_1, \cdots, \theta_M$ and $S(X) = X + \mathfrak{U}^\circ$. This fact is immediate from the foregoing consistency conditions.

Hence μ is a finite *measure of cylinder sets*; i.e., a function on the class of cylinder sets whose projection $\mu_{\mathcal{S}}$ into each $\mathfrak{X}/\mathcal{S}^\circ$ is a measure. A finite measure of cylinder sets extends to a unique measure in $\mathfrak{X}$ precisely when it is continuous on cylinder sets, a condition equivalent with continuity of its Fourier transform at $\varphi \equiv 0$. Since $g = \mu^\wedge$ is continuous at $\varphi \equiv 0$, μ extends uniquely to the promised measure.

3. The central convergence criterion (CCC). The *convolution* of two finite measures μ and ν in $\mathfrak{X}$ is defined as follows: For each finite-dimensional subspace $\mathcal{S}$ of $\mathfrak{X}'$ let $\mathfrak{N}^{\mathcal{S}} = \mu_{\mathcal{S}} * \nu_{\mathcal{S}}$ define the measure $\mathfrak{N}^{\mathcal{S}}$ in $\mathfrak{X}/\mathcal{S}^\circ$. If $\mathfrak{I} \supseteq \mathcal{S}$ is another finite-dimensional subspace and P the natural projection $P(X + \mathfrak{I}^\circ) = X + \mathcal{S}^\circ$ from $\mathfrak{X}/\mathfrak{I}^\circ$ onto $\mathfrak{X}/\mathcal{S}^\circ$, then $\mathfrak{N}^{\mathcal{S}} = \mathfrak{N}^{\mathfrak{I}} \circ P^{-1}$. As in the Lèvy continuity theorem this is seen from the computation of Fourier transforms, plus the fact that $\mu_{\mathcal{S}}^\wedge$ (respectively, $\nu_{\mathcal{S}}^\wedge$) and $\mu_{\mathfrak{I}}^\wedge$ ($\nu_{\mathfrak{I}}^\wedge$) agree on $\mathcal{S}$. As before, this implies that for each cylinder set $\mathcal{C} = \{(X(\varphi_1), \cdots, X(\varphi_N) \varepsilon C\}$, the relation

$$\eta(\mathcal{C}) = \eta^{\mathcal{S}}(\{X + \mathcal{S}^\circ : (X(\varphi_1), \cdots, X(\varphi_N)) \varepsilon C\}),$$

where $\mathcal{S}$ is the span of the φ_i, well-defines a measure η of cylinder sets. Given φ in $\mathfrak{X}'$ and its span $\mathcal{S}$,

$$\eta^\wedge(\varphi) = \eta^{\mathcal{S}}{}^\wedge(\varphi) = \mu_{\mathcal{S}}^\wedge(\varphi)\nu_{\mathcal{S}}^\wedge(\varphi) = \mu^\wedge(\varphi)\nu^\wedge(\varphi)$$

shows η has continuous Fourier transform on $\mathfrak{X}$, with value $\mu(\mathfrak{X})\nu(\mathfrak{X})$ at $\varphi \equiv 0$. Hence η extends uniquely to a measure in $\mathfrak{X}$, which we define to be $\mu * \nu$. The relation $(\mu * \nu)^\wedge = \mu^\wedge \nu^\wedge$ is immediate. This notion of convolution seems natural, in view of the easily verified fact that, if X and Y are independent $\mathfrak{X}$-valued random variables with probability distributions μ and ν respectively, then $\mu * \nu$ is the probability distribution of $X + Y$. $\prod_{k=1}^n \nu_k$ will denote the convolution of three or more measures, whether in $\mathfrak{X}$ or in some associated finite-dimensional space. μ will be called *infinitely divisible* if for all n $\mu = (\mu_n)^n$ in the above sense, for some μ_n. A double sequence μ_{nk} ($n \geqq 1$, $k = 1, 2, \cdots, k_n$) of measures is said to have *bounded variances* if for each converging sequence φ_α in $\mathfrak{X}'$,

$$\sup_\alpha \max_k \int X(\varphi)^2\, d\mu_{nk}(X) \to_n 0, \quad \text{and} \quad \sup_\alpha \sup_n \sum_k \int X(\varphi_\alpha)^2\, d\mu_{nk}(X) < \infty.$$

THEOREM 3.1 (Central Convergence Criterion.) *If the probability measures μ_{nk} have bounded variances and*

$$\prod_k \mu_{nk} \to_n \mu, \tag{1}$$

then there exist a unique measure ν in $\mathfrak{X}$, $A \varepsilon \mathfrak{X}$, and continuous seminorm $\| \ \|$ where $\|\varphi\|^2 = \langle\varphi, \varphi\rangle$ for an inner product $\langle\cdot, \cdot\rangle$ continuous on $\mathfrak{N} \times \mathfrak{N}$ for each nuclear subspace $\mathfrak{N}$ of $\mathfrak{X}'$, such that for each φ and θ in $\mathfrak{X}'$

$$\log \mu^\wedge(\varphi) = iA(\varphi) - \tfrac{1}{2}\|\varphi\|^2 + \int [e^{iN(\varphi)} - 1 - iX(\varphi)]\, d\nu(X), \tag{2}$$

$\nu(\{X(\varphi) = 0\}) = 0$ and $\int X(\varphi)^2\, d\nu(X) < \infty$;

(3) *$\sum_k \mu_{nk}(\cdot \cap \mathcal{C}) \to_n \nu(\cdot \cap \mathcal{C})$ for every closed cylinder set $\mathcal{C}$ omitting some finite intersection of closed hyperplanes;*

(4) $\langle\varphi, \theta\rangle = \lim_{\epsilon\to 0} \lim_{n\to\infty} \sum_k [\int_{D(\epsilon)} X(\varphi)X(\theta)\, d\mu_{nk}(X)$
$- \int_{D(\epsilon)} X(\varphi)\, d\mu_{nk}(X) \int_{D(\epsilon)} X(\theta)\, d\mu_{nk}(X)]$,

where $D(\epsilon) = \{|X(\varphi)|^2 + |X(\theta)|^2 < \epsilon\}$; *and*

(5) $$\sum_k \int X\, d\mu_{nk}(X) \to_n A \quad \text{in} \quad \mathfrak{X}.$$

Conversely, if measures μ_{nk} *have bounded variances and* (3), (4) *and* (5) *hold for some* $A \in \mathfrak{X}$ *and prenorm* $\|\varphi\| = \langle\varphi, \varphi\rangle^{1/2}$, *then* (1) *and* (2) *hold for a unique* ν.

