The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1439–1448

DISTRIBUTION OF A PRODUCT AND THE STRUCTURAL SET UP OF DENSITIES[1]

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1. Introduction and summary. This paper is a study of statistical distributions by analyzing the structure of density functions. This work was motivated by the paper [10] in which the distribution of the product of two non-central chi-square variables is obtained. Their problem arose from a problem in the physical sciences connected with the theory of spinstabilized rockets. It is pointed out in [10] that the original problem was to obtain the distribution of the product of a central Raleigh and a non-central Raleigh variable. By examining the structure of the density function of non-central Raleigh or non-central chi-square it is apparent that the density is the product of the special cases of two special functions. In [8, chapter 2] a lengthy treatment of Raleigh and associated distributions is given. We do not know any other particular problem in the physical sciences, but from the structural property of the problem pointed out in [10] it is quite likely that there may be a number of such problems in the physical sciences or in other disciplines, which are to be tackled. Hence we will give the distribution of the product of two independent stochastic variables whose density functions can be expressed as the product of any two special functions. Products of H-functions are used so that almost all classical density functions (central or non-central) will be taken care of, since H-functions are the most generalized special functions. Since a number of types of factors can be absorbed inside the H-function we think that all the distributions which are frequently used in the statistical theory of distributions will be included in this problem that we discuss here, with some modifications in some cases. Several special cases are pointed out so that one can easily get the distribution of the product or ratio of independent stochastic variables whose density functions are products of special functions. The result in [10] is obtained as a special case and further, the distributions of the products of two non-central F simple and multiple correlation coefficients are pointed out for the sake of mathematical interest because, structurally, the density functions in these cases belong to different categories. Since several properties of H-functions are available in the literature, it is easy to study other properties or to compute percentage points of the product distribution discussed in this paper. The approach of examining the structure of densities may simplify the problem of obtaining distributions of several statistics.

2. Some definitions and basic results. In this section some definitions and some preliminary results which are used in the derivation of the distribution of a

Received 11 June 1968.

[1] This work was done at the summer research institute of the Canadian Mathematical Congress, Montreal branch. The authors acknowledge with thanks the grant by the National Research Council of Canada which made the collaboration possible.

product of independent stochastic variables are given. Even though a general definition of H-functions is given here, only special forms are used due to restriction of non-negativity of the functions entering into a density function.

2.1. *H-Function.* The H-function is defined as [1]

$$(1)\qquad H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_q, B_q)\end{matrix}\right.\right] = (1/2\pi i)\int_C h(s)x^s\, ds,$$

where x is not equal to zero and,

$$x^s = \exp\{s(\log|x| + i\arg x)\}$$

in which $\log|x|$ denotes the natural logarithm of $|x|$ and $\arg x$ is not necessarily the principal value;

$$h(s) = [\textstyle\prod_{j=1}^m \Gamma(b_j - sB_j)\prod_{j=1}^n \Gamma(1 - a_j + sA_j)]/ [\prod_{j=m+1}^q \Gamma(1 - b_j + sB_j)\prod_{j=n+1}^p \Gamma(a_j - sA_j)]$$

where p, q, m, n are integers such that $0 \leqq n \leqq p$, $1 \leqq m \leqq q$, $A_j (j = 1, 2, \cdots, p)$, $B_j (j = 1, 2, \cdots, q)$ are positive integers and $a_j (j = 1, 2, \cdots, p)$, $b_j (j = 1, 2, \cdots, q)$ are complex numbers such that

$$A_j(b_h + v) \neq B_h(a_j - 1 - \lambda) \qquad \text{for } v, \lambda = 0, 1, \cdots; \quad h = 1, \cdots, m; j = 1, \cdots, n;$$

C is a contour in the complex s-plane such that the points,

$$s = (b_j + v)/B_j \qquad (j = 1, \cdots, m; v = 0, 1, \cdots)$$

and

$$s = (a_j - 1 - v)/A_j \qquad (j = 1, \cdots, n; v = 0, 1, \cdots)$$

lie to the right and left of C respectively, while C runs from $s = \infty - ik$ to $s = \infty + ik$ where k is a constant ($k > |\text{Im}\, b_j|/B_j$, $j = 1, 2, \cdots, m$). Whenever there is no confusion the following simplified notations will be used;

$$(2)\qquad H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_q, B_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[x\left|\begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right.\right]$$

$$(3)\qquad (a)_n = a(a+1)\cdots(a+n-1)$$

$$(4)\qquad \textstyle\prod_1^n \Gamma(a_i) = \Gamma(a_1)\cdots\Gamma(a_n).$$

Empty sums are to be interpreted as zero and empty products as unity.

