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ON THE ASYMPTOTIC DISTRIBUTION OF A CERTAIN FUNCTIONAL OF THE WIENER PROCESS

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0. Summary. Let the random variable Y_N be defined by

$$Y_N = \sum_{k=1}^N W^2(k)/k^2,$$

where $W(t)$ is the Wiener process, the Gaussian random process with mean zero and covariance $EW(s)W(t) = \min(s, t)$. Note that $EY_N \sim \log N$. We show that for $a > 1$

$$\Pr[Y_N \geqq a \log N] = N^{-(8a)^{-1}(a-1)^2(1+\epsilon_N)},$$

where $\epsilon_N \to 0$ as $N \to \infty$.

1. Introduction. The Wiener Process $W(t)$ is the Gaussian random process with mean zero and covariance

$$E[W(s)W(t)] = \min(s, t). \tag{1.1}$$

In this paper the random variable

$$Y_N = \sum_{k=1}^N W^2(k)/k^2 \tag{1.2}$$

is studied. Y_N is the signal energy in the celebrated feedback communication scheme of Schalkwijk and Kailath [3, 6]. Its expectation is

$$E(Y_N) = \sum_{k=1}^N 1/k \sim \log N, \qquad \text{as } N \to \infty. \tag{1.3}$$

We are concerned with

$$P_N = P_N(a) = \Pr[Y_N \geqq a \log N], \qquad a > 1. \tag{1.4}$$

It will be shown[1] that as $N \to \infty$,

$$P_N = \exp\{[-(a-1)^2/(8a)] \log N + o(\log N)\} = N^{-(8a)^{-1}(a-1)^2(1+\epsilon_N)} \tag{1.5}$$

where $\epsilon_N \to 0$.

In Sections 2 and 3 of this paper upper and lower bounds respectively on P_N are obtained, each bound of the form of (1.5).

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[1] The anonymous referee pointed out that we can write $Y_N = \sum_{i,j=1}^N A_{ij}\xi_i\xi_j$ where $\xi_1, \xi_2, \cdots, \xi_N$ are standard Gaussian variates, and $A_{ij} = \sum_{\min(i,j)}^N k^{-2}$. Thus Y_N is a quadratic form in normal variates. The relevent literature, e.g. Varberg, *Ann. Math. Statist.* **37** (1966), and Grenander, Pollak, Slepian, *SIAM J.*, **7** (1959), does not appear to facilitate the solution of our problem, however.

2. Upper bound on P_N. In this section we show that (for $a > 1$)

$$P_N = \Pr[Y_N \geqq a \log N]$$

$$\leqq \left[\frac{2a}{1+a}\right]^{\frac{1}{2}} \left[1 - \frac{a-1}{4aN}\right]^{-\frac{1}{2}} N^{-(a-1)^2/(8a)} \tag{2.1}$$

$$\sim [2a/(1+a)]^{\frac{1}{2}} N^{-(a-1)^2/(8a)},$$

which is of the same form as (1.5).

Let us consider the random variable

$$Y_N^* = \sum_{k=1}^N f_k W^2(k), \tag{2.2}$$

where $W(t)$ is as above the Wiener process, and f_k is arbitrary. Note that when $f_k = 1/k^2$, $Y_N^* = Y_N$. We now calculate $\varphi(\lambda) = Ee^{\lambda Y_N^*}$. From (1.1) it is easy to show that the N-fold density function for the samples $W(1), W(2), \cdots, W(N)$ of the Wiener process is $(2\pi)^{-N/2} \exp[-\frac{1}{2}\boldsymbol{w}B\boldsymbol{w}^t]$, where $\boldsymbol{w} = (w_1, w_2, \cdots, w_N)$ and B is the $N \times N$ matrix with i, jth entry b_{ij} given by

$$b_{ij} \begin{array}{ll} = 2 & i = j = 1, 2, \cdots, N-1, \\ = 1 & i = j = N, \\ = -1 & |i-j| = 1, \\ = 0 & \text{otherwise.} \end{array} \tag{2.3}$$