REMARKS. If $\mathfrak{X}$ has dimension one over the reals $\mathfrak{R}$, the above assertions constitute precisely the classical Levy-Khintchine form of the CCC for bounded variances. An extended CCC for measures in $\mathfrak{R}^N$ is presented in [6]. As in the classical $\mathfrak{R}^1$ work, the measures μ_{nk} in [6] are only assumed to be uniformly asymptotically negligible; however, if they in fact have bounded variances then the extended CCC in $\mathfrak{R}^N$ reduces to the above assertions, for the special case $\mathfrak{X} = \mathfrak{R}^N$. Inasmuch as the extended CCC in $\mathfrak{R}^N$ involves essentially the same work as in $\mathfrak{R}^1$, its reduction to the bounded variances case is the same as the reduction of the extended CCC in $\mathfrak{R}^1$. The details will not be reproduced here; however, we shall repeatedly use the bounded variance CCC for $\mathfrak{R}^N$. More precisely, we shall assume the bounded variance CCC for finite-dimensional spaces, just as in the statement of Theorem 3.1, considering the special case of finite-dimensional $\mathfrak{X}$.

PROOF OF THE DIRECT PART. Given φ in $\mathfrak{X}'$ let $\lambda_B(B) = \lambda(\{X(\varphi) \in B\})$ define the projection λ_φ of λ into $\mathfrak{R}$, for any measure λ in $\mathfrak{X}$. Then the double sequence $\mu_{nk\varphi}$ has bounded variances and $\prod_k \mu_{nk\varphi} \to_n \mu_\varphi$. Applying the CCC to the case $\mathfrak{X} = \mathfrak{R}$ yields a unique measure ν^φ in $\mathfrak{R}$ and real numbers A_φ and $\sigma_\varphi^2 > 0$ such that for all real u

(2′) $$\log \mu_\varphi{}^\wedge(u) = iA_\varphi u - \tfrac{1}{2}\sigma_\varphi{}^2 u^2 + \int_{-\infty}^{\infty} [e^{itu} - 1 - itu]\, d\nu^\varphi(t),$$
$$\nu^\varphi(\{0\}) = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} t^2\, d\nu^\varphi(t) < \infty;$$

(3′) $\sum_k \mu_{nk\varphi}(\cdot \cap Y) \to_n \nu^\varphi(\cdot \cap Y)$ for closed Y omitting 0;

(4′) $$\sigma_\varphi{}^2 = \lim_{\epsilon\to 0} \lim_{n\to\infty} \sum_k [\int_{|t|<\epsilon} t^2\, d\mu_{nk\varphi}(t) - (\int_{|t|<\epsilon} t\, d\mu_{nk\varphi}(t))^2];$$

and

(5′) $$\sum_k \int t\, d\mu_{nk\varphi}(t) \to A_\varphi.$$

Since $\int_{-\infty}^{\infty} t\, d\mu_{nk\varphi}(t) = \int_{\mathfrak{X}} X(\varphi)\, d\mu_{nk}(X)$ and $\log \mu_\varphi{}^\wedge(1) = \log \mu^\wedge(\varphi)$, conditions (2), (4) and (5) in the conclusion of the direct part of the CCC will be established once it is shown that there exist a unique measure ν in $\mathfrak{X}$, $A \in \mathfrak{X}$, and inner product $\langle\cdot, \cdot\rangle$ continuous on $\mathfrak{N} \times \mathfrak{N}$ for each nuclear subspace $\mathfrak{N}$ of $\mathfrak{X}'$, such that for all φ in $\mathfrak{X}'$,

(i) $A_\varphi = A(\varphi)$ and $\sum_k \int X\, d\mu_{nk}(X) \to_n A$ in $\mathfrak{X}$,
(ii) $\sigma_\varphi{}^2 = \langle\varphi, \varphi\rangle$, and
(iii) $\nu^\varphi = \nu_\varphi$.

The Banach-Steinhaus theorem for barrelled spaces implies that A_φ, viewed as a function of φ, is continuous and linear on $\mathfrak{X}':A(\varphi) = A_\varphi$ defines $A \varepsilon \mathfrak{X}$. From the fact that $\mathfrak{X}'$ is a Montel space, it is known that the point-wise convergence of $\sum_k \int X\, d\mu_{nk}(X)$ to A is in fact convergence in $\mathfrak{X}$. This proves (i).

It is worth noting here a rôle of the bounded variance hypothesis, as distinct from the weaker classical hypothesis of uniform asymptotic negligibility: In the latter case, i.e., in the extended one-dimensional CCC, condition (5′) is weakened in such a way that A_φ, as a function of φ, does not appear to be necessarily a limit of *linear* functions on $\mathfrak{X}'$. In this case A could not be defined as an element of $\mathfrak{X}$.

To establish (ii) define

$$\langle \varphi, \theta \rangle = \tfrac{1}{2}(\sigma^2_{(\varphi+\theta)} - \sigma_\varphi^2 - \sigma_\theta^2)$$

for each φ and θ in $\mathfrak{X}'$. Using (4′) and a change of variable to compute $\langle \varphi, \theta \rangle$ yields the relations (4) and (ii). Temporarily fix θ and ϵ in this computation. Since each summand in (4) is continuous and linear in φ, the Banach-Steinhaus theorem implies that the limit of the sum in (4) as $n \to \infty$ is a continuous linear function of $\varphi \varepsilon \mathfrak{X}'$. Letting $\epsilon \to 0$ this new limit is also, for the same reason, a continuous linear function of $\varphi \varepsilon \mathfrak{X}'$. Now fixing φ and letting θ vary, the same argument shows $\langle \varphi, \theta \rangle$ is a continuous linear function of $\theta \varepsilon \mathfrak{X}'$; i.e., $\langle \varphi, \theta \rangle$ is a separately continuous bilinear function on $\mathfrak{X}' \times \mathfrak{X}'$. Since any nuclear subspace $\mathfrak{N}$ of $\mathfrak{X}'$ is metrizable and barrelled, the separate continuity implies joint continuity on each $\mathfrak{N} \times \mathfrak{N}$.