2.2. *Some special properties of the H-function.* The following special properties are of importance when the distributions of some transformed variables are under consideration.

$$(5)\qquad H_{p,q}^{m,n}\left[x^\mu\left|\begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right.\right] = (1/\mu)H_{p,q}^{m,n}\left[x\left|\begin{matrix}((a_\lambda, A_\lambda/\mu))\\((b_\lambda, B_\lambda/\mu))\end{matrix}\right.\right]$$

for $\mu > 0$.

$$(6)\qquad H^{m,n}_{p,q}\left[(1/x)\middle|\begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right] = H^{n,m}_{q,p}\left[x\middle|\begin{matrix}((1-b_\lambda, B_\lambda))\\((1-a_\lambda, A_\lambda))\end{matrix}\right]$$

$$(7)\qquad x^\sigma H^{m,n}_{p,q}\left[x\middle|\begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right] = H^{m,n}_{p,q}\left[x\middle|\begin{matrix}((a_\lambda+\sigma A_\lambda, A_\lambda))\\((b_\lambda+\sigma B_\lambda, B_\lambda))\end{matrix}\right]$$

These results are the direct consequences of the definition of the H-function. Since the Mellin transform of the H-function is the coefficient of x^{-s} in the integrand of (1), we therefore have

$$\int_0^\infty x^{s-1} H^{m,n}_{p,q}\left[x\middle|\begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right] dx$$

$$(8)\qquad = \prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\Big/ \left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right]$$

where $R(b_j+sB_j)>0$ for $j=1,2,\cdots,m$ and $R(1-a_j-A_js)>0$, for $j=1,\cdots,n$; where $R(\cdot)$ means the real part of $(\cdot)$.

2.3. *Special cases.* The special cases of the H-function include the functions such as Meijer's G-functions, Bessel, Whittaker, Struve Generalized hypergeometric functions, Boersma functions, MittagLeffler functions, E-functions, and several others ([1], [7]). Some of these which occur frequently in the density functions will be pointed out here.

Meijer's G-function:

$$(9)\qquad G^{m,n}_{p,q}\left[x\middle|\begin{matrix}a_1,\cdots,a_p,\\b_1,\cdots,b_q,\end{matrix}\right].$$

Put $A_1=A_2=\cdots=A_p=1=B_1=B_2=\cdots=B_q$, in the H-function, then a G-function is obtained. A general G-function is present in the density function of the exact distribution of the likelihood ratio criterion [3].

Generalized hypergeometric function: ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$.

$$(10)\qquad {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x) = \left[\prod_{j=1}^{q}\Gamma(b_j)\Big/\prod_{j=1}^{p}\Gamma(a_j)\right]$$

$$\cdot H^{1,p}_{p,q+1}\left[-x\middle|\begin{matrix}(1-a_1,1),\cdots(1-a_p,1)\\(0,1),(1-b_1,1),\cdots,(1-b_q,1)\end{matrix}\right].$$

From this relationship one can obtain the generalized hypergeometric distribution associated with ${}_2F_1(a, b; c; dx)$ and several other interesting statistical distributions associated with the special cases ${}_1F_1(a;b;cx)$ and ${}_0F_1(a;bx)$, (see [7]).

$$(11)\qquad x^{\alpha} \exp(-px^{\beta}) = p^{-\alpha/\beta} H_{0,1}^{1,0}[px^{\beta} \mid (\alpha/\beta, 1)].$$

$$(12)\qquad x^{\alpha}/(1 + ax^{\beta}) = x^{\alpha}{}_1F_0(1; -ax^{\beta}) = x^{\alpha} H_{1,1}^{1,1}\left[ax^{\beta} \,\middle|\, \begin{matrix}(0, 1)\\(0, 1)\end{matrix}\right]$$

$$= a^{-\alpha/\beta} H_{1,1}^{1,1}\left[ax^{\beta} \,\middle|\, \begin{matrix}(\alpha/\beta, 1)\\(\alpha/\beta, 1)\end{matrix}\right].$$

Several more special functions can be easily obtained. So, structurally, if a density function contains products of two H-functions then apart from the simple classical densities, the densities of non-central chi-square, non-central t, non-central F, and naturally the corresponding central ones, multiple and simple correlation coefficients, correlation ratio, the likelihood ratio criterion, special cases of the largest and smallest eigenvalues of certain generalized variances [9], are all contained in the product of H-functions. The distributions obtained in this article will include the distribution of the product or ratio of any two independent stochastic variables whose densities are any of the ones mentioned above or which are representable in terms of product of two special functions, as special cases.