Letting M be the $N \times N$ diagonal matrix with i, ith entry $-2\lambda f_i (i = 1, 2, \cdots, N)$, we have

$$\varphi(\lambda) = Ee^{\lambda Y_N^*} = (2\pi)^{-N/2} \int \exp[-\tfrac{1}{2}\boldsymbol{w}(M+B)\boldsymbol{w}^t]\, d\boldsymbol{w}, \tag{2.4}$$

where the integral is taken over all of N-space. Thus, provided $M + B$ is positive definite,

$$\varphi(\lambda) = |M+B|^{-\frac{1}{2}} \int (2\pi)^{-N/2} |M+B|^{\frac{1}{2}} \exp[-\tfrac{1}{2}\boldsymbol{w}(M+B)\boldsymbol{w}^t]\, d\boldsymbol{w} = |M+B|^{-\frac{1}{2}}. \tag{2.5}$$

Consider the difference equation

$$h_k = (2 - 2\lambda f_k) h_{k-1} - h_{k-2}, \qquad k = 2, 3, \cdots, \tag{2.6a}$$

subject to the initial conditions

$$h_0 = 1, \qquad h_1 = (2 - 2\lambda f_1). \tag{2.6b}$$

Note that h_k, $1 \leqq k \leqq N - 1$, equals the determinant of the submatrix of $M + B$ consisting of the first k rows and columns. Further,

$$|M + B| = h_N - h_{N-1}. \tag{2.7}$$

Also note that the condition that $M + B$ be positive definite is equivalent to $h_k > 0 (k = 1, 2, \cdots, N-1)$ and $|M + B| > 0$.

Although for the case $f_k = 1/k^2$, it is possible to express the generating function $\sum_{k=0}^{\infty} h_k x^k$ of the solution to (2.6) in terms of hypergeometric functions [2], this approach does not appear to facilitate the solution of our problem. We therefore take the following approach: with $\lambda \geqq 0$ fixed we "guess" at an $h_k(k = 0, 1, 2, \cdots)$ and employ (2.6) backwards to find $f_k(k = 1, 2, \cdots)$. If we are lucky and $f_k \geqq 1/k^2(k = 1, 2, \cdots)$, then

$$Y_N^* \geqq Y_N, \tag{2.8}$$

so that

$$P_N = \Pr[Y_N \geqq a \log N] \leqq \Pr[Y_N^* \geqq a \log N]. \tag{2.9}$$

We then make use of the well known Chebyshev-type inequality

$$\begin{aligned} \Pr[Y_N^* \geqq a \log N] &\leqq \exp(-\lambda a \log N) E e^{\lambda Y^*} \\ &= \exp(-\lambda a \log N)\varphi(\lambda) \quad (\lambda > 0) \end{aligned} \tag{2.10}$$

to obtain a bound on P_N. To get the tightest bound we minimize the right member of (2.10) with respect to λ (making sure, of course, that f_k remains $\geqq 1/k^2$ so that (2.8) will hold).

Let λ, $0 < \lambda < \frac{1}{8}$, be fixed (it will be shown below that the tightest bound in (2.10) is always obtained with λ in this range). Let us take

$$h_k = (k + 1)^\alpha, \qquad \alpha = \tfrac{1}{2}(1 + (1 - 8\lambda)^{\frac{1}{2}}. \tag{2.11}$$

Note that with h_k so defined, $M + B$ is positive definite. Certainly $h_0 = 1$, and if

$$f_1 = (2 - 2^\alpha)/(2\lambda), \tag{2.12}$$

then the second initial condition (2.6b) is also satisfied. Finally, if for $k \geqq 2$,

$$\begin{aligned} f_k &= (h_k - 2h_{k-1} + h_{k-2})/((-2\lambda)h_{k-1}) \\ &= [(1 + 1/k)^\alpha - 2 + (1 - 1/k)^\alpha]/(-2\lambda), \end{aligned} \tag{2.13}$$

the difference equation (2.6a) is also satisfied. We now show that $f_k \geqq 1/k^2(k = 1, 2, \cdots,)$. First consider f_1:

$$\begin{aligned} f_1 &= (2 - 2^\alpha)/(2\lambda) = (1 - \exp[-(1 - \alpha) \log 2])/\lambda \\ &\geqq [(1 - \alpha) \log 2]\, (\alpha(1 - \alpha)/2)^{-1}\, (1 - [(1 - \alpha) \log 2]/2) \\ &= 2 \log 2[\alpha^{-1}(1 - (\log 2)/2) + (\log 2)/2] \\ &\geqq 2 \log 2 = 1.38 > 1, \end{aligned} \tag{2.14}$$

where the first inequality follows from $1 - e^{-x} \geqq x - x^2/2$, and the second from the fact that $\alpha < 1$. Now consider $f_k(k \geqq 2)$ as given by (2.13). Using the binomial formula for $(1 \pm 1/k)^\alpha$ we obtain

$$\begin{aligned} f_k = 2/(-2\lambda)[&\alpha(\alpha - 1)/(2!k^2) + \alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)/ \\ &(4!k^4) + \alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)(\alpha - 5)/ \\ &(6!k^6) + \cdots]. \end{aligned} \tag{2.15}$$