To finish conclusions (2), (4) and (5) it remains only to construct the measure ν for the generalized Poisson component of μ. This is the work of the remaining sections of the paper; we take ν to be the ν constructed in Theorem 5.1. Condition (3) and the uniqueness of ν are proved in Lemma 5.9. The remaining assertions of uniqueness are simply the assertions of uniqueness (for each φ) of A_φ and σ_φ^2 for the one-dimensional CCC.

Proof of the Converse. If (3), (4) and (5) hold for every φ and θ in $\mathfrak{X}'$ then $\prod_k \mu_{nk\varphi} \to_n \mu_\varphi$ by the corresponding converse of the one-dimensional CCC. This implies

$$\prod_k \hat{\mu}_{nk}(\varphi) = \prod_k \hat{\mu}_{nk}(1) \to_n \mu_\varphi^\wedge(1) = \mu^\wedge(\varphi)$$

and hence $\prod_k \mu_{nk} \to_n \mu$ in view of Theorem 2.1.

4. ν as a measure of cylinder sets. In this section the first phase of the construction of the Poisson component of μ is performed: A measure ν of cylinder sets is constructed which in Section 5 will be extended to the ν promised in Theorem 3.1.

Lemma 4.1 *Let $\mathfrak{X}$ be a finite-dimensional space with Euclidean norm $|x|$; let λ be a measure in $\mathfrak{X}$ such that the measure $|x|^2\lambda$ is finite, and for each $\varphi \varepsilon \mathfrak{X}'$, $e^{ix(\varphi)} - 1 - ix(\varphi)$ is λ-summable. Then the function g, where*

$$g(\varphi) = \int_{\mathfrak{X}} [e^{ix(\varphi)} - 1 - ix(\varphi)]\, d\lambda(x),$$

uniquely determines the measure $|x|^2\lambda$.

PROOF. As in the case $\mathfrak{X} = \mathfrak{R}$ [4], continuity of g follows from the bounded convergence theorem. By applying Theorem 2.1 to an approximating sequence of Riemann sums it can be seen that e^g is the Fourier transform of a probability measure in $\mathfrak{X}$, with finite variance $\int |x|^2 \, d\lambda(x)$. Relative to a given basis in $\mathfrak{X}$ orthonormal for the norm $|\ |$, g can be regarded as a function on $\mathfrak{R}^N$; as such it has a Laplacian with value $-\int_{\mathfrak{X}} e^{ix(\varphi)}|x|^2 \, d\lambda(x)$ at each $\varphi \in \mathfrak{X}'$. The proof now follows from uniqueness of the Fourier transform.

To construct ν as a measure of cylinder sets, a consistent family of measures $\nu^{\mathfrak{U}}$ on the corresponding quotient spaces $\mathfrak{X}/\mathfrak{U}^\circ$ is needed, as $\mathfrak{U}$ runs through the finite-dimensional subspaces of $\mathfrak{X}'$. Since $\prod_k \mu_{nk} \to_n \mu$, for each $\mathfrak{U}$ $\prod_k \mu_{nk\mathfrak{U}} \to_n \mu_{\mathfrak{U}}$ is immediate from the notions of convolution and convergence for measures in $\mathfrak{X}$. Since the double sequence $\mu_{nk\mathfrak{U}}$ has bounded variances, the finite-dimensional CCC applies. Let the element $A_{\mathfrak{U}}$ of $\mathfrak{X}/\mathfrak{U}^\circ$, the seminorm $\|\varphi\|_{\mathfrak{U}}$ on $\mathfrak{U}$, and the measure $\nu^{\mathfrak{U}}$ in $\mathfrak{X}/\mathfrak{U}^\circ$, be the entities guaranteed by the finite-dimensional CCC corresponding respectively to the A, $\|\varphi\|$, and ν of (2) through (5). We shall denote the conditions in $\mathfrak{X}/\mathfrak{U}^\circ$ corresponding to (2) through (5), by ($2'''$) through ($5'''$) respectively. The family of measures $\nu^{\mathfrak{U}}$ enjoys the following consistency conditions:

THEOREM 4.2 *If* $\mathfrak{S}$ *and* $\mathfrak{J}$ *are finite-dimensional subspaces of* $\mathfrak{X}'$ *and* $\mathfrak{S} \subseteqq \mathfrak{J}$, *and if* $P: y = X + \mathfrak{J}^\circ \to x = X + \mathfrak{S}^\circ$, *then* $\nu^{\mathfrak{S}} = \nu^{\mathfrak{J}} \circ P^{-1}$.

PROOF. By conditions ($5'''$) for $\mathfrak{U} = \mathfrak{S}$ and then $\mathfrak{U} = \mathfrak{J}$,

$$A_{\mathfrak{U}}(\varphi) = \lim_n \sum_k \int_{\mathfrak{X}/\mathfrak{U}^0} z(\varphi) \, d\mu_{nk\mathfrak{U}}(z)$$

for all φ in $\mathfrak{S}$, $\mathfrak{U} = \mathfrak{S}$ and $\mathfrak{U} = \mathfrak{J}$. The relation

$$\int_{\mathfrak{X}/\mathfrak{S}^0} x(\varphi) \, d\mu_{nk\mathfrak{S}}(x) = \int_{\mathfrak{X}/\mathfrak{J}^0} y(\varphi) \, d\mu_{nk\mathfrak{J}}(y)$$

(due to the condition $\mu_{nk\mathfrak{S}} = \mu_{nk\mathfrak{J}} \circ P^{-1}$) thereby implies $A_{\mathfrak{J}}(\varphi) = A_{\mathfrak{S}}(\varphi)$ for all φ in $\mathfrak{S}$. Similar consideration of condition ($4'''$), first in $\mathfrak{X}/\mathfrak{J}^\circ$ and then in $\mathfrak{X}/\mathfrak{S}^\circ$, leads to the conclusion that $\|\varphi\|_{\mathfrak{J}}$ and $\|\varphi\|_{\mathfrak{S}}$ also agree on $\varphi \in \mathfrak{S}$. Combining this information with the fact that

$$(2''') \quad \log \mu_{\mathfrak{U}}(\varphi) = iA_{\mathfrak{U}}(\varphi) - \tfrac{1}{2}\|\varphi\|_{\mathfrak{U}}^2 + \int_{\mathfrak{Y}/\mathfrak{U}^0} [e^{iz(\varphi)} - 1 - iz(\varphi)] \, d\nu^{\mathfrak{U}}(z)$$