3. The distribution of the product. Let $W = XY$, where X and Y are two independent stochastic variables whose densities are each products of two H-functions. The density function of W is obtained in a simplified form by using the properties of Mellin transforms. Since the mathematical expectation of a product of two independent stochastic variables is the product of the expectations, when the expectations exist, the Mellin transform of a product is the product of the Mellin transforms when the variables are statistically independent. After obtaining the Mellin transform of W the density function of W is obtained by the inverse Mellin transform. In order to simplify the results, an expansion of the H-function, which is quoted as lemma 1 in [1], is used. The density function of W is obtained with the help of a number of lemmas which are given in this section. Since there are a number of parameters at our choice, the H-functions which occur in the density functions, without loss of generality, are assumed to be non-negative and defined for the appropriate ranges of values of the variables involved.

LEMMA 1. (See [1], (6.5).)

$$(13)\qquad H_{p,q}^{m,n}\left[ax \,\middle|\, \begin{matrix}((a_{\lambda}, A_{\lambda}))\\((b_{\lambda}, B_{\lambda}))\end{matrix}\right] = \chi(a_{\lambda}, b_{\lambda}, \nu_i)x^{(b_j+\nu_1)/B_j}, \qquad j \neq \lambda,$$

where

$$\chi(a_{\lambda}, b_{\lambda}, \nu_1) = \textstyle\sum_{j=1}^{m} \sum_{\nu_1=0}^{\infty} \{\prod_1^{*m} \Gamma[b_{\lambda} - B_{\lambda}(b_j + \nu_1)/B_j]\}/$$
$$\{\textstyle\sum_{m+1}^{q} \Gamma[1 - b_{\lambda} + B_{\lambda}(b_j + \nu_1)/B_j]\}$$
$$\times \{\textstyle\prod_1^{n} \Gamma[1 - a_{\lambda} + A_{\lambda}(b_j + \nu_1)/B_j](-1)^{\nu_1} a^{(b_j+\nu_1)/B_j}\}/$$
$$\{\textstyle\prod_{n+1}^{p} \Gamma[a_{\lambda} - A_{\lambda}(b_j + \nu_1)/B_j]\nu_1!\,B_j\}, \qquad j \neq \lambda.$$

$\prod^*$ means the product of the factors with $\lambda = 1, 2, \cdots, m$ save $\lambda = j$ where the following conditions hold:
$B_j(b_\lambda + \lambda_1) \neq B_\lambda(b_j + \nu_1)$ for $\lambda \neq j, j = 1, \cdots, m; \lambda_1, \nu_1 = 0, 1, 2, \cdots; M > 0$ for $x \neq 0$ and $M = 0$ for $0 < |x| < 1/\epsilon$, $\epsilon = \prod_1^p A_j^{A_j} \prod_1^q B_j^{-B_j}$ and $M = \sum_1^q B_j - \sum_1^p A_j$.

LEMMA 2.

$$\int_0^\infty x^{\sigma-1} H_{p,q}^{m,n}\left[\alpha x \,\middle|\, \begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right] H_{r,s}^{k,l}\left[\beta x \,\middle|\, \begin{matrix}((c_\lambda, C_\lambda))\\((d_\lambda, D_\lambda))\end{matrix}\right] dx$$

$$\begin{aligned}(14)\quad &= \chi(a_\lambda, b_\lambda, \nu_1)\beta^{-\sigma-(b_j+\nu_j)/B_j}\\ &\times \{\textstyle\prod_1^k \Gamma[d_\mu + \sigma D_\mu + (b_j + \nu_1)D_\mu/B_j]\}/\\ &\{\textstyle\prod_{k+1}^s \Gamma[1 - d_\mu - \sigma D_\mu - D_\mu(b_j + \nu_1)/B_j]\}\\ &\times \{\textstyle\prod_1^l \Gamma[1 - c_\mu - C_\mu\sigma - (b_j + \nu_1)C_\mu/B_j]\}/\\ &\{\textstyle\prod_{l+1}^r \Gamma[c_\mu + \sigma C_\mu + C_\mu(b_j + \nu_1)/B_j]\}\end{aligned}$$

$$(15)\quad = \alpha^{-\sigma} H_{q+r,p+s}^{k+n,l+m}\left[\beta/\alpha \,\middle|\, \begin{matrix}((1 - b_\lambda - B_\lambda\sigma, B_\lambda)), ((c_\lambda, C_\lambda))\\((1 - a_\lambda - A_\lambda\sigma, A_\lambda)), ((d_\lambda, D_\lambda))\end{matrix}\right].$$