Now by definition of α (2.11), $\alpha(\alpha - 1) = -2\lambda$ so that

$$(2.16)\quad f_k = 1/k^2 + [2(\alpha-2)(\alpha-3)]/(4!k^4) + [2(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)]/(6!k^6) + \cdots .$$

Since $\frac{1}{2} < \alpha < 1$, the coefficients of $1/k^{2j} (j \geqq 1)$ are all positive, so $f_k \geqq 1/k^2$ (for any $\lambda (0 < \lambda < \frac{1}{8})$). For this choice of f_k,

$$(2.17)\quad \begin{aligned} |M + B| &= h_N - h_{N-1} = (1+N)^\alpha - N^\alpha \\ &= N^\alpha[(1 + 1/N)^\alpha - 1] \\ &\geqq N^\alpha[\alpha/N + \alpha(\alpha-1)/(2N^2)]. \end{aligned}$$

Thus

$$(2.18)\quad \varphi(\lambda) = |M+B|^{-\frac{1}{2}} \leqq N^{(1-\alpha)/2}[\alpha + (\alpha(\alpha-1))/(2N)]^{-\frac{1}{2}}.$$

To determine the best choice of λ we minimize the upper bound of (2.10):

$$(2.19)\quad \varphi(\lambda) \exp(-\lambda a \log N) = N^{[(1-\alpha)/2]-\lambda a+\epsilon_N}.$$

where $\epsilon_N \to 0$ as $N \to \infty$. Writing $\lambda = \frac{1}{2}\alpha(1-\alpha)$ we have immediately that the minimum is obtained when $\alpha = (1+a)/(2a)$. Note that with α so chosen, $0 < \lambda < \frac{1}{8}$ as required. Substitution of (2.18) and (2.10) into (2.9) yields the desired bound on P_N (2.1).

3. Lower bound on P_N. In this section we show that

$$(3.1a)\quad P_N = \Pr[Y_N \geqq a \log N] \geqq \exp[-E_0(a)\log N + o(\log N)],$$

where

$$(3.1b)\quad E_0(a) = (a-1)^2/(8a).$$

3.1. *Outline of derivation.* For $K = 1, 2, \cdots, N$, let us define the random variable

$$(3.2)\quad Y_{N,K} = \textstyle\sum_{k=K}^{N} W^2(k)/k^2.$$

Since $Y_{N,K} \leqq Y_N$,

$$(3.3)\quad P_N = \Pr[Y_N \geqq a \log N] \geqq \Pr[Y_{N,K} \geqq a \log N],$$

so that it will suffice to lower bound this later quantity. We will also consider the random variable (for $K = 1, 2, \cdots, N$)

$$(3.4)\quad \hat{Y}_{N,K} = \textstyle\int_K^N W^2(t)/t^2\, dt.$$

Our strategy is to first show that $\Pr[\hat{Y}_{N,K} \geqq a \log N] \geqq \exp[-E_0(a)\log N + o(\log N)]$. We then show that $\Pr[Y_{N,K} \geqq a \log N]$ is "close" to $\Pr[\hat{Y}_{N,K} \geqq a \log N]$ so that (3.1) follows from (3.3). Specifically we shall prove the following:

LEMMA 1. *Let $\hat{Y}_{N,K}$ be defined by* (3.4). *Then with K arbitrary but fixed, and $a > 1$,*

$$(3.5)\quad \Pr[\hat{Y}_{N.K} \geqq a \log N] \geqq \exp[-E_0(a)\log N + o(\log N)], \text{ as } N \to \infty,$$

where $E_0(a)$ is defined by (3.1b).