for all φ in $\mathfrak{S}$, $\mathfrak{U} = \mathfrak{S}$ and $\mathfrak{U} = \mathfrak{J}$, it is seen that the two integrals here, corresponding to $\mathfrak{U} = \mathfrak{S}$ and $\mathfrak{U} = \mathfrak{J}$, coincide for all φ in $\mathfrak{S}$. But this integral for $\mathfrak{U} = \mathfrak{J}$ can be written

$$\int_{\mathfrak{X}/\mathfrak{S}^0} [e^{ix(\varphi)} - 1 - ix(\varphi)] \, d(\nu_{\mathfrak{J}} \circ P^{-1})(x),$$

since $P(y)(\varphi) = y(\varphi)$ for $y \in \mathfrak{X}/\mathfrak{J}^\circ$, $\varphi \in \mathfrak{S}$. If $|x|$ is a Euclidean norm on $\mathfrak{X}/\mathfrak{S}^\circ$ then by condition ($2'''$) the measure $|x|^2\nu^{\mathfrak{S}}$ is finite. Let $\|y\|$ be a Euclidean norm on $\mathfrak{X}/\mathfrak{J}^\circ$ such that $\|y\| = |P(y)|$ for y in $\mathfrak{X}/\mathfrak{J}^\circ$. Then $|x|^2\nu^{\mathfrak{J}} \circ P^{-1}$ is a finite measure on $\mathfrak{X}/\mathfrak{S}^\circ$, since $\|y\|^2\nu^{\mathfrak{J}}$ is a finite measure on $\mathfrak{X}/\mathfrak{J}^\circ$ by condition ($2'''$). Thus Lemma 4.1 applies, proving $|x|^2\nu^{\mathfrak{S}} = |x|^2\nu^{\mathfrak{J}} \circ P^{-1}$. Since by condition ($2'''$)

$$\nu^{\mathfrak{S}}(\{\mathfrak{S}^\circ\}) = \nu^{\mathfrak{J}}(\{\mathfrak{J}^\circ\}) = 0, \qquad \nu^{\mathfrak{S}} = \nu^{\mathfrak{J}} \circ P^{-1}.$$

If $\mathcal{C}$ is any cylinder set, say $\mathcal{C} = \{(X(\varphi_1), \cdots, X(\varphi_N)) \in X\}$, let $\mathcal{S}$ be the span of $\varphi_1, \cdots, \varphi_N$, define $Q(X) = X + \mathcal{S}^\circ$, and define $\nu(\mathcal{C}) = \nu^{\mathcal{S}}(Q(\mathcal{C}))$. Exactly as in the proof of Theorem 2.1, the consistency condition just established implies that $\nu(\mathcal{C})$ is well-defined. ν is now a measure of cylinder sets, since for each $\mathcal{S}$ $\nu_{\mathcal{S}} = \nu^{\mathcal{S}}$; i.e., the projections $\nu_{\mathcal{S}}$ are measures.

5. The Poisson component of μ. In order to construct a measure ν in $\mathfrak{X}$ such that $\nu_\varphi = \nu^\varphi$ for φ $\mathfrak{X}'$, the following extension theorem will be needed:

THEOREM 5.1. *Let ν be a measure of cylinder sets such that for each closed cylinder set $\mathcal{C}$ omitting a finite intersection of closed hyperplanes, the measure $\nu(\cdot \cap \mathcal{C})$ of cylinder sets is countably additive on cylinder sets; then ν extends to a unique measure in $\mathfrak{X}$ such that $\nu(\{0\}) = 0$.*

PROOF. Consider the ring $\bar{R}$ of cylinder sets contained in a $\mathcal{C}$ of the above kind. By hypothesis, ν is a σ-finite measure on $\bar{R}$, in the sense of Halmos [2]. The Borel subsets of $\mathfrak{X} \sim \{0\}$ form the σ-ring generated by $\bar{R}$; this is seen from the fact that $\mathfrak{X} \sim \{0\} = \bigcup_j \{X(\varphi_j) \neq 0\}$ if $\varphi_1, \varphi_2, \cdots$ is a countable dense subset of $\mathfrak{X}'$. It follows that ν has a unique extension to a σ-finite measure (again in Halmos' sense) on the Borel subsets of $\mathfrak{X} \sim \{0\}$. We further define $\nu(\{0\}) = 0$. This extension will henceforth be denoted ν. To finish the proof, it remains only to show ν is regular. Given any $\mathcal{B}$ omitting $\{0\}$, $\nu(\mathcal{E}) \leqq \inf \nu(\mathcal{Z})$ is obvious, where $\mathcal{Z}$ runs through the countable unions, containing $\mathcal{B}$, of closed cylinder sets. To establish the opposite inequality it suffices to show that the class $\bar{M}$ of Borel sets $\mathcal{B}$ omitting $\{0\}$ for which it (the inequality $\geqq$) holds, is a monotone class: for $\bar{M}$ contains the ring $\bar{R}$. Consider, then, an increasing sequence $\mathcal{B}_k$ of Borel sets omitting $\{0\}$, for which

$$\nu(\mathcal{B}_k) \geqq \inf \{\nu(\mathcal{Z}) : \mathcal{Z} \supseteqq \mathcal{B}_k\}$$

(with $\mathcal{Z}$ running only through countable unions of closed cylinder sets). Without loss of generality assume $\nu(\bigcup_k \mathcal{B}_k) < \infty$. Given $\epsilon > 0$, choose for each k a countable union $\mathcal{Z}_k \supseteqq \mathcal{B}_k$ of closed cylinder sets, such that $\nu(\mathcal{Z}_k) - \nu(\mathcal{B}_k) < \epsilon/2^k$. Then $\nu(\bigcup_k \mathcal{Z}_k) - \nu(\bigcup_k \mathcal{B}_k) < \epsilon$ implies

$$\nu(\bigcup_k \mathcal{B}_k) \geqq \inf \{\nu(\mathcal{Z}) : \mathcal{Z} \supseteqq \bigcup_k \mathcal{B}_k\} - \epsilon.$$

Since ϵ is arbitrary, it can be ignored; so the inequality $\geqq$ is preserved under increasing sequences. On the other hand if $\mathcal{B}_k$ is a decreasing sequence of such sets with $\nu(\bigcap_k \mathcal{B}_k) < \infty$, then for arbitrary $\epsilon > 0$ there exists a K such that $\nu(\mathcal{B}_K) - \nu(\bigcap_k \mathcal{B}_k) < \epsilon$. Thus $\nu(\mathcal{B}_K) = \inf \{\nu(\mathcal{Z}) : \mathcal{Z} \supseteqq \mathcal{B}_K\}$ implies

$$\nu(\bigcap_k \mathcal{B}_k) \geqq \inf \{\nu(\mathcal{Z}) : \mathcal{Z} \supseteqq \bigcap_k \mathcal{B}_k\} - \epsilon,$$

and ϵ can be ignored. Since $\bar{M}$ is a monotone class containing the ring $\bar{R}$, it contains the $\{0\}$-omitting Borel sets. Since $\nu(\{0\}) = 0$, ν is regular.