PROOF. To prove (15) apply the formula (8) twice. The conditions on the parameters are as follows: $R(\sigma + b_j/B_j + d_i/D_i) > 0$ for $j = 1, \cdots, m; i = 1, \cdots, k$; $R(\sigma + (a_j - 1)/A_j + (c_j - 1)/C_j) < 0$ for $j = 1, \cdots, n; i = 1, \cdots, l$; $|\arg \alpha| < \lambda_1\pi/2$; $|\arg \beta| < \lambda_2\pi/2$ where,

$$\lambda_1 = \textstyle\sum_1^m B_j - \sum_{m+1}^q B_j + \sum_1^n A_j - \sum_{n+1}^p A_j$$
$$\lambda_2 = \textstyle\sum_1^k D_j - \sum_{k+1}^s D_j + \sum_1^l C_j - \sum_{l+1}^r C_j; \qquad \lambda_1, \lambda_2 > 0.$$

From this lemma, the normalizing factors for the density, defined in terms of the product of two H-functions, can be obtained by putting $\sigma = 1$.

The technique used to prove (14) is to expand one of the H-functions by lemma 1 and then apply the result (8). That is

$$(16)\quad \int_0^\infty x^{\sigma-1} H_{p,q}^{m,n}\left[\alpha x \,\middle|\, \begin{matrix}((a_\lambda, A_\lambda))\\((b_\lambda, B_\lambda))\end{matrix}\right] H_{r,s}^{k,l}\left[\beta x \,\middle|\, \begin{matrix}((c_\lambda, C_\lambda))\\((d_\lambda, D_\lambda))\end{matrix}\right] dx$$

$$= \chi(a_\lambda, b_\lambda, \nu_1) \int_0^\infty x^{\sigma+(b_j+\nu_1)/B_j-1} H_{r,s}^{k,l}\left[\beta x \,\middle|\, \begin{matrix}((c_\lambda, C_\lambda))\\((d_\lambda, D_\lambda))\end{matrix}\right] dx$$

$$\begin{aligned}(17)\quad &= \chi(a_\lambda, b_\lambda, \nu_1)\beta^{-\sigma-(b_j+\nu_1)/B_j}\\ &\times \{\textstyle\prod_1^k \Gamma[d_\mu + \sigma D_\mu + (b_j + \nu_1)D_\mu/B_j]\}/\\ &\{\textstyle\prod_{k+1}^s \Gamma[1 - d_\mu - \sigma D_\mu - D_\mu(b_j + \nu_1)/B_j]\}\\ &\times \{\textstyle\prod_1^l \Gamma[1 - c_\mu - C_\mu\sigma - (b_j + \nu_1)C_\mu/B_j]\}/\\ &\{\textstyle\prod_{l+1}^r \Gamma[c_\mu + \sigma C_\mu + C_\mu(b_j + \nu_1)/B_j]\}.\end{aligned}$$

LEMMA 3.

$$\int_0^\infty x^{\sigma-1} H_{p,q}^{m,n}\left[\alpha x \,\middle|\, \begin{matrix} ((a_\lambda, A_\lambda)) \\ ((b_\lambda, B_\lambda)) \end{matrix}\right] H_{r,s}^{k,l}\left[\beta x \,\middle|\, \begin{matrix} ((c_\lambda, C_\lambda)) \\ ((d_\lambda, D_\lambda)) \end{matrix}\right] dx$$

$$\times \int_0^\infty x^{\sigma-1} H_{p',q'}^{m',n'}\left[\gamma x \,\middle|\, \begin{matrix} ((a_\tau', A_\tau')) \\ ((b_\tau', B_\tau')) \end{matrix}\right] H_{r',s'}^{k',l'}\left[\delta x \,\middle|\, \begin{matrix} ((c_\eta', C_\eta')) \\ ((d_\eta', D_\eta')) \end{matrix}\right] dx \tag{18}$$

$$= \chi(a_\lambda, b_\lambda, t)\chi(a_\tau', b_\tau', \nu_1 - t)\beta^{-b_j/B_j - t/B_j}\delta^{-(b_\tau' - t + \nu_1)/B_\tau'}\ \xi(\sigma),$$

where,

$$\xi(\sigma) = (\beta\delta)^{-\sigma}$$

$$\times \frac{\prod_1^k \Gamma(d_\mu + \sigma D_\mu + (b_j + t)D_\mu/B_j) \cdot \prod_1^l \Gamma[1 - c_\mu - \sigma C_\mu - (b_j + t)C_\mu/B_j]}{\prod_{k+1}^s \Gamma[1 - d_\mu - \sigma D_\mu - D_\mu(b_j + t)/B_j] \cdot \prod_{l+1}^r \Gamma[c_\mu + \sigma C_\mu + c_\mu\,(b_j + t)/B_j]} \tag{19}$$