LEMMA 2. *Let* $Z_{N,K} = \hat{Y}_{N,K} - Y_{N,K}$, *where* $\hat{Y}_{N,K}$ *is defined by* (3.4) *and* $Y_{N,K}$ *is defined by* (3.2). *Then for any* $\delta, \Lambda > 0$ *there exists a* $K = K(\delta, \Lambda)$ *sufficiently large and a constant* $c_0 = c_0(\delta, \Lambda)$ *such that for* $N \geqq K$,

$$\Pr[Z_{N,K} \geqq \delta \log N] \leqq c_0 \exp(-\Lambda \log N). \tag{3.6}$$

LEMMA 3. *Let* $Y_{N,K}$, $\hat{Y}_{N,K}$, $Z_{N,K}$ *be as in Lemmas* 1 *and* 2. *Then for any* a, $\delta > 0$,

$$\Pr[Y_{N,K} \geqq a \log N] \geqq \Pr[\hat{Y}_{N,K} \geqq (a + \delta) \log N] - \Pr[Z_{N,K} > \delta \log N]. \tag{3.7}$$

Our final goal (3.1) now follows directly from these lemmas. Let $\delta > 0$ be fixed. From Lemma 2 we choose K large enough so that

$$\Pr[Z_{N,K} \geqq \delta \log N] \leqq c_0 \exp[-2E_0(a + \delta) \log N], \tag{3.8}$$

where $E_0(a + \delta)$ is defined by (3.1b). With K so chosen we have from Lemma 1

$$\Pr[\hat{Y}_{N,K} \geqq (a + \delta) \log N] \geqq \exp[-E_0(a + \delta) \log N + o(\log N)]. \tag{3.9}$$

Thus as $N \to \infty$, the entire right member of (3.7) is dominated by the first term, so that Lemma 3 yields

$$P_N \geqq \Pr[Y_{N,K} \geqq a \log N] \geqq \exp[-E_0(a + \delta) \log N + o(\log N)]. \tag{3.10}$$

Since this is true for all $\delta > 0$, we have

$$\liminf_{N\to\infty} \log P_N/\log N \geqq -E_0(a + \delta) \to -E_0(a) \quad \text{as } \delta \to 0, \tag{3.11}$$

from which (3.1) follows immediately.

Thus it remains to prove Lemmas 1–3. Before doing so we will state two additional lemmas due respectively to L. A. Shepp[2] and C. E. Shanon.[3] We prove Lemmas 1–3 in Section 3.2.

LEMMA 4. (Shepp): *Let* μ *be a signed measure (i.e., the difference of two measures) on* $[0, T]$ *such that*[4] $\int_0^T d|\mu(t)| < \infty$, *and let*

$$A = E(\exp[-\tfrac{1}{2}\textstyle\int_0^T W^2(t)\, d\mu(t)]).$$

Consider the solution $g(x)$ *(which always exists) of the integral equation*[5]

$$g(x) = 1 + \int_x^T (t - x) g(t)\, d\mu(t), \qquad 0 \leqq x \leqq T. \tag{3.12}$$

If $g(x) > 0$, $0 \leqq x \leqq T$, *then* $A = (g(0))^{-\frac{1}{2}}$.

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[2] Reference [5], Section 18.

[3] Reference [4]. Similar results can be found in Ref. [1]. Since this lemma is not available in the literature, a proof is given in the appendix.

[4] Let $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are measures. Then $d|\mu| = d\mu^+ + d\mu^-$.

[5] $\int_a^b f(t)\, d\mu(t)$ will be taken as $\int_{a-}^{b+} f(t)\, d\mu(t)$ throughout this paper.

Lemma 4 immediately yields two corollaries.

COROLLARY 1. *Let* $\lambda > 0$ *and let*

$$B = E\{\exp [\lambda \textstyle\int_K^N W^2(t)\, d\mu(t)]\},$$

where $1 \leqq K \leqq N$, *and* μ *is a signed measure such that* $\int_K^N d|\mu(t)| < \infty$. *Consider the solution* $g(x)$ *(which always exists) of the integral equation*

$$g(x) = 1 - 2\lambda \textstyle\int_x^N (t - x) g(t)\, d\mu(t), \qquad K \leqq x \leqq N \tag{3.13}$$