The remainder of this section is devoted to showing that the measure ν of cylinder sets constructed at the end of the preceding section, satisfies the conditions of Theorem 5.1. These conditions are established in Theorem 5.4, whose hypothesis is shown in Theorem 5.8 to be satisfied by ν.

Two preliminary lemmas are needed, as well as the following notation: For any measure η of cylinder sets define the measure $\eta_{(\varphi)}$ by

$$\eta_{(\varphi)}(B) = \eta(\{(X(\varphi_0), \cdots, X(\varphi_N)) \varepsilon B\})$$

for any given $(\varphi) = (\varphi_0, \cdots, \varphi_N)$ in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$. In these terms, the condition that η be continuous on cylinder sets is simply continuity of the map $(\varphi) \rightarrow \eta_{(\varphi)}$ (relative to the product topology in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$ and weak convergence of measures). The notion of weak-star (w^*) convergence of measures in finite-dimensional spaces will be used; this condition for a sequence of measures, $\lambda_\alpha \rightarrow_\alpha \lambda(w^*)$, may be defined in either of the following two equivalent ways: For every continuity point $t = (t_0, \cdots, t_N)$ of the map $t \rightarrow \lambda(\{s \mid s_j \leqq t_j, j = 0, \cdots, N)$,

$$\lambda_\alpha(\{s \mid s_j \leqq t_j, j = 0, \cdots, N\}) \rightarrow_\alpha \lambda(\{s \mid s_j \leqq t_j, j = 0, \cdots, N\});$$

equivalently, for every continuous g on $\mathfrak{R}^N$ vanishing at ∞,

$$\int g \, d\lambda_\alpha \rightarrow \int g \, d\lambda.$$

In particular the w^*-compactness theorem will be used: If λ_α is a sequence of measures in $\mathfrak{R}^N$ such that $\sup_\alpha \lambda_\alpha(\mathfrak{R}^N) < \infty$, then it has a w^*-converging subsequence.

LEMMA 5.2 *If* $\mathcal{S}$ *is a finite-dimensional subspace of* $\mathfrak{X}'$ *with basis* $\varphi_1, \cdots, \varphi_N$, *then the map* $X \rightarrow (X(\varphi_1), \cdots, X(\varphi_N)) \varepsilon \mathfrak{R}^N$ *is open.*

PROOF. Let $\mathfrak{N}$ be a nuclear subspace of $\mathfrak{X}'$, containing $\varphi_1, \cdots, \varphi_N$; let the dual of $\mathfrak{N}$ have the strong topology. Then the operation on $\mathfrak{X}$ that restricts each X to $\mathfrak{N}$ is an open map. But the map $Y \rightarrow (Y(\varphi_1), \cdots, Y(\varphi_N))$ on the dual of $\mathfrak{N}$ is open, since $\mathfrak{N}$ is a Frechet-Schwartz space and $\mathfrak{R}^N$ is a reflexive Frechet space ([3], Proposition 3.17.18). Hence the map $X \rightarrow (X(\varphi_1), \cdots, X(\varphi_N))$ on $\mathfrak{X}'$ is the composition of two open maps.

LEMMA 5.3 *Let* ν *be a measure of cylinder sets such that the measure-valued map*

$$(\varphi) \rightarrow |t|^2 \nu_{(\varphi)} \qquad (t = (t_0, \cdots, t_N), (\varphi) = (\varphi_0, \cdots, \varphi_N))$$

is w^**-continuous on* $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$. *Fix* $\varphi_1, \cdots, \varphi_N$ *and let* φ_0 *vary. There exist arbitrarily small neighborhoods* V *of* $(0, \cdots, 0)$ *in* $\mathfrak{R}^N$ *such that the map* $\varphi_0 \rightarrow f_V(t)\ |t|^2 \nu_{(\varphi)}$ *is* w^**-continuous at* $0 \varepsilon \mathfrak{X}'$, *where* f_v *is the indicator function on* $\mathfrak{R}^{N+1}$ *for the set* $\mathfrak{R} \times (\mathfrak{R}^N \sim V)$.

PROOF. Let V be an N-dimensional rectangle with $(0, \cdots, 0)$ in its interior, and vertices at continuity points of the function

$$(\alpha) \qquad t \rightarrow \nu(\textstyle\bigcap_{j=1}^N \{X(\varphi_j) \leqq t_j\}) \ (t = (t_1, \cdots t_N)).$$

Since this function has at most countably many points of discontinuity, V can be taken arbitrarily small. For any sequence φ_{0k} converging in $\mathfrak{X}'$ to 0, write $(\varphi)_k = (\varphi_{0k}, \varphi_1, \cdots, \varphi_N)$. Given any continuous g on $\mathfrak{R}^{N+1}$ vanishing at ∞, it follows that

$$\lim_k \int_{R \times V} g(t)|t|^2 \, d\nu_{(\varphi)_k}(t) = \int_V g(0, t_1, \cdots, t_N) \, d\nu_{(\varphi_1, \cdots, \varphi_N)}(t).$$

Exactly as in the one-dimensional Helley-Bray lemma [4], this can be proved by approximating these integrals with Riemann-Stieltjes sums formed from rectangles whose vertices are continuity points of the map (α). Since $\delta_0 \times \nu_{(\varphi_1, \cdots \varphi_N)} = \nu_{(0, \varphi_1, \cdots, \varphi_N)}$ the lemma is proved.