$$\times \frac{\prod_1^{k'} \Gamma[d_\eta' + \sigma D_\eta' + (b_\tau' + \nu_1 - t)\, D_\eta'/B_\tau'] \cdot \prod_1^{l'} \Gamma[1 - c_\eta' - \sigma C_\eta' - (b_\tau' + \nu_1 - t)C_\eta'/B_\tau']}{\prod_{k'+1}^{s'} \Gamma[1 - d_\eta' - \sigma D_\eta' - D_\eta'\,(b_\tau' + \nu_1 - t)/B_\tau'] \cdot \prod_{l'+1}^{r'} \Gamma[c_\eta' + \sigma C_\eta' + C_\eta'\,(b_\tau' + \nu_1 - t)]/B_\tau']}.$$

The inverse Mellin transform of (18) can be obtained by taking the inverse Mellin transform of each term of the series (18). Hence the following lemma can be given.

LEMMA 4.

$$\chi(a_\lambda, b_\lambda, t)\chi(a_\tau', b_\tau', \nu_1 - t)\beta^{-(b_j+t)/B_j}\delta^{(t-\nu_1-b_\tau')/B_\tau'}$$

$$\times\ (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} \xi(\sigma)(\beta\delta)^{-\sigma}\omega^{-\sigma}\, d\sigma$$

$$= \chi(a_\lambda, b_\lambda, t)\chi(a_\tau', b_\tau', \nu_1 - t)\beta^{-(b_j+t)/B_j}\delta^{(t-\nu_1-b_\tau')/B_\tau'} \tag{20}$$

$$\times H_{r+r',s+s'}^{k+k',l+l'}\left[\beta\delta\omega \,\middle|\, \begin{matrix} ((c_\mu + (b_j + t)(C_\mu/B_j, C_\mu)), ((c_\eta' + (b_\tau' + \nu_1 - t)C_\eta'/B_\tau'), c_\eta')) \\ ((d_\mu + (b_j + t)D_\mu/B_j, D_\mu)), ((d_\eta' + (b_\tau' + \nu_1 - t)D_\eta'/B_\tau', D_\eta')) \end{matrix}\right]$$

by virtue of (1).

THEOREM 1. *Let X and Y be two independent stochastic variables with density functions*

$$\alpha H_{p,q}^{m,n}\left[\alpha x \,\middle|\, \begin{matrix} ((a_\lambda, A_\lambda)) \\ ((b_\lambda, B_\lambda)) \end{matrix}\right] H_{r,s}^{k,l}\left[\beta x \,\middle|\, \begin{matrix} ((c_\mu, C_\mu)) \\ ((d_\mu, D_\mu)) \end{matrix}\right]$$

$$\div\ H_{q+r,p+s}^{k+n,l+m}\left[\beta/\alpha \,\middle|\, \begin{matrix} ((1 - b_\lambda - B_\lambda, B_\lambda)), ((c_\mu, C_\mu)) \\ ((1 - a_\lambda - A_\lambda, A_\lambda)), ((d_\mu, D_\mu)) \end{matrix}\right], \qquad x > 0$$

and

$$\gamma H_{p',q'}^{m',n'}\left[\gamma y \,\middle|\, \begin{matrix}((a_\tau', A_\tau'))\\((b_\tau', B_\tau'))\end{matrix}\right] H_{r',s'}^{k',l'}\left[\delta y \,\middle|\, \begin{matrix}((c_\eta', C_\eta'))\\((d_\eta', D_\eta'))\end{matrix}\right]$$

$$\div\, H_{q'+r',p'+s'}^{k'+n',l'+m'}\left[\delta/\gamma \,\middle|\, \begin{matrix}((1-b_\tau'-B_\tau', B_\tau')),\, ((c_\eta', C_\eta'))\\((1-a_\tau'-A_\tau', A_\tau')),\, ((d_\eta', D_\eta'))\end{matrix}\right], \qquad y > 0.$$