If

$$g(x) > 0, \qquad K \leqq x \leqq N,$$

and

$$\textstyle\int_K^N g(t)\, d\mu(t) < g(K)/(2\lambda K),$$

then

$$B = [g(K) - 2\lambda K \textstyle\int_K^N g(t)\, d\mu(t)]^{-\frac{1}{2}}.$$

COROLLARY 2. *Let* $\hat{\varphi}_{N,K} = E \exp (\lambda \hat{Y}_{N,K})$ *where* $\hat{Y}_{N,K}$ *is defined by* (3.4), *and* $0 < \lambda < \frac{1}{8}$. *Then*

$$\hat{\varphi}_{N,K}(\lambda) = [(\alpha_+ - \alpha_-)^{-1}(\alpha_+^{\,2}(K/N)^{\alpha_-} - \alpha_-^{\,2}(K/N)^{\alpha_+})]^{-\frac{1}{2}}, \tag{3.14}$$

where $\alpha_\pm = \frac{1}{2}(1 \pm (1 - 8\lambda)^{\frac{1}{2}})$.

Note that Corollary 2 follows directly from Corollary 1 on substituting $d\mu(t) = t^{-2}\, dt$ and observing that the integral equation (3.13) is equivalent to the differential equation $g''(x) = -2\lambda t^{-2} g(x)$, $K \leqq x \leqq N$, subject to $g(N) = 1$, $g'(N) = 0$.

Finally we state the Shannon result:

LEMMA 5. (Shannon). *Let* X *be a random variable and let* $\gamma(\lambda) = \log E e^{\lambda X} (\lambda > 0)$. *Then for any* $\xi > 0$,

$$\begin{aligned} \Pr [X \geqq \gamma'(\lambda) - \xi(\gamma''(\lambda))^{\frac{1}{2}}] \\ \geqq (1 - \xi^{-2}) \exp [\gamma(\lambda) - \lambda\gamma'(\lambda) - \lambda\xi(\gamma''(\lambda))^{\frac{1}{2}}]. \end{aligned} \tag{3.15}$$

3.2. *Proofs of Lemmas* 1–3.

3.2.1. *Proof of Lemma* 1. We shall use Corollary 2 to Lemma 4 which gives $\hat{\varphi}_{N,K}(\lambda) = E \exp (\lambda \hat{Y}_{N,K})$, and then use Lemma 5 to obtain Lemma 1. Letting the random variable X in Lemma 5 be $\hat{Y}_{N,K}$, a direct computation yields for $0 < \lambda < \frac{1}{8}$ and for fixed K (as $N \to \infty$):

$$\begin{aligned} \gamma(\lambda) &= \log \hat{\varphi}_{N,K}(\lambda) = \tfrac{1}{4}[1 - (1 - 8\lambda)^{\frac{1}{2}}] \log N + o(\log N), \\ \gamma'(\lambda) &= (1 - 8\lambda)^{-\frac{1}{2}} \log N + o(\log N), \\ \gamma''(\lambda) &= O(\log N). \end{aligned} \tag{3.16}$$

Thus from Lemma 5, for fixed $\xi > 0$ (with $\beta = (1 - 8\lambda)^{\frac{1}{2}}$,

$$\begin{aligned} \Pr [\hat{Y}_{N,K} \geqq \beta^{-1} \log N - o(\log N)] \\ \geqq (1 - 1/\xi^2) \exp \{-[(1 - \beta)^2/(8\beta)] \log N + o(\log N)\}. \end{aligned} \tag{3.17}$$

If we set $1/\beta = a > 1$ (which corresponds to $\lambda = ((a^2 - 1)/a^2)\frac{1}{8} < \frac{1}{8}$ as required) (3.17) becomes

$$(3.18)\quad \Pr[\hat{Y}_{N,K} \geqq a \log N - o(\log N)] \geqq \exp[-(a-1)^2/(8a)] \log N + o(\log N).$$

To obtain Lemma 1, rewrite (3.18) as

$$(3.19)\quad \Pr[\hat{Y}_{N,K} \geqq a \log N(1 + \epsilon_{1N})] \geqq \exp[-E_0(a) \log N(1 + \epsilon_{2N})],$$

where ϵ_{1N}, $\epsilon_{2N} \to 0$ as $N \to \infty$. Lemma 1 then follows on replacing a by $a/(1 + \epsilon_{1N})$, and observing that $E_0(a/(1 + \epsilon_{1N})) = E_0(a)(1 + \epsilon_{3N})$ (where $\epsilon_{3N} \to 0$ as $N \to \infty$).