THEOREM 5.4. *Let $\mathcal{C}$ be a closed cylinder set that omits some finite intersection of closed hyperplanes; Let ν be a measure of cylinder sets such that the measure-valued maps $(\varphi) \to |t|^2 \nu_{(\varphi)}$ are w^*-continuous on $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$. Then the measure $\nu(\cdot \cap \mathcal{C})$ of cylinder sets is countably additive on cylinder sets.*

PROOF. By [1], Chapter IV, Section 2, Theorem 6, it is enough to show $\eta = \nu(\cdot \cap \mathcal{C})$ is continuous on cylinder sets; i.e., that the map $(\varphi) \to \eta_{(\varphi)}$ is continuous (relative to the produce topology in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$ and weak convergence of the measures). A sequence $(\varphi)_\alpha$ converges in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$ if and only if for some nuclear subspace $\mathfrak{N}$ of $\mathfrak{X}'$, $(\varphi)_\alpha$ is eventually in $\mathfrak{N} \times \cdots \times \mathfrak{N}$ and converges there. Hence one need only show $(\varphi) \to \eta_{(\varphi)}$ is continuous on each $\mathfrak{N} \times \cdots \times \mathfrak{N}$. Given $\mathfrak{N}$ let λ be the projection of $\eta = \nu(\cdot \cap \mathcal{C})$ into $\mathfrak{N}'$; i.e., for each Borel set $\mathcal{B}$ of continuous linear forms on $\mathfrak{N}$ define

$$\lambda(\mathcal{B}) = \eta(\{X \in \mathfrak{X} : X \mid \mathfrak{N} \in \mathcal{B}\}).$$

Then $\eta_{(\varphi)} = \lambda_{(\varphi)}$ whenever $(\varphi) \in \mathfrak{N} \times \cdots \times \mathfrak{N}$; thus η is continuous on cylinder sets if and only if for every nuclear subspace $\mathfrak{N}$ of $\mathfrak{X}'$, the corresponding λ is continuous on cylinder sets. In the dual of a nuclear space $\mathfrak{N}$ any measure λ of cylinder sets is continuous on cylinder sets, provided its Fourier transform is continuous at the null element of $\mathfrak{N}$. For each $\mathfrak{N}$, $\lambda^\wedge$ agrees with $\eta^\wedge$ on $\mathfrak{N}$; thus the lemma will be proved once $\eta^\wedge$ is shown to be continuous at $0 \in \mathfrak{X}'$:

Let $\varphi_1, \cdots, \varphi_k$ be such that $\mathcal{C}$ misses $\bigcap_{i=1}^k \{X(\varphi_i) = 0\}$; then $\mathcal{C}$ has the form $\{(X(\varphi_1), \cdots, X(\varphi_N)) \in C\}$ for some $N \geqq k$. The $\varphi_1, \cdots, \varphi_N$ can be assumed to be linearly independent. Let $\varphi_0 \in \mathfrak{X}'$ be variable, and let $\mathcal{S} = \mathcal{S}_{\varphi_0}$ be the (variable) span of $\varphi_0, \varphi_1, \cdots, \varphi_N$; then

$$\eta^\wedge(\varphi_0) = \int_{P(\mathcal{C})} e^{ix(\varphi_0)} \, d\nu_{\mathcal{S}}(x),$$

where $P(X) = X + \mathcal{S}^\circ$. Thus $\eta^\wedge(\varphi_0) = \int_{\mathcal{R} \times \mathcal{C}} e^{it_0} \, d\nu_{(\varphi)}(t)$; $t = (t_0, \cdots, t_N)$; $(\varphi) = (\varphi_0, \varphi_1, \cdots, \varphi_N)$. Clearly $(0, \cdots, 0) \notin C$, and Lemma 5.2 implies C is closed. By Lemma 5.3 there exists a measurable set $D \supseteqq C$ with $(0, \cdots, 0) \notin \bar{D}$, such that for any sequence φ_{0k} converging to 0 in $\mathfrak{X}'$, writing $(\varphi)_k = (\varphi_{0k}, \varphi_1, \cdots \varphi_N)$,

$$\lim_k \int_{\mathcal{R} \times D} e^{it_0} \, d\nu_{(\varphi)_k}(t) = \nu(\{(X(\varphi_1), \cdots, X(\varphi_N)) \in D\}),$$

since $e^{it_0}/|t|^2$ is continuous and bounded on $\mathcal{R} \times \bar{D}$ and vanishes as $|t| \to \infty$. Thus the Fourier transform of $\nu(\cdot \cap \{(X(\varphi_1), \cdots, X(\varphi_N)) \in D\})$ is continuous at $\varphi_0 \equiv 0$. This finite measure of cylinder sets is therefore countably additive on cylinder sets. The conclusion follows from $D \supseteqq C$.

It now remains to show that the measure ν of cylinder sets satisfies the continuity condition of Theorem 5.4.

Given $(\varphi) = (\varphi_1, \cdots, \varphi_N)$ in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$, let $\mu_{nk(\varphi)}(B) = \mu_{nk}(\{(X(\varphi_1), \cdots,$

$X(\varphi_N)) \varepsilon B\}$ define the measures $\mu_{nk(\varphi)}$ in $\mathcal{R}^N$; define $\mu_{(\varphi)}$ analogously. Then the measures $\mu_{nk(\varphi)}$ have bounded variances and $\prod_k \mu_{nk(\varphi)} \to_n \mu_{(\varphi)}$, so that the finite-dimensional CCC applies. In this context, the CCC guarantees a unique measure $\nu^{(\varphi)}$ in $\mathcal{R}^N$, a linear form $A_{(\varphi)}$ on $\mathcal{R}^N$ and non-negative definite bilinear form $\|u\|_{(\varphi)} = \langle u, u\rangle_{(\varphi)}$ in $\mathcal{R}^{N'}$, such that for all u and v in $\mathcal{R}^{N'}$,

$$(2'') \qquad \log \mu^{\wedge}{}_{(\varphi)}(u) = iA_{(\varphi)}(u) - \tfrac{1}{2} \|u\|^2_{(\varphi)} + \int_{\mathcal{R}^N} [e^{it(u)} - 1 - it(u)]\, d\nu^{(\varphi)}(t),$$

together with the other analogues (3″), (4″), and (5″) of (3′), (4′) and (5′) respectively, obtained by replacing the symbol φ with (φ), $A_\varphi u$ with $A_{(\varphi)}(u)$, tu with $t(u) = \sum_{j=1}^N t_j u_j$, $|t|^2$ with $\sum_{j=1}^N t_j^2$, and $\sigma^2{}_\varphi u^2$ with $\|u\|^2{}_{(\varphi)}$.