Then $W = XY$ *has the density*

$$f(w) = \chi(a_\lambda, b_\lambda, t)\chi(a_\tau', b_\tau', \nu_1 - t)\beta^{-(b_j+t)/B_j}\delta^{(t-\nu_1-b_\tau')/B_\tau'}$$

$$\times\, H_{r+r',s+s'}^{k+k',l+l'}\left[\beta\delta\omega \,\middle|\, \begin{matrix}((c_\mu + (b_j+t)(c_\mu/B_j), C_\mu)),\\ ((c_\eta' + (b_\tau' + \nu_1 - t)(c_\eta'/B_j), C_\eta'))\\ ((d_\mu + (b_j+t)(D_\mu/B_j), D_\mu)),\\ ((d_\eta' + (b_\tau' + \nu_1 - t)(D_\eta'/B_\tau'), D_\eta'))\end{matrix}\right]$$

(21)

$$\div\, (\alpha\gamma)^{-1} H_{q+r,p+s}^{k+n,l+m}\left[\beta/\alpha \,\middle|\, \begin{matrix}((1-b_\lambda-B_\lambda, B_\lambda)),\, ((c_\mu, C_\mu))\\((1-a_\lambda-A_\lambda, A_\lambda)),\, ((d_\mu, D_\mu))\end{matrix}\right]$$

$$\times\, H_{q'+r',p'+s'}^{k'+n',l'+m'}\left[\delta/\gamma \,\middle|\, \begin{matrix}((1-b_\tau'-B_\tau', B_\tau')),\, ((c_\eta', C_\eta'))\\((1-a_\tau'-A_\tau', A_\tau')),\, ((d_\eta', D_\eta'))\end{matrix}\right]$$

for $\omega > 0$.

The proof follows from the application of Lemmas 1–4. The normalizing factor is obtained from Lemma 2 by putting $\sigma = 1$ and then taking the corresponding products.

4. Special product distributions. As a special case of (21) we will obtain the distribution of the product of non-central chi-square variables given in [10], namely,

$$(22)\quad P(w) = e^{-\frac{1}{2}(\Delta_1^2+\Delta_2^2)} \sum_{m=0}^{\infty} \sum_{j=0}^{m} [\Delta_1^{2j}\Delta_2^{2(m-j)} 2^{1-\frac{1}{2}K_1-\frac{1}{2}K_2-2m+\frac{1}{4}K_2+\frac{1}{2}m-1}$$
$$\times\, K_{\frac{1}{2}K_1-\frac{1}{2}K_2-m+2j}^{(W^{\frac{1}{2}})}/j!(m-j)!\Gamma(j+\tfrac{1}{2}K_1)\Gamma(m-j+\tfrac{1}{2}K_2)],$$

where $W = Y_1Y_2$ and Y_1, Y_2 are independent non-central chi-square variables. The density function of the non-central chi-square variable is given by

$$(23)\quad P_j(y_j) = \tfrac{1}{2}(y_j/\Delta_j^2)^{\frac{1}{4}(K_j-2)} e^{-\frac{1}{2}(y_j+\Delta_j^2)} I_{\frac{1}{2}(K_j-2)}(\Delta_j y_j^{\frac{1}{2}}), \qquad j = 1, 2;\, y_j > 0$$

where Δ_1, Δ_2 are the non-centrality parameters and k_1 and k_2 are the degrees of freedom respectively. (22) can be obtained from (21) by putting $m = 1, n = p = 0, q = 2, k = 1, l = r = 0, s = 1, B_1 = B_2 = D_1 = 1 = B_1' = B_2' = D_1', b_1 = 0 = b_1', b_2 = 1 - k_1/2, b_2' = 1 - k_2/2, d_1 = k_1/2 - 1, d_1' = k_2/2 - 1, \alpha = -\Delta_1^2/4, \gamma = -\Delta_2^2/4, \beta = \frac{1}{2} = \delta$, and then using the formulae

$$(24)\qquad x^v e^{-x} = H_{0,1}^{1,0}[x \mid (v, 1)], \qquad G_{0,2}^{2,0}[x \mid v/2, -v/2] = 2K_v(2x^{\frac{1}{2}})$$

where $K_v(x)$ is the modified Bessel function of the third kind, defined by

$$K_v(x) = \pi(I_{-v}(x) - I_v(x))/2 \sin v\pi, \tag{25}$$

and

$$H_{0,2}^{1,0}[-x \mid (0, 1), (1 - k, 1)] = {}_0F_1(k; x)/\Gamma(k) = x^{(1-k)/2} I_{k-1}(2x^{\frac{1}{2}}).$$

Under these conditions the density functions of X and Y reduce to the density functions of non-central chi-square variables and consequently (21) reduces to (22).