3.2.2. *Proof of Lemma* 2. Let $\delta, \Lambda > 0$ be given. Let $\lambda = \Lambda/\delta$. We will show that there exists a $K = K(\lambda)$ sufficiently large and a $c_0 < \infty$ such that for all $N \geqq K$, $E \exp(\lambda Z_{N,K}) \leqq c_0$. Thus, (as in (2.10))

$$(3.20)\quad \Pr[Z_{N,K} \geqq \delta \log N] \leqq \exp[-\lambda(\delta \log N)] E \exp(\lambda Z_{N,K}) \leqq c_0 \exp(-\Lambda \log N),$$

and we have proved the lemma.

Now note that

$$(3.21)\quad Z_{N,K} = \hat{Y}_{N,K} - Y_{N,K} = \int_K^N W^2(t)t^{-2}\, dt - \sum_{k=K}^N W^2(k)k^{-2} = \int_K^N W^2(t)\, d\mu_0(t),$$

where the signed measure $\mu_0 = \mu_0^+ - \mu_0^-$, where $d\mu_0^+(t) = t^{-2}\, dt(1 \leqq t < \infty)$, and μ_0^- assigns measure $1/k^2$ to $t = k$ and zero elsewhere ($k = 1, 2, \cdots$). Thus we can use Corollary 1 to Lemma 4 to estimate $E \exp(\lambda Z_{N,K})$. We will now prove a proposition about the solution to (3.13).

PROPOSITION. *Let μ be a signed measure on the interval $(1, \infty)$ which is the difference of two finite measures, and for which there exists constants a_0, b_0, $k_0 \geqq 0$ such that for $x \geqq k_0$ and all $T(x \leqq T < \infty)$:*

$$(3.22a)\quad \text{(i)}\quad \int_x^T d|\mu(t)| < a_0/x,$$

$$(3.22b)\quad \text{(ii)}\quad \left|\int_x^T d\mu(t)\right| < b_0/x^2.$$

Let $g(x) = g_{N,K}(x)$ be the solution (which always exists) of the integral equation (3.13), i.e.,

$$(3.23)\quad g(x) = 1 - 2\lambda \int_x^N (t - x)g(t)\, d\mu(t), \qquad K \leqq x \leqq N.$$

Then for any $\epsilon > 0$, there exists a $K = K(\epsilon) > 0$ sufficiently large so that for all $N \geqq K$,

$$(3.24a)\quad \text{(i)}\quad g_{N,K}(x) \geqq 1 - \epsilon, \qquad K \leqq x \leqq N,$$

$$(3.24b)\quad \text{(ii)}\quad g_{N,K}(K) - 2\lambda K \int_K^N g(t)\, d\mu(t) \geqq 1 - \epsilon.$$

It is readily verified that the signed measure μ_0 satisfies the hypotheses of the proposition. Hence, the proposition and Corollary 1 to Lemma 4 imply that for any $\lambda > 0$, $E \exp(\lambda Z_{N,K}) \to 1$ as $K \to \infty$ (uniformly in $N \geqq K$). Thus (3.20) is valid with any $c_0 > 1$ and Lemma 2 is proved.

Proof of the Proposition. First note that from (3.23), $g(x)$ is differentiable and

$$g'(x) = 2\lambda \int_x^N g(t)\, d\mu(t), \qquad K \leqq x \leqq N. \tag{3.25}$$

Next, define for $N > 0$ and $1 \leqq t \leqq N$

$$\alpha_N(t) = \int_t^N d\mu(\tau), \tag{3.26}$$

and for $N > 0$ and $1 \leqq x \leqq t \leqq N$,

$$\beta_N(t, x) = \int_t^N (\tau - x)\, d\mu(\tau). \tag{3.27}$$

Integrating (3.27) by parts we have

$$\beta_N(t, x) = -\int_t^N (\tau - x)\, d\alpha_N(\tau) = (t - x)\alpha_N(t) + \int_t^N \alpha_N(\tau)\, d\tau.$$

Thus from (3.22b), for $t \geqq k_0$,

$$|\alpha_N(t)| \leqq b_0/t^2, \qquad \text{and} \quad |\beta_N(t, x)| \leqq 2b_0/t. \tag{3.28}$$