LEMMA 5.5 *For each u, as a function of $(\varphi) = (\varphi_1, \cdots, \varphi_N)$, $\int_{\mathcal{R}^N} [e^{it(u)} - 1 - it(u)]\, d\nu^{(\varphi)}(t)$ is continuous relative to the product topology in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$.*

PROOF. This follows from (2″), from the identity $\mu^{\wedge}{}_{(\varphi)}(u) = \mu^{\wedge}(\sum_{j=1}^N u_j\varphi_j)$, and the continuity of $\mu^{\wedge}$, once it is shown that $A_{(\varphi)}(u)$ and $\|u\|^2{}_{(\varphi)}$ are continuous in the variable $(\varphi) = (\varphi_1, \cdots, \varphi_N)$.

From (5″),

$$A_{(\varphi)}(u) = \lim_n \sum_k \int_{\mathfrak{X}} X(\sum_{j=1}^N u_j\varphi_j)\, d\mu_{nk}(X).$$

As in the initial argument in the CCC, continuity and linearity in (φ) is assured by the Banach-Steinhaus theorem, applied this time to the Barrelled space $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$.

For each u, (4″) can be used to compute $\tfrac{1}{2}(\|u\|^2_{(\varphi)+(\theta)} - \|u\|^2_{(\varphi)} - \|u\|^2_{(\theta)})$ for the variables (φ) and (θ) in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$. Further repeating the argument for $N = 0$, a double application of the Banach-Steinhaus theorem shows that this quantity is continuous and linear (separately) in each of the variable n-tuples (φ) and (θ). Taking $(\theta) = (\varphi)$, $\|u\|^2_{(\varphi)}$ is continuous on $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$; this finishes the proof.

In the following result we use the uniformness over converging sequences, in the notion of bounded variances.

LEMMA 5.6. *For any sequence $(\varphi)_\alpha = (\varphi_{0\alpha}, \cdots, \varphi_{N\alpha})$ converging in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$ the sequence $|t|^2\, \nu^{(\varphi)\alpha}$ of measures in $\mathcal{R}^{N+1}$ is w^*-compact.*

PROOF. By the w^*-compactness theorem, it suffices to prove

$$\sup_\alpha \int |t|^2\, d\nu^{(\varphi)\alpha}(t) < \infty.$$

$-\int |t|^2\, d\nu^{(\varphi)\alpha}(t)$ is the value at $u = (0, \cdots, 0)$ of the Laplacian of g, where

$$g(u) = \int [e^{it(u)} - 1 - it(u)]\, d\nu^{(\varphi)\alpha}(t).$$

Fix α and let $\mathcal{S}$ be the span of $\varphi_{0\alpha}, \cdots, \varphi_{N\alpha}$. With a change of variable, (2″) yields

$$g(u) = \log \mu^{\wedge}(\textstyle\sum_j u_j\varphi_{j\alpha}) - iA_{\mathcal{S}}(\sum_j u_j\varphi_{j\alpha}) + \tfrac{1}{2}\langle \sum_j u_j\varphi_{j\alpha}, \sum_j u_j\varphi_{j\alpha}\rangle_{\mathcal{S}},$$

where $A_{\mathcal{S}}$ is continuous and linear on $\mathfrak{X}/\mathcal{S}^\circ$ and $\langle\cdot, \cdot\rangle$ is separately continuous and bilinear on $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$. Using this to compute the Laplacian of g and evaluating at $u = (0, \cdots, 0)$ produces the value

$$\sum_{j=1}^{N} [\int X(\varphi_{j\alpha})^2 \, d\mu(X) - (\int X(\varphi_{j\alpha}) \, d\mu(X))^2] + \sum_{j=0}^{N} \|\varphi_{j\alpha}\|_{\mathcal{S}}^2 .$$

Since for each j $\varphi_{j\alpha}$ converges in $\mathfrak{X}'$ as $\alpha \to \infty$,

$$\sup_\alpha \int X(\varphi_{j\alpha})^2 \, d\mu(X) < \infty ;$$

thus the first sum is bounded uniformly in α. As for the second sum, (4″) with a change of variable yields

$$\|\varphi_{j\alpha}\|_{\mathcal{S}}^2 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sum_k [\int_{D(\epsilon)} X(\varphi_{j\alpha})^2 \, d\mu_{nk}(X) - (\int_{D(\epsilon)} X(\varphi_{j\alpha}) \, d\mu_{nk}(X))^2],$$

where $D(\epsilon) = \{|X(\varphi_{j\alpha})| < \epsilon\}$. The condition

$$\sup_{\alpha, n} \sum_k \int X(\varphi_{j\alpha})^2 \, d\mu_{nk}(X) < \infty$$

for each j implies $\sup_\alpha \|\varphi_{j\alpha}\|_{\mathcal{S}}^2 < \infty$; so the second sum is also bounded uniformly in α. Thus

$$\sup_\alpha \int |t|^2 \, d\nu^{(\varphi)\alpha}(t) < \infty,$$

finishing the proof.

LEMMA 5.7. *The map* $(\varphi) \to |t|^2 \nu^{(\varphi)}$ *from* $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$ *is continuous, relative to* w^**-convergence of the measures* $|t|^2 \nu^{(\varphi)}$.

PROOF. Let $(\varphi)_\alpha \to$ be a converging sequence in $\mathfrak{X}' \times \cdots \times \mathfrak{X}'$, say $(\varphi)_\alpha \to_\alpha (\varphi) = (\varphi_0, \cdots, \varphi_N)$. By Lemma 5.6 there exists a w^*-converging subsequence $|t|^2 \nu^{(\varphi)\alpha}$ of the sequence $|t|^{2(\varphi)\alpha}$, say

$$|t|^2 \nu^{(\varphi)\alpha} \to_\alpha |t|^2 \eta(w^*).$$

If V is a rectangle containing $(0, \cdots, 0)$ in its in interior, with vertices at points t where $\eta(\{t\}) = 0$, then

$$\lim_\alpha \int_{t \notin V} [e^{it(u)} - 1 - it(u)] \, d\nu^{(\varphi)\alpha}(t) = \int_{t \notin V} [e^{it(u)} - 1 - it(u)] \, d\eta(t),$$

since $(e^{it(u)} - 1 - it(u))/|t|^2$ is bounded and continuous on $t \neq 0$ and vanishes at ∞. By taking V so that $\int_V |t|^2 \, d\eta(t)$ becomes arbitrarily small, the same convergence can be seen, of these integrals over all of $\mathfrak{R}^{N+1}$. But by Lemma 5.5

$$\lim_\alpha \int [e^{it(u)} - 1 - it(u)] \, d\nu^{(\varphi)\alpha}(t) = \int [e^{it(u)} - 1 - it(u)] \, d\nu^{(\varphi)}(t),$$

so by Lemma 4.1, $|t|^2 \eta = |t|^2 \nu^{(\varphi)}$. Since any converging subsequence of $|t|^2 \nu^{(\varphi)\alpha}$ thus converges (w^*) to $|t|^2 \nu^{(\varphi)}$ and the sequence is w^*-compact, it also converges (w^*) to $|t|^2 \nu^{(\varphi)}$. This finishes the proof.