4.1. *Product of likelihood ratios.* Put $m = p_1$, $n = 0$, $p = p_1 = q$, $a_1 = (n - p_1 - 1)/2$, $a_2 = (n - p_1)/2, \cdots, a_p = (n - 2)/2$, $A_1 = \cdots = A_p = 1 = B_1 = \cdots = B_q$, $b_1 = (n - p_1 - p_2)/2, \cdots, b_q = (n - p_2 - 2)/2$. Under these conditions a particular case of the distribution of a particular likelihood ratio criterion, as given in ([3], p. 1162), namely

$$\begin{aligned} P_3(x) &= \prod_{r=1}^{p_1} [\Gamma\{(n + 1 - r)/2\}/\Gamma\{(n + 1 - p_2 - r)/2\}] \\ &\times x^{(n-p_1-p_2-1)/2} G_{p_1,p_1}^{p_1,0}\left[x \,\middle|\, \begin{matrix} p_2/2, (p_2 + 1)/2, \cdots, (p_2 + p_1 - 1)/2 \\ 0, \frac{1}{2}, \cdots, (p_1 - 1)/2 \end{matrix}\right], \end{aligned} \tag{26}$$

$x > 0$, is obtained. Hence the distribution of the product of two independent likelihood ratios with the parameters n, p_1, p_2 and n', p_1', p_2' can be written as follows:

$$g(w) = \text{const.}\, H_{r+r',s+s'}^{k+k',l+l'}\left[\beta\delta w \,\middle|\, \begin{matrix} ((c_\mu, C_\mu)), ((c_\eta', C_\eta')) \\ ((d_\mu, D_\mu)), ((d_\eta', D_\eta')) \end{matrix}\right], \tag{27}$$

where $k = p_1$, $l = 0$, $r = p_1 = s$, $c_1 = (n - p_1 - 1)/2, \cdots, c_r = (n - 2)/2$, $C_1 = \cdots = C_r = 1 = D_1 = \cdots = D_s$, $d_1 = (n - p_1 - p_2 - 1)/2, \cdots, d_s = (n - p_2 - 2)/2$, and the second set of parameters are given by the corresponding values. This density can be simplified as

$$\begin{aligned} g(w) = \beta\delta &\textstyle\prod_{i=1}^{p_1} \{\Gamma(\frac{1}{2}(n - p_1 + i))/\Gamma(\frac{1}{2}(n - p_1 - p_2 + i))\} \cdot \\ &\textstyle\prod_{i=1}^{p_1'} \{\Gamma(\frac{1}{2}(n' - p_1' + i))/\Gamma(\frac{1}{2}(n' - p_1' - p_2' + i))\} \\ &\times G_{p_1+p_1',p_1+p_1'}^{p_1+p_1',0}\left[\beta\delta\omega \,\middle|\, \begin{matrix} \frac{1}{2}(n - p_1 - 1), \cdots, \frac{1}{2}(n - 2), \\ \frac{1}{2}(n' - p_1' - 1), \cdots, \frac{1}{2}(n' - 2) \\ \frac{1}{2}(n - p_1 - p_2 - 1), \cdots, \frac{1}{2}(n - p_2 - 2), \\ \frac{1}{2}(n' - p_1' - p_2' - 1), \cdots, \frac{1}{2}(n' - p_2' - 2) \end{matrix}\right]. \end{aligned} \tag{28}$$

4.2. *Multiple correlation coefficient.* The density function can be written in the form

$$\begin{aligned} f_1(y) = {} & [(1 - \rho^2)^{\frac{1}{2}(n-1)}\, \Gamma(\tfrac{1}{2}(n - 1))]/[\Gamma(\tfrac{1}{2}(p - 1))\Gamma(\tfrac{1}{2}(n - p))]\tfrac{1}{2} y^{\frac{1}{2}(p-1)} \\ & \cdot (1 - y)^{\frac{1}{2}(n-p)}\, {}_2F_1(\tfrac{1}{2}(n - 1), \tfrac{1}{2}(n - 1); \tfrac{1}{2}(p - 1); \rho^2 y), \end{aligned} \tag{29}$$

for $0 < y < 1$. The functional part can be considered to be the product of two H-functions which will reduce into a single H-function. Hence the density of the product of two multiple correlation coefficients with different sets of parameters can be obtained directly from (21). The result is