Now rewriting the integral equation (3.23) as

$$g(x) = 1 + 2\lambda \int_x^N g(t)[\partial\beta_N(t, x)/(\partial t)]\, dt,$$

and integrating parts, we obtain

$$g(x) = 1 - 2\lambda g(x)\beta_N(x, x) - 2\lambda \int_x^N \beta_N(t, x) g'(t)\, dt. \tag{3.29}$$

Thus

$$|g(x)| \leqq 1 + 2\lambda |g(x)||\beta_N(x, x)| + 2\lambda \int_x^N |\beta_N(t, x)||g'(t)|\, dt. \tag{3.30}$$

But from (3.25)

$$|g'(t)| \leqq 2\lambda \int_t^N |g(t)|\, d|\mu(t)| \leqq 2\lambda M \int_t^N d|\mu(t)|, \tag{3.31}$$

where $M = \sup_{K \leqq x \leqq N} g(x)$. Combining (3.30) and (3.31) we have

$$|g(x)| \leqq 1 + 2\lambda M|\beta_N(x, x)| + 4\lambda^2 M \int_x^N |\beta_N(t, x)| \left[\int_t^N d|\mu(\tau)|\right] dt.$$

Finally from (3.22a) and (3.28) we have, if $K \geqq k_0$.

$$M \leqq 1 + 4\lambda M b_0/K + 8\lambda^2 M b_0 a_0/K = 1 + \gamma M/K, \tag{3.32}$$

where $\gamma = 4\lambda b_0 + 8\lambda^2 b_0 a_0$. Solving (3.32) for M yields

$$M = \sup_{K \leqq x \leqq N} |g(x)| \leqq (1 - \gamma/K)^{-1} =_{\text{def}} B(K), \tag{3.33}$$

provided $K \geqq \gamma, k_0$.

Returning to (3.29) we can write

$$|g(x) - 1| \leqq 2\lambda |g(x)||\beta_N(x, x)| + 2\lambda \int_x^N |\beta_N(t, x) g'(t)|\, dt.$$

Repeating the same steps as in the derivation of (3.32) we obtain when $K \geqq k_0$,

$$|g(x) - 1| \leqq 4\lambda M b_0/K + 8\lambda^2 M b_0 a_0/K = \gamma M/K.$$

If in addition $K \geqq \gamma$ we can apply (3.33) to obtain

$$|g(x) - 1| \leqq \gamma B(K)/K \to 0, \quad \text{as } K \to \infty.$$

This implies (3.24a) and the first part of the proposition is proved.

To establish (3.24b), write

$$g(K) - 2\lambda K \int_K^N g(t)\, d\mu(t) = 1 - 2\lambda \int_K^N t g(t)\, d\mu(t).$$

Using $\int_t^N \tau d\mu(\tau)$ instead of $\beta_N(t, x)$ and paralleling the derivation of (3.32) we have (if $K \geqq \gamma, k_0$)

$$|g(K) - 2\lambda K \int_K^N g(t)\, d\mu(t) - 1| \leqq \gamma M/K \leqq \gamma B(K)/K \to 0, \qquad \text{as} \quad K \to \infty,$$

which implies (3.24b) and the proposition.

3.2.3. *Proof of Lemma* 3. For any random variables $Y_{N,K}$, $\hat{Y}_{N,K}$, and for any $a, \delta > 0$

$$\begin{aligned}\Pr[\hat{Y}_{N,K} \geqq (a + \delta) \log N] &= \Pr[\hat{Y}_{N,K} \geqq (a + \delta) \log N,\ Y_{N,K} \geqq a \log N] \\ &\quad + \Pr[\hat{Y}_{N,K} \geqq (a + \delta) \log N,\ Y_{N,K} < a \log N] \\ &\leqq \Pr[Y_{N,K} \geqq a \log N] + \Pr[Y_{N,K} - Y_{N,K} \\ &\geqq \delta \log N].\end{aligned}$$

Setting $Z_{N,K} = \hat{Y}_{N,K} - Y_{N,K}$, this is Lemma 3.

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APPENDIX

A.1. Proof of Lemma 5.