THEOREM 5.8. *For each closed cylinder set* $\mathfrak{C}$ *omitting some finite intersection of closed hyperplanes, the measure* $\nu(\cdot \cap \mathfrak{C})$ *of cylinder sets is countably additive on cylinder sets.*

PROOF. In view of Theorem 5.4 and Lemma 5.7, it will be enough to show $\nu_{(\varphi)} = \nu^{(\varphi)}$ for any $(\varphi) = (\varphi_1, \cdots, \varphi_N)$. Since $\nu^{(\varphi)}(\{0\}) = 0$ by (2″) and

$\nu_{(\varphi)}(\{0\}) = 0$ by construction, the condition $\nu_{(\varphi)}(\cdot \cap F) = \nu^{(\varphi)}(\cdot \cap F)$ for each closed F omitting $\{0\}$, will suffice. By (3″)

$$\nu^{(\varphi)}(\cdot \cap F) = \lim_n \sum_k \mu_{nk(\varphi)}(\cdot \cap F).$$

On the other hand, letting $\mathcal{S}$ denote the span of $\varphi_1, \cdots, \varphi_N$,

$$\begin{aligned}\nu_{(\varphi)}(\cdot \cap F) &= \nu^{\mathcal{S}}(\cdot \cap \{X + \mathcal{S}^0 : (X(\varphi_1), \cdots, X(\varphi_N)) \varepsilon F\}) \\ &= \lim_n \sum_k \mu_{nk\mathcal{S}}(\cdot \cap \{X + \mathcal{S}^0 : (X(\varphi_1), \cdots, X(\varphi_N)) \varepsilon F\}) \\ &= \lim_n \sum_k \mu_{nk(\varphi)}(\cdot \cap F),\end{aligned}$$

by (3‴) and a change of variable. This finishes the proof.

LEMMA 5.9. *If the μ_{nk} have bounded variances and $\prod_k \mu_{nk} \to \mu$, then*

$$\sum_k \mu_{nk}(\cdot \cap \mathcal{C}) \to_n \nu(\cdot \cap \mathcal{C})$$

for every closed cylinder set $\mathcal{C}$ omitting some finite intersection of closed hyperplanes.

PROOF. By Theorem 2.1, it is enough to show the Fourier transforms converge point-wise; i.e., that

$$\sum_k \int_{\mathcal{C}} e^{iX(\varphi)}\, d\mu_{nk}(X) \to_n \int_{\mathcal{C}} e^{iX(\varphi)}\, d\mu(X)$$

for each φ in $\mathfrak{X}'$. As in the proof of Theorem 5.4 we may write $\mathcal{C} = \{(X(\varphi_1), \cdots, X(\varphi_N)) \varepsilon C\}$ where $\mathcal{C}$ misses $\bigcap_{i=1}^{k} \{X(\varphi_i) = 0\}$ for some $k \leqq N$ and C is closed in $\mathcal{R}^N$. Letting $\mathcal{S}$ denote the span of $\varphi_1, \cdots, \varphi_N$ and any given φ, the above convergence is equivalent with

$$\sum_k \int_{P(\mathcal{C})} e^{ix(\varphi)}\, d\mu_{nk\mathcal{S}}(x) \to_n \int_{P(\mathcal{C})} e^{ix(\varphi)}\, d\mu(X),$$

where $P(X) = x = X + \mathcal{S}^\circ$. $P(\mathcal{C})$ is closed by Lemma 5.2, and omits $\{\mathcal{S}^\circ\}$, hence condition (3‴) of the finite-dimensional CCC applies:

$$\sum_k \mu_{nk\mathcal{S}}(\cdot \cap P(\mathcal{C})) \to_n \nu(\cdot \cap P(\mathcal{C})).$$

The Fourier transforms therefore converge, completing the proof.

In conclusion, some remarks on infinitely divisible measures are in order: If μ is infinitely divisible and has finite covariances $\int_{\mathfrak{X}} X(\varphi)\, X(\theta)\, d\mu(X)$, there exist measures μ_n $(n \geqq 1)$ such that $\mu = (\mu_n)^n$. The double sequence μ_{nk}, where $\mu_{nk} = \mu_n$ $(k = 1, \cdots, n)$ then enjoys the two conditions comprising boundedness of variances: The first of these conditions in this case can be expressed

$$\sup_\alpha \int X(\varphi_\alpha)^2\, d\mu_n(X) \to_n 0$$

for every converging sequence $\{\varphi_\alpha\}$. This is satisfied, since $n \int X(\varphi_\alpha)^2\, d\mu_n(X) = \int X(\varphi_\alpha)^2\, d\mu(X)$ and $\int X(\varphi)^2\, d\mu(X)$, being continuous in φ, is bounded on the relatively compact range of $\{\varphi_\alpha\}$. The second of these conditions becomes $\sup_\alpha \int X(\varphi_\alpha)^2\, d\mu(X) < \infty$. Hence for any such μ, $\log \mu^\wedge(\varphi)$ has the form indicated in condition (2) of Theorem 3.1. Conversely, it is clear that any μ for which $\log \mu^\wedge(\varphi)$ has that form is necessarily infinitely divisible with finite covariances.

REFERENCES

[1] GEL'FAND, I. M. and VILENKIN, N. YA. (1964). *Generalized Functions* **4**. Academic Press, New York. (Translation)

[2] HALMOS, P. R. (1950). *Measure Theory*. Van Nostrand, New York.

[3] HORVATH, J. H. (1966). *Topological Vector Spaces and Distributions* **1**. Addison-Wesley, Reading.

[4] LOÈVE, M. (1960). *Probability Theory* (2nd Edition). Van Nostrand, New York.

[5] PARTHASARATHY, K. R., RAO R. RANGA and VARADHAN, S. R. S. (1963). Probability distributions on locally compact Abelian groups. *Illinois J. Math.* **7** 337–369.

[6] TAKANO, K. (1956). Central convergence criterion in the multidimensional case. *Ann. Inst. Statist. Math. Tokyo* **7** 95–102.