$$
\begin{aligned}
g_1(w) = {} & [(1-\rho^2)^{\frac{1}{2}(n-1)}(1-\rho'^2)^{\frac{1}{2}(n'-1)}\Gamma(\tfrac{1}{2}(n-1))\Gamma(\tfrac{1}{2}(n'-1))]/ \\
& [\Gamma(\tfrac{1}{2}(p-1))\Gamma(\tfrac{1}{2}(p'-1))] \\
& \times \textstyle\sum_{m=0}^{\infty}\sum_{j=0}^{m}[\tfrac{1}{2}(n-1)_j\,\tfrac{1}{2}(n-1)_j\,\tfrac{1}{2}(n'-1)_{m-j}\,\tfrac{1}{2}(n'-1)_{m-j}]/ \\
& [\tfrac{1}{2}(p-1)_j\tfrac{1}{2}(p'-1)_{m-j}\,j!\,(m-j)!] \\
& \times G_{22}^{20}\left[\omega\,\middle|\,\begin{matrix}\tfrac{1}{2}(n-3)+j,\ \tfrac{1}{2}(n'-3)+m-j\\ \tfrac{1}{2}(p-3)+j,\ \tfrac{1}{2}(p'-3)+m-j\end{matrix}\right],
\end{aligned} \tag{30}
$$

for $0 < w < 1$.

4.3. *Product of independent non-central Beta variates.* Since the non-central F can be obtained from non-central Beta and vice versa, we will discuss only the non-central Beta variates. The density function of a non-central Beta variate can be written as

$$
\begin{aligned}
f_2(x) = {} & [\exp(-\tfrac{1}{2}\lambda^2)\Gamma((k+m)/2)]/[\Gamma(k/2)\Gamma(m/2)]x^{k/2-1}(1-x)^{m/2-1} \\
& \times {}_1F_1((k+m)/2;\, k/2;\, \lambda^2x/2).
\end{aligned} \tag{31}
$$

The density function of W, which is the product of two independent non-central Beta variates with different parameters, can be obtained from (21) by making suitable changes in the parameters in the form

$$
\begin{aligned}
h_2(w) = {} & \exp[-\tfrac{1}{2}(\lambda^2+\lambda_1^2)][\Gamma\,\tfrac{1}{2}(k+m)\Gamma\,\tfrac{1}{2}(k'+m')]/ \\
& [\Gamma(k/2)\Gamma(k'/2)] \\
& \times \sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{\{(k+m)/2\}_j\,\tfrac{1}{2}(k'+m')_{l-j}\,\tfrac{1}{2}(\lambda^2)^j\tfrac{1}{2}(\lambda_1^2)^{l-j}}{(k/2)_j(k'/2)_{l-j}\,j!\,(l-j)!} \\
& \times G_{22}^{20}\left[\omega\,\middle|\,\begin{matrix}\tfrac{1}{2}(k+m)-1+j,\ \tfrac{1}{2}(k'+m')-1+l-j\\ \tfrac{1}{2}(k)-1+j,\ \tfrac{1}{2}(k')+l-j-1\end{matrix}\right],
\end{aligned} \tag{32}
$$

where $0 < w < 1$.

4.4. *Products of independent simple correlation coefficients.* The density function of the square of a simple correlation coefficient y can be written in the form

$$
\begin{aligned}
f_4(y) = {} & (2^{n-4}/\pi(n-3)!)(1-\rho^2)^{\frac{1}{2}(n-1)}y^{-\frac{1}{2}}(1-y)^{\frac{1}{2}(n-4)} \\
& \times \textstyle\sum_{s=0}^{\infty}\Gamma^2\{(n+s-1)/2\}\,(2\rho y^{\frac{1}{2}})^s/s!, \qquad 0 < y < 1.
\end{aligned} \tag{33}
$$

The density function of W, which is the product of two independent simple correlation coefficients with different parameters, follows as a particular case of

(21) and can be put in the form

$$h_3(w) = 2^{n+n'-\delta}\Gamma\, \tfrac{1}{2}(n-2)\Gamma\, \tfrac{1}{2}(n'-2)(1-\rho^2)^{\frac{1}{2}(n-1)}(1-\rho_1^{\,2})^{\frac{1}{2}(n'-1)}/ [\pi^2(n-3)!(n'-3)!]$$

$$\times \sum_{m=0}^{\infty} \sum_{j=0}^{m} \cdot\Gamma^2(n+j-1/2)\Gamma^2(n'+m-j-1/2)(2\rho)^j(2\rho_1)^{m-j}/ [m!(m-j)!] \tag{34}$$

$$\times G_{22}^{20}\left[\omega \middle| \begin{matrix} \tfrac{1}{2}(n-3+j), \tfrac{1}{2}(m+n'-j-3) \\ \tfrac{1}{2}(j-1), \tfrac{1}{2}(m-j-1) \end{matrix}\right],$$

where $0 < w < 1$.

The authors are thankful to the referee for giving some useful suggestions in the improvement of this paper.

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