LEMMA 5. *Let X be a random variable and $\gamma(\lambda) = \log Ee^{\lambda X} (\lambda > 0)$. Then for any $\xi > 0$,*

$$\Pr[X \geqq \gamma'(\lambda) - \xi(\gamma'(\lambda))^{\frac{1}{2}}] \geqq (1 - \xi^{-2}) \exp[\gamma - \lambda\gamma'(\lambda) - \lambda\xi(\gamma''(\lambda))^{\frac{1}{2}}].$$

PROOF. (Shannon). Let X have distribution function $F(x)$ and let $\varphi(\lambda) = Ee^{\lambda X}$. Define a new random variable $\hat{X}$ with distribution function $G(x) = G(x, \lambda)$ given by

$$G(x) = \varphi(\lambda))^{-1} \int_{-\infty}^{x} e^{\lambda y}\, dF(y), \tag{A1}$$

for all $\lambda > 0$ such that $\varphi(\lambda) < \infty$. Note that $dG(x)/dF = e^{\lambda x}/\varphi(\lambda)$. Let $\Psi(s) = E \exp(s\hat{X})$ be the moment generating function for $\hat{X}$. Then

$$\Psi(s) = \int_{-\infty}^{\infty} e^{sx}\, dG(x) = \int_{-\infty}^{\infty} (\varphi(\lambda))^{-1} e^{sx} e^{\lambda x}\, dF(x) = \varphi(s + \lambda)/\varphi(\lambda). \tag{A2}$$

We can then compute the moments of $\hat{X}$:

$$\text{(A3)} \qquad E\hat{X} = \Psi'(0) = \varphi'(\lambda)/\varphi(\lambda) = d \log \varphi(\lambda)/d\lambda = \gamma'(\lambda),$$

$$\text{(A4)} \qquad E\hat{X}^2 = \Psi''(0) = \varphi''(\lambda)/\varphi(\lambda).$$

Hence the variance of $\hat{X}$ is

$$\text{(A5)} \quad \begin{aligned} \sigma^2\hat{X} &= E(\hat{X} - E\hat{X})^2 = E\hat{X}^2 - (E\hat{X})^2 \\ &= \varphi''(\lambda)/\varphi(\lambda) - [\varphi'(\lambda)/\varphi(\lambda)]^2 = d(\varphi'(\lambda)/\varphi(\lambda))/d\lambda = \gamma''(\lambda). \end{aligned}$$

Thus Chebycheff's inequality applied to $\hat{X}$ yields

$$\text{(A6)} \qquad \Pr [\beta_1 \leqq \hat{X} < \beta_2] \geqq 1 - \xi^{-2},$$

where

$$\text{(A7a)} \qquad \beta_1 = E\hat{X} - \xi\sigma\hat{X} = \gamma' - \xi(\gamma'')^{\frac{1}{2}},$$

$$\text{(A7b)} \qquad \beta_2 = E\hat{X} + \xi\sigma\hat{X} = \gamma' + \xi(\gamma'')^{\frac{1}{2}},$$

and $\xi > 0$. Thus

$$\text{(A8)} \quad \begin{aligned} \Pr [X > \beta_1] &= \textstyle\int_{\beta_1}^{\infty} dF(x) = \varphi(\lambda)\int_{\beta_1}^{\infty} e^{-\lambda x}\, dG(x) \\ &\geqq \varphi(\lambda)\textstyle\int_{\beta_1}^{\beta_2} e^{-\lambda x}\, dG(x) \geqq \varphi(\lambda)e^{-\lambda\beta_2} \Pr [\beta_1 \leqq \hat{X} \leqq \beta_2] \\ &\geqq \exp(\gamma - \lambda\beta_2)(1 - \xi^{-2}). \end{aligned}$$

Substitution of (A7) into (A8) yields the lemma.

REFERENCES

[1] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications,* **2** 517–520. Wiley, New York.

[2] Pollak, H. O. Private communication.

[3] Schalkwijk, J. and Kailath, T. (1966). A coding scheme for additive channels with feedback—part I: no bandwidth constraint, *IEEE Trans. Information Theory* **IT-12** 172–182.

[4] Shannon, C. E. (1956). Unpublished seminar notes in information theory.

[5] Shepp, L. A. (1966). Radon-Nikodym derivatives of gaussian measures. *Annals Math. Statist.* **37** 321–354.

[6] Wyner, A. D. (1968). On the Schalwijk-Kailath coding scheme with a peak energy constraint. *IEEE Trans. Information Theory.* **IT-14** 128–